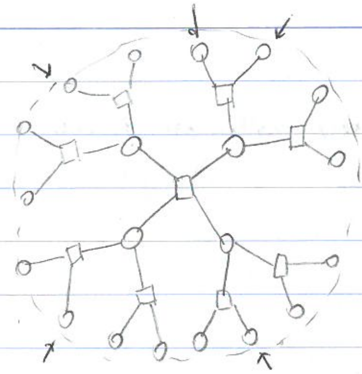


locall neighborhood R:

propagate boundary messages $\rightarrow$ measure $\mu_R(\underline{x}_R)$

↳ assumption of looking at neighborhood of distance $R-1$, entering boundary computed from μ_R through BP — $\equiv$ messages at boundaries indep.

↳ assumption, internal measure well approximated by tree.

WRONG ASSUMPTION when: → loopy graphs (Yedidia) - GRAPHICAL MODEL APPROX - CLUSTER VARIATIONAL METHODS (Kikuchi)

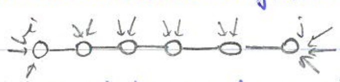
→ long range correlations (long loops)

SPARSE RANDOM GRAPHS

When getting long range correlations, what should we do with BP?

First, what correlations to measure?

→ connected correlations from distant variables: $\langle x_i x_j \rangle_c \equiv \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$



in 1d correlations always decay: $C(r) = \langle x_i x_j \rangle_c \big|_{|i-j|=r} \sim e^{-r/\xi}$

But the number of neighbors increases with distance: $\chi_F = \sum_i \chi_{i0} = \sum_r C(r)\, d(d-1)^{r-1}$ (d random regular graph)

$$= \frac{d}{d-1} \sum_r \exp\left(-\frac{r}{\xi} + r \ln(d-1)\right)$$

↳ criticality for finite correlation length: $\xi_c = \frac{1}{\ln d-1}$ — !

20/07/2017

Define measure of how i and j are correlated:

$\chi^{(2)} = \frac{1}{N} \sum_{ij} \| \mu_{ij}(\cdot,\cdot) - \mu_i(\cdot)\mu_j(\cdot) \|$ → norm over probability measures

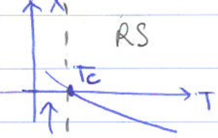
$\Rightarrow \chi^{(n)} = \frac{1}{N^{n-1}} \sum_{(i_1 \dots i_n) \in V} \| \mu_{i_1 \dots i_n}(\,) - \mu_{i_1} \cdots \mu_{i_n} \|$ eg: ℓ_1-"Frobenius" $\| A(x_i, x_j) \| = 1/2 \sum_{x_i x_j} |A(x_i, x_j)|$.

The model (BP) is stable to small perturbation if for all finite n, $\frac{\chi^{(n)}}{N} \to 0$ as $N \to \infty$

→ divergence of susceptibility ↔ instability of BP fixed points

In practice, should look at all $\chi^{(n)}$, but instead typically look at $\chi^{(2)}$ only (instability featured in pairwise models mainly)

↳ perturbating the fixed point: $\nu^* + \delta\nu$ → rate of decay? $\lambda = \lim_{t \to \infty} \frac{1}{t} \| \delta\nu \|$ ← exponential decay



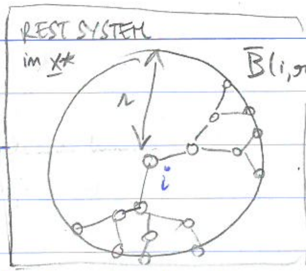
BP stops converging

can be measured by population dynamics

Instability making the RS ansatz incorrect.

