Renormalization Group on Fermions (III) [Shankar]

Recall in 2D we have 2 marginal interaction terms:

\[ U(\theta_1 - \theta_2) \quad \text{and} \quad V(\theta_3 - \theta_4) \]

Similarly, out of the 3 diagrams for \( V \), only one contributes:

\[ \int \frac{v(\theta_1 - \theta_2) v(\theta_3 - \theta_4)}{(i\omega - \xi(k_1))(i\omega - \xi(k_2))} \]

\[ \propto \int d\omega \int d\theta v(\theta_1) v(\theta_3 - \theta_1) d\theta \]

\[ \Rightarrow \frac{dV}{dt}(\theta_1 - \theta_2) = -\int v(\theta_3 - \theta_2) v(\theta_3 - \theta_1) d\theta \]

Expand \( v(\theta) = \sum_m v_m e^{im\theta} \), \[ \frac{dv_m}{dt} = -c_r^{n=1} \ln (k_{ab}) \]

This gives rise to Cooper instability.

Now we have the theory at fixed pt. Go on to solve for physical quantity \( g \) of the theory. E.g.,

\[ \mathcal{X}(q, \omega) = \langle \hat{p}(q, \omega) \hat{p}(-q, -\omega) \rangle \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 d\omega_1}{(i\omega - \xi(k))(i\omega - \xi(k + q))} \]

\[ \propto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega d\theta}{q \cos \omega} \]

\[ \square = U_0 \chi^2_0 \]
\[ \mathcal{L}(q, \omega=0) = \frac{k_0}{1 - i\omega \tau_0} \]

**NOTE:** Diagrams such as \( \mathcal{L} \) does not contribute in narrow bandwidth (\( \Delta \ll k_F \)) limit.

Now, another tool: large \( N \) expansion.

Consider:

\[ L = \sum_{i<j}^{N} \mathcal{F}_i \beta \mathcal{F}_j + \frac{q}{N} \sum_{ij} (\mathcal{F}_i \mathcal{F}_j) (\mathcal{F}_i \mathcal{F}_j) \]

We can relate large-\( N \) and narrow bandwidth via: \( N \sim \frac{k_F}{\Delta} \)

**REMARK:** The conclusion continues to hold if the Fermi surface is not circular. Cooper instability is present as long as \( \mathcal{E}(k) = \mathcal{E}(-k) \).

Mathematically, change variable \( (k, \omega, \theta) \mapsto (\epsilon, \omega, \theta) \)

\[ S_0 = \int \mathcal{F}(\epsilon, \omega, \theta) (i\omega - \epsilon) \mathcal{F}(\epsilon, \omega, \theta) J(\theta) d\omega d\epsilon \]

The only change is that \( u(\theta, \theta_2) \neq u(\theta, -\theta_2) \)

For non-circular Fermi surface, shape of Fermi surface can change. One can compute the 1-loop self-energy (\( \sim Q^2 \)) can recompute Fermi surface order to order...
However, one also need to worry about nesting:

\[ W(k_1, k_2; k_1 + (\pi, \pi), k_2 + (\pi, \pi)) \]

\[ \chi \Rightarrow \frac{dW(\theta_1, \theta_2)}{dt} = \int W \cdot W \, d\theta \]

This give rise to CDW instability.

**REMARK:** The RG is implemented perturbatively. Thus if the flow is towards large coupling, RG cannot determine the properties of the resulting state. However, practically there will be a temperature associate with such instability, and these temperature can be very small and hence unimportant.

**REMARK:** Since all interactions happen on a thin shell, Fermi liquid in any dimension has the generic form

\[ S = \int_{-\infty}^{\infty} \int_{-\Lambda}^{\Lambda} \int_{-\Lambda}^{\Lambda} \chi(w, k, \theta)(iw - k) \varphi(w, k, \theta) \, dk \, d2dw \]

However, something different can happen if Fermi "surface" has a different co-dimension, e.g., Fermi line in 3D.

At Gaussian (zero-coupling) fixed pt, the above scale as

\[ s^8/s^3 = s^{-1} \]

Thus, 4-Fermion interaction is still irrelevant in Gaussian fixed point. But at strong coupling different co-dimension model may flow to different fixed points.
To handle the problem, we may do expansion in $d = 1 + \varepsilon$ dimensions. From this we get, e.g.: 

$$\beta = \frac{d\gamma}{d\epsilon} = -\epsilon v - v^2$$

By scaling, the singular part $A_\varepsilon$ energy scales as:

$$E(\varepsilon v, h) = s^{-\varepsilon} E(\varepsilon v s^\varepsilon, h s^\varepsilon)$$

Heutz-Millis approach:

Instead of 4-fermion term, use Hubbard-Strokovitch:

$$S_0 \rightarrow \int \bar{\psi} (\ldots) \psi + \bar{\psi} \psi \Delta + h.c. + |\Delta|^2 = e^{S(\omega)} + |\Delta|^2$$

For fermion without gap, integrate out $\Delta$ generates singular behavior

$$e^{S} \rightarrow \int \Delta (\xi + (\xi^2 + \omega^2)^{\varepsilon/2}) \Sigma d\omega d\Sigma d\Omega$$

The scaling seemingly is different from conventional way. But ultimately these will agree.

Lesson: Why we need RG:

1. It allows us to understand why (qualitatively) universal classes exist.

2. Traditional methods (e.g., high/low temperature series) faces singularities at critical points. But for RG, critical point is a natural starting point for expansion (as $\beta$-function = 0 at fixed point).

3. RG allows one to obtain continuum theory in a non-perturbative way from a lattice model.