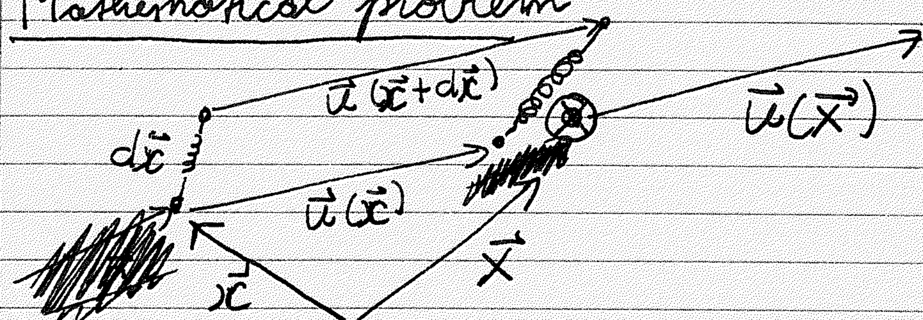
Finite Strain Elasticity Theory

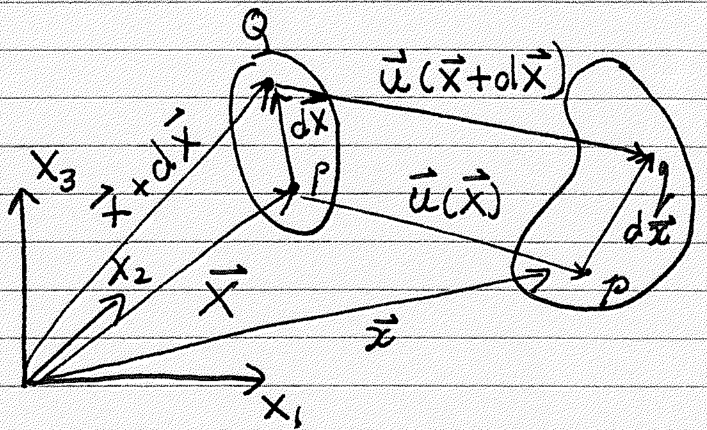
* If the strain tensor u_{ij} is of the order of one or larger, then classical elasticity theory is mathematically incorrect. This is to be distinguished from physical incorrectness due to the fact that higher order invariants must be included

* Mathematical problem



The stress $\overleftrightarrow{\sigma}$ in the region marked by x is not related to gradients of $\vec{u}(\vec{X})$ but instead to gradients of $\vec{u}(\vec{x})$ originating in mass points originally located far from $\vec{X}$.

* Define $\vec{u}(\vec{X})$ as the displacement field in the reference frame of the undeformed body



P : material point in undeformed body, located at $\vec{X}$

p : same material point in deformed body, located $\vec{X} + \vec{u}(\vec{X}) = \vec{x}$

Q : material point in undeformed body, located at $\vec{X} + d\vec{X}$

q : same material point in deformed body, located at $\vec{X} + d\vec{X} + \vec{u}(\vec{X} + d\vec{X}) = \vec{x} + d\vec{x}$

Use $\vec{x} = \vec{X} + \vec{u}(\vec{x})$. Define $d\vec{u} = \vec{u}(\vec{X} + d\vec{X}) - \vec{u}(\vec{X})$

$$d\vec{x} = d\vec{X} + \underbrace{d\vec{u}}_{\text{relative displacement}}$$

Expand

$$\vec{u}(\vec{X} + d\vec{X}) \cong \vec{u}(\vec{X}) + \underbrace{\overbrace{\nabla_X \vec{u}}}_{\frac{\partial u_i}{\partial X_J} \sim \text{ strain tensor}} \cdot d\vec{X}$$

$$d\vec{x} = d\vec{X} + d\vec{u}$$

$$= d\vec{X} + \vec{\nabla}_X \vec{u} \cdot d\vec{X}$$

$$= (\overleftrightarrow{I} + \vec{\nabla}_X \vec{u}) \cdot d\vec{X}$$

$$d\vec{x} = \overleftrightarrow{F}\, d\vec{X}$$

Deformation Gradient Tensor ~~metric tensor~~

Right Cauchy-Green deformation tensor

$$\boxed{\overleftrightarrow{C} = \overleftrightarrow{F}^T \overleftrightarrow{F}} \qquad C_{IJ} = \underbrace{\frac{\partial x_k}{\partial X_I} \frac{\partial x_k}{\partial X_J}}_{\text{Metric Tensor}}$$

$$\vec{x} \longrightarrow \vec{X}$$

$$d\vec{x}^2 = d\vec{X}\, \overleftrightarrow{C}\, d\vec{X}$$

Invariants of $\overleftrightarrow{C}$ $(d=2)$

$\mathrm{Tr}\,\overleftrightarrow{C}$, $\mathrm{Det}\,\overleftrightarrow{C}$, $\frac{1}{2}\left(\mathrm{Tr}\,\overleftrightarrow{C}^2 - (\mathrm{Tr}\,\overleftrightarrow{C})^2\right)$ ...

* $\overleftrightarrow{E} = \frac{1}{2}(\overleftrightarrow{C} - \overleftrightarrow{I})$ strain Tensor * ("Green - Lagrange" Strain Tensor

$$E_{KL} = \frac{1}{2}\left(\frac{\partial u_K}{\partial x_L} + \frac{\partial u_L}{\partial x_K} + \frac{\partial u_M}{\partial x_K}\frac{\partial u_M}{\partial x_L} \right)$$

Let λ_1^2 & λ_2^2 be the eigenvalues of $\overleftrightarrow{C}$

* Show that $(\mathrm{Det}\,\overleftrightarrow{C})^{1/2} - 1 = \lambda_1\lambda_2 - 1$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{J}$

measures area changes. (In general $dA' = \det \overleftrightarrow{F}\, dA$)

* Show that $\dfrac{\mathrm{Tr}\,\overleftrightarrow{C}}{\mathrm{Det}\,\overleftrightarrow{C}^{1/2}} - 2 = \dfrac{\lambda_1}{\lambda_2} + \dfrac{\lambda_2}{\lambda_1} - 1$

measures shear.

Cauchy Stress Tensor

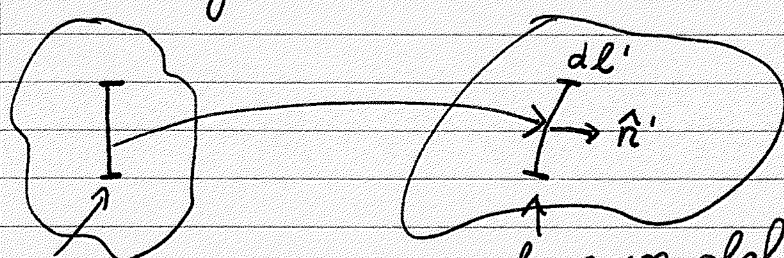
Elastic Energy

$F = 2\mu E_{ij}^2 + \lambda E_{ll}^2$ "Saint - Venant"

E_{ij} = Green - Lagrange Strain Analog of usual strain tensor

$S_{ij} = \partial F / \partial E_{ij} = 2\mu E_{ij} + \lambda E_{ll} \delta_{ij}$

"Cauchy Stress Tensor"



line in reference system

line in deformed system

$\overleftrightarrow{S} \cdot \hat{n}'$ is the force/unit length in deformed configuration on deformed line element