↳ POINT-TO-SET CORRELATION FUNCTION

Even if BP fixed points are stable, the system can be creating some kind of order in a different way

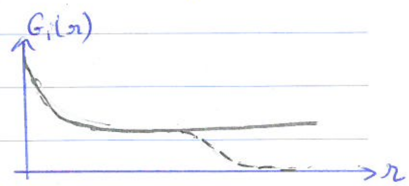


$\bar{B}(i,r) = \{ j : |i-j| \geq r \}$

Even if the root is not correlated with any finite set of variables, could be with the entirety of the boundary at $R \to +\infty$: divergent number of variables.

we want to look at: $G_i(r) = \| \mu_{i,\bar{B}(i,r)}(\cdot,\cdot) - \mu_i(\cdot)\mu_{\bar{B}(i,r)}(\cdot) \|$

And seperate the cases where it decays or not with distance:



first experiment: Take system outside ball fixed x^*, then look inside the ball $\mu(\underline{x}_B | \underline{x}_{B^*})$
$\rightarrow$ if $x_i \sim \mu(x_B | x_{B^*}) \simeq x_i^*$?

second experiment: Take again equilibrium x^*, delete everything inside Ball, can you guess x_i^* ?

in practice the same ...

The model is called non-reconstructible or extremal, if $G_i(r) \xrightarrow{r \to +\infty} 0 \ \forall i$.

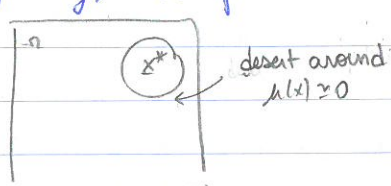
So in the end we have two different long range correlations types $\rightarrow$ different phases?

Phases: REPLICA SYMMETRIC (RS): both correlations $\chi^{(n)}$ and $G_i(r)$ decay!

DYNAMICAL (d1RSB) $T_K < T < T_d$: stable to small perturbation, $\chi^{(n)} < +\infty$, stable BP fixed point. reconstructible $G_i(r) \to$ cste.

csq: $\rightarrow$ any local algorithm will have great trouble jumping barriers: size of rea(rr)angments diverging!
$\hookrightarrow$ breaking of ergodicity, stuck dynamics (slow to visit phase space..)

SPACE OF CONFIGURATIONS



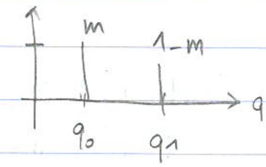
$\rightarrow$ as the system is stable under small perturbation $\rightarrow$ can show that there must be exponentially many state: add one constraint
- killing finite fraction of solutions
- must have been exponentially many for thermodynamic observables to remain OK.

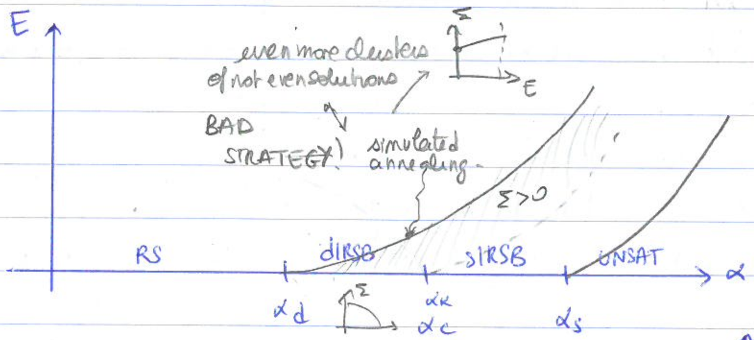
replica jargon: 1RSB $m=1$. $q = \begin{cases} q_0 & \text{with proba } 1-e^{-N\Sigma} \\ q_1 & \text{with proba } e^{-N\Sigma} \end{cases}$ $\sim P(q) \sim \delta(q-q_0)$

$\hookrightarrow$ weights of clusters: $W_\gamma = e^{-N\Sigma}$ $\rightarrow \sum_\gamma W_\gamma^2 = e^{-N\Sigma} \leftrightarrow$ proba $e^{-N\Sigma}$

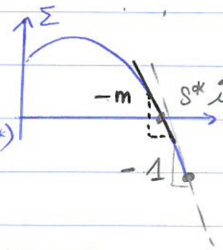
STATIC 1RSB (s1RSB) $T < T_K$: both correlations do not decay
BP is making wrong prediction csq $\rightarrow \Sigma = 0$
measure concentrates on $O(1)$ clusters.

