July 21, 2013

# Real space RG of disordered 1d systems.

## I.   OVERVIEW

The ultimate goal of these lectures is to first and foremost, teach you how to use real space renormalization group to solve strongly disordered systems.

If you think about problems you have so far encountered as researchers and students, the vast majority of them had translational symmetry. Therefore the number one hit of mathematical methods you learned over and over again was Fourier analysis. Regardless of what the problem is - say diagonalizing a tight binding Hamiltonian in 1d:

$$\hat{\mathcal{H}} = -t \sum_i (c_i^\dagger c_{i+1} + H.C.) \tag{1}$$

we use Fourier transform, and diagonalize the Hamiltonian:

$$c_k = \sum_j c_j e^{ikx_j} \tag{2}$$

and as we all know, this gives:

$$\hat{\mathcal{H}} = -2t \sum_k \cos(ka) c_k^\dagger c_k \tag{3}$$

with $a$ the lattice constant.

But what happens when we have disorder present? For instance, what happens if the hoppings are random?

$$\hat{\mathcal{H}} = - \sum_i J_i (c_i^\dagger c_{i+1} + H.C.) \tag{4}$$

Which means that there is a distribution of hopping strengths, and we can assume no correlation between hopping at different places. (Fig. 1)

Of course the number one answer is - plug it into a computer and see what you get! You should do that regardless, but in this series I will teach you a very powerful method which tackles this problem analytically. Today I will discuss a problem related to the random hopping of fermions, or, hard core bosons - the random 1d Heisenberg model:

$$\hat{\mathcal{H}} = \sum_i J_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j. \tag{5}$$

The solution for this problem was developed by Ma and Dasgupta[1,2] and Daniel Fisher[3]. Using real-space Renormalization group, Fisher found the ground state and low lying excitations of this model. What I will show you applies equally well to the random hopping problem of hard core bosons, or, of non-interacting fermions. So part of your job for tonight or this weekend would be to plug in the random hopping problem to Mathematica or Matlab, and see if what I'm telling you is correct.

Tomorrow, we will apply the same methods to a problem of current interest: The superfluid insulator transition in 1d in the presence of strong randomness:

$$\hat{\mathcal{H}} = \sum_i \left[ -J_i \cos(\phi_{i+1} - \phi_i) + U_i \hat{n}_i^2 \right]. \tag{6}$$

Which is the Hamiltonian describing a random array of superconducting grains. This will be complementary to Thierry's lectures, which considered the same model at weak disorder.

The last part of the lectures will be about another work horse of quantum critical phenomena - the transverse field Ising model:

$$\hat{\mathcal{H}} = - \sum_i J_i \sigma_i^z \sigma_{i+1}^z - \sum_i h_i \sigma_i^x. \tag{7}$$

This model has recently become very relevant in the current effort to understand the type of criticallity involved in the many-body localization transition.
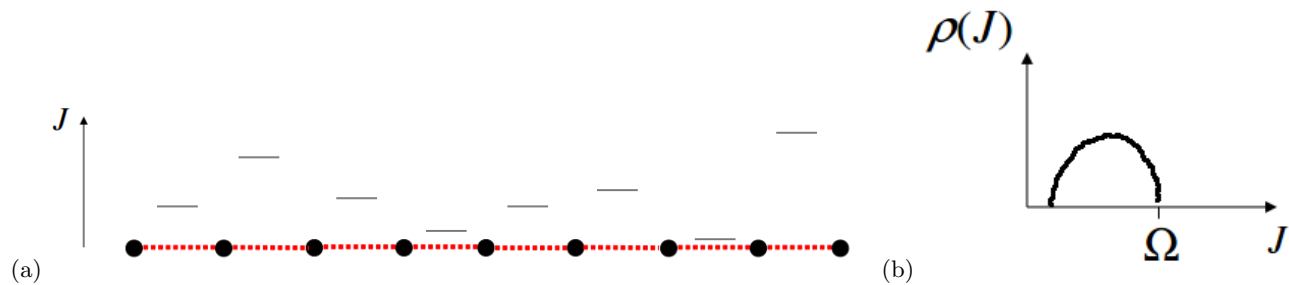
FIG. 1. (a) The bond strength $J$ varies from cite to cite as if each $J$ is an independent random variable. (b) Each Hamiltonian with random $J$'s is described by a distribution of $J$'s, say, the probability density $\rho(J)$.