→ weights of each cluster γ: $\sum_\gamma W_\gamma^2 = 1-m$ →





generic picture, can change according to model



↳ interpretation of m: $W(\#sol = e^{NS}) = e^{N\Sigma(s)}$ s^* in phase sRSB –

$\mathcal{Z}(m) = \sum_{\gamma} e^{NS_\gamma m} = \int ds\, e^{Nsm + N\Sigma(s)} = e^{Nms^* + N\Sigma(s^*)}$

clusters ↗

⇒ total number of solutions $= \mathcal{Z}(1)$ → dominated by $\begin{cases} \Sigma(s^*) = 0 \\ \rho(s) \sim e^{m(s-s_0^*)} \end{cases}$

$\sum_\gamma W_\gamma^2 = \int ds\, e^{2Ns}\, e^{Nm(s-s_0^*)}$

ENSEMBLE RANDOM (GRAPHICAL) MODEL

We can introduce randomness → $G(V, E)$

→ $\psi_i \quad e^{\beta H_i x_i} \quad H_i \sim p_0(H_i)$

→ $\psi_a \quad e^{\beta J_{ij} x_i x_j} \quad J_{ij} \sim p_0(J_{ij})$

↳ this disorder is then incorporated into the BP algorithm: $\begin{cases} \nu_{i\to a} \overset{d}{=} f_i[\{\hat\nu\}; H_i] \\ \hat\nu_{a\to i} \overset{d}{=} \hat f_a[\{\nu_{i\to a}\}, J_{ij}] \end{cases}$

→ How can we compute ensemble average without sampling different problems and converging BP on each?

Density evolution equation → compute directly the disordered average:

- one way to see that: $\nu_{i\to a}$ R.V. of H_i, and solve in terms of distributions
- another way: "self averaging", statistical treatment by drawing messages at random and iterating from a huge sample of messages (~ experiments) [10^5 is enough]
  ↳ random incoming messages $P(\nu)$ ⟶ random outcoming messages

⇒ population dynamics ≡ density evolution:

equations in probability $\begin{cases} P(\nu) = \mathbb{E}_{H,J}\left(\int \prod_{i=1}^{\ell} d\hat P(\hat\nu_1)\, d\hat P(\hat\nu_2) \dots d\hat P(\hat\nu_\ell)\, \mathbb{1}\left(\nu = f[\hat\nu_1, \dots \hat\nu_\ell; H_i]\right)\right) \\ \hat P(\hat\nu) = \end{cases}$

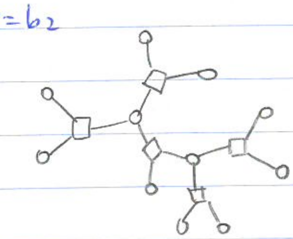
ANDOM XORSAT: Kautmann = SAT.

THE MODEL N binary variables: $x_i \in \{0,1\}$; $M = \alpha N$ constraints in the form of linear equations mod 2.

$k=3$ → number of variables per constraints: $x_2 + x_4 + x_7 = b_1$ $\equiv$ $H\underline{x} = \underline{b}$ $H_{ij} \in \{0,1\}$

$x_{13} + x_{24} + x_5 = b_2$

↳ all constraints chosen at random (M k-tuples at random). → $|\partial i| = \text{Poisson}(\alpha K)$, $|\partial a| = K$.



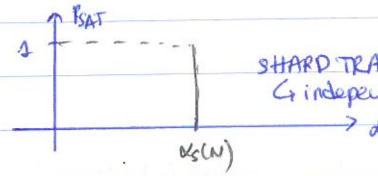
THE MEASURE: $\mu(\underline{x}) = \frac{1}{Z} \prod_a \psi_a(\underline{x}_{\partial a})$

RK: percolating ratio from branching ratio: $\underbrace{(k-1)}_{\text{nodes}}\overbrace{\alpha K}^{\text{constraints}} = 1$ work way above percolation threshold not to be trivial (at least one loop).

↳ with the change of variable: $s_i = (-1)^{x_i}$ $J_a = (-1)^{b_a}$ → $H = \sum_a \left(1 - J_a \prod_{i \in \partial a} s_i\right)$ $\equiv$ Sparse k-spin model! → cavity rather than replica.

THE NUMBER OF SOLUTIONS: $Z = \begin{cases} 2^{N - \text{rank}(H)} & \text{if } b \in \text{Image}(H) \\ 0 & \text{otherwise} \end{cases}$ → $\#b = 2^{\text{rank}(H)}$

↳ one can show that in the thermodynamic limit →



SHARP TRANSITION
↳ independent of b

PLANTED MODEL: As we do not care for the actual choice of b, we'll choose $\underline{b} = 0$.

Indeed, $S = \{\underline{x} : H\underline{x} = \underline{b}\}$ ⇒ $\forall x^* \in S$ $S = x^* + S_0$ → equivalent under one translation
$S_0 = \{\underline{x} : H\underline{x} = \underline{0}\}$ ↳ same properties for S and S_0.

↳ constraint generated at random → H full rank typically $|S| = |S_0| = 2^{N-M}$.

<u>Important property</u>: 2 classes of variables → frozen $x_i = 0$ $\forall x \in S_0$
↓ prove as EXERCISE → free $x_i = \begin{cases} 0 & \text{proba } 1/2 \\ 1 & \text{proba } 1/2 \end{cases}$

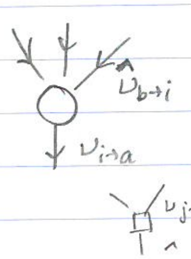
EASY BOUNDS: $\overbrace{\frac{1}{k(k-1)}}^{\text{percolation, loop, frustration}} < \alpha_s < \overbrace{1}^{\text{inversion of } H}$

BELIEF PROPAGATION EQUATIONS:

We'll denote the uniform $(1/2, 1/2)$ marginal by $*(x) = 1/2\, \delta(x_i) + 1/2\, \delta(x_i)$

and the delta on 0 marginal by $0(x) = \delta(x_i)$

property: $\{\nu^{(0)}, \hat{\nu}^{(0)}\} \in \{0, *\}^2 \Rightarrow \{\nu^{(t)}, \hat{\nu}^{(t)}\} \in \{0, *\}^2$ (stable under BP equations)



$\nu_{i \to a} = \begin{cases} 0 & \text{if at least 1 } \hat{\nu}_{b \to i} = 0 \\ * & \text{otherwise (every incoming} = *) \end{cases}$

$\hat{\nu}_{a \to i} = \begin{cases} 0 & \text{if all } \nu_{j \to a} = 0 \\ * & \text{otherwise} \end{cases}$

"has more zero than $\tilde{\nu}$"

Rk: there is a partial ordering w.r.t to the messages: $\nu \geqslant \tilde{\nu}$ if $\tilde{\nu}=0 \Rightarrow \nu=0$

$\rightarrow$ stable under BP: if $\nu^{(t)} \geqslant \tilde{\nu}^{(t)} \Rightarrow \nu^{(s)} \geqslant \tilde{\nu}^{(s)} \quad \forall s>t$.

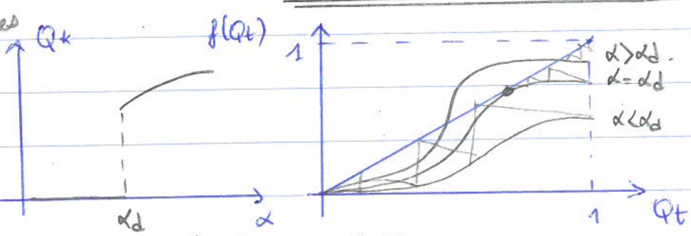
↳ BP FIXED POINTS:

1st case $\{\nu^{(0)}, \hat{\nu}^{(0)}\} = *$ $\rightarrow$ already a fixed point

2nd case $\{\nu^{(0)}; \hat{\nu}^{(0)}\} = 0$ $\rightarrow$ will follow the evolution with Q_t = fraction of $\nu=0$ messages

$Q_0 = 1$

$$Q_{t+1} = \sum_{\ell=0}^{\infty} \underbrace{e^{-\alpha K} \frac{(\alpha K)^\ell}{\ell!}}_{\text{degree}} \Big[1 - \underbrace{\hat{Q}_t^\ell}_{\text{probability all incoming messages are trivial}}\Big] \quad \text{with } \hat{Q}_t = Q_t^{K-1} \Rightarrow \boxed{\begin{cases} Q_{t+1} = 1 - e^{-\alpha K Q_t^{K-1}} = f(Q_t) \\ Q_0 = 1 \end{cases}}$$



$\begin{cases} K=3 \quad \alpha_d = 0.8185 \\ \alpha_d \sim \ln K / K \end{cases}$

$\Rightarrow$ we initialized BP to very special conditions $\rightarrow$ measure concentrated on the planted configuration $\rightarrow$ if it belongs to a finite cluster, will sp ad over and only over this planted cluster $\rightarrow Q^* > 0$.

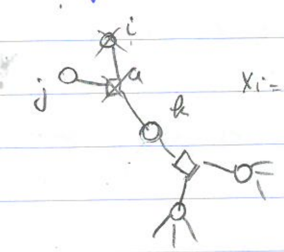
Rk: Reconstruction problem with boundaries $\nu=0$ $\rightarrow$ the number of zeros while going toward the center/root of the ball/tree $\Rightarrow$ less and less 0 according to $Q^{t+1} = f(Q^t)$.

Phases $\begin{cases} \alpha < \alpha_d: \text{ strong correlation decay } Q \to 0: \text{ Replica symmetric} \\ \alpha > \alpha_d: \text{ clusters, can we understand them better?} \end{cases}$

Idea: Try to keep the dense part of the tree only (should be driving this transition!)

$\Rightarrow$ Remove leaf by keeping track of the value ~~remaining~~ deleted variables should take

$\rightarrow$ As long as there are leaves remove them + corresponding clause $\equiv$ smart gaussian elimination



$x_i = x_j + x_k + b_a$

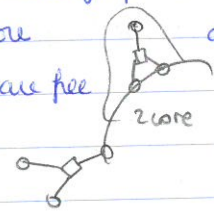
↳ if nothing left eventually $\rightarrow$ go back constructing graph from random assignment to last eliminated variable

→ start existing for $\alpha \geq \alpha_d$.

→ if something left ⇒ 2-core ⇒ subgraph $|\partial i| \geq 2 \ \forall i$ ⇔ all frozen $\hat{\nu}, \nu = 0$

↳ directly implied by the core — also frozen → BACKBONE (of cluster)

↳ the remaining variables are free



⇒ one to one correspondance between BP messages ⇔ topology.

In this special CSP core = label for the cluster → all solutions have same core configurations.

↳ so that counting clusters = counting core configurations.

SIZE OF THE CORE:

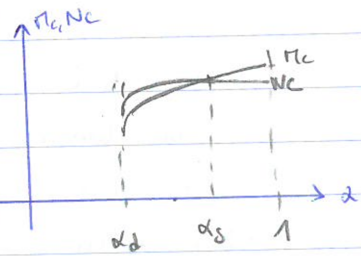
$$\begin{cases} \mathbb{P}[\hat{\nu} = 0] = \hat{Q}^* \\ \mathbb{P}[\nu = 0] = Q^* \end{cases} \quad \text{with } Q^* = 1 - e^{\alpha k \hat{Q}_*}, \quad \hat{Q}^* = Q_*^{k-1}$$

↳ expected number of variables in core $N_c = N[1 - e^{-k\hat{Q}_*} - k\alpha\hat{Q}_* e^{-k\alpha\hat{Q}_*}]$

$$M_c = N\alpha\hat{Q}_* (1 - e^{-k\alpha\hat{Q}_*})$$

solution on the core: $2^{N_c - M_c} = e^{\Sigma N} \to \Sigma = \frac{N_c - M_c}{N} \ln 2$

analytic expression for complexity



SOLUTION IN SPACE OF CONFIGURATIONS FOR $\alpha_d < \alpha < \alpha_s$

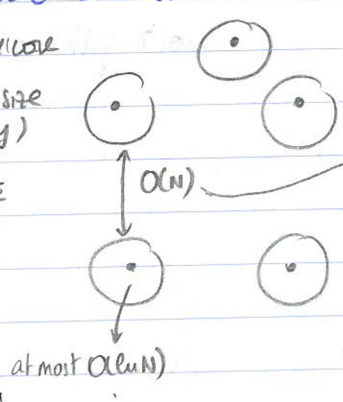
$e^{N\Sigma}$ configuration of core

clusters of equal size (given special topology) of internal entropy

$e^{NS} = 2^{N-M} e^{-N\Sigma}$

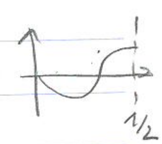
↓

EXACT PICTURE

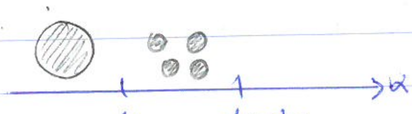


→ proof in the book / Guilhem exercise?

configurations at distance d $W(d) \to \frac{\ln W(d)}{N}$



PHASE DIAGRAM:



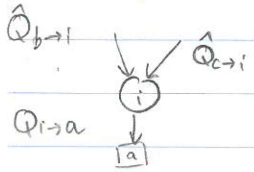
→ The simplest CSP, that can be solved with basic graph structure argument (leaf removal) straightforwardly. Yet still feature clusters + sat sharp.

CAVITY METHOD WITH LONG RANGE CORRELATIONS:

Decompose the measure over different states: $\mu(\underline{x}) = \sum_{\gamma=1}^{e^{N\Sigma}} \underbrace{W_\gamma}_{\text{weights}} \underbrace{\mu_\gamma(\underline{x})}_{\text{Bethe measure}}$

→ meta messages, 1RSB cavity method: statistical distribution of fixed points of BP.

$$\begin{cases} Q_{i\to a}(\nu) = \mathbb{P}\left[\nu_{i\to a}^{(\gamma)} = \nu\right] \\ \hat{Q}_{a\to i}(\hat\nu) = \mathbb{P}\left[\nu_{a\to i}^{(\gamma)} = \hat\nu\right] \end{cases}$$ → make them locally consistent!



now the messages $\hat Q_{b\to i}$ and $\hat Q_{c\to i}$ are not entirely independent, they need to be compatible!

e.g. for 3-XORSAT → core: $\hat Q = \frac{1}{2}\mathbb{1}(\hat\nu = 0) + \frac{1}{2}\mathbb{1}(\hat\nu = 1)$,

but getting $\begin{cases}\hat\nu_{b\to i} = 0 \\ \hat\nu_{c\to i} = 1\end{cases}$ cannot be possible → $\hat\nu$ not a BP fixed point!

↳ XORSAT 1RSB: $Q_{i\to a}(\nu) \propto \sum_{\{\hat\nu_{b\to i}\}_{b\in\partial i\setminus a}} \prod \hat Q_{b\to i}(\hat\nu_{b\to i})\, \mathbb{1}(\nu = f(\{\hat\nu_{b\to i}\}))\, \mathbb{1}(\{\hat\nu_{b\to i} \text{ are compatible}\})$.

XORSAT

at least one solution allowed.

GENERIC MODEL 1RSB

CAVITY EQUATIONS: $$\begin{cases} Q_{i\to a}(\nu) \propto \sum_{\{\hat\nu_{b\to i}\}} \mathbb{1}(\nu = f_i(\{\hat\nu_{b\to i}\}))\, Z_{i\to a}^{m}(\{\hat\nu_{b\to i}\}) \prod_{b\in\partial i/a} \hat Q_{b\to i}(\{\hat\nu_{b\to i}\}) \\ \hat Q_{a\to i}(\hat\nu) \propto \sum_{\{\nu_{j\to a}\}} \mathbb{1}(\hat\nu = \hat f_a(\{\nu_{j\to a}\}))\, Z_{a\to i}^{m}(\{\nu_{j\to a}\}) \prod_{j\in\partial a/i} Q_{j\to a}(\{\nu_{j\to a}\}) \end{cases}$$

$Z_{i\to a}$ ≡ proportional to the entropy of solutions when merging branches

↳ corresponds to look at statistics of clusters according to $Z(m) = \sum_{\gamma} e^{N s_\gamma m}$

↳ $\Sigma(s)$ legendre transform