## A.  Basics of RSRG

The method we will develop is the real-space renormalization group. For those of you who have nightmares about epsilon-expansions - don't worry, we will do everything from scratch, and by the end (hopefully...) you'll have a good intuition about the process of renormalization. Our work plan will be: Decimation:

- Identify the highest energy piece in the Hamiltonian.

- Eliminate the highest energies by solving the highest energy piece as if no other terms in the Hamiltonian exist.

- Incorporate the rest of the Hamiltonian using perturbation theory, or other means. For the method to work, the resulting Hamiltonian will have a lower maximum energy scale. Flow:

- Iterate the decimation steps until the Hamiltonian is completely solved.

In the last step, we will take a conceptual leap and instead of following how one Hamiltonian evolves, we'll consider how an ensemble of Hamiltonians evolves. That means - we will look at distribution functions of couplings, and ask how they evolve during our RG treatment.

## II.  RANDOM HEISENBERG MODEL

So the model we would attack today is the random antiferromagnet in 1d:

$$\hat{\mathcal{H}} = \sum_i J_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j. \tag{8}$$

The motivation to study this model came from experiments done on quasi-1d organic salts, mostly $Qn(TCNQ)_2$[4,5], which showed a remarkable susceptibility:

$$\chi \sim T^{-\alpha} \tag{9}$$

with $\alpha < 1$ and varying from sample to sample. These salts have chains of stacked double benzene rings, with each pair having one excess spin-1/2. (Fig. 2)

Recall that the Currie susceptibility of free spins is $\chi \sim T^{-1}$.

## A.  Pure Heisenberg chain?

What are the low energy properties of a uniform Heisenberg chain? Well, without going to much into details, a good rule of thumb is that the Heisenberg model is in the same phase as x-y chain, where the sz-sz coupling is missing:

$$\hat{\mathcal{H}} = \sum_i J\left(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - \hat{S}_i^z \hat{S}_{i+1}^z\right) = \sum_i J\hat{S}_i^+ \hat{S}_{i+1}^-. \tag{10}$$

Which is in turn, mappable to the fermionic chain model in Eq. (1), where a site with spin up maps to a full site, and a spin-down site maps to an empty site in the fermionic chain.
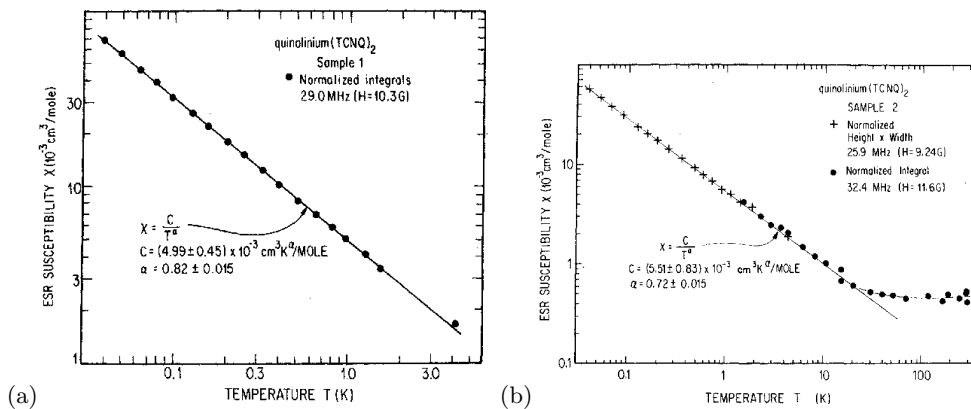
It is easy to verify that:

$$_{n,n+1}\langle s_{tot} = 0| V | s_{tot} = 0\rangle_{n,n+1} = 0 \tag{16}$$

So no first order contribution. The next contribution we can get from a neat way of writing the second order perturbation:

$$E^{(2)} = \langle \psi| \frac{V \sum_m |m\rangle \langle m| V}{E_\psi - E_m} |\psi\rangle \tag{17}$$

where the energy denominators are calculated according to $\hat{\mathcal{H}}_n$ only. Also:

$$|\psi\rangle = |\{s_i\}_{i<n}\rangle |s_{tot} = 0\rangle_{n,n+1} |\{s_i\}_{i>n+1}\rangle \tag{18}$$

so and $|m\rangle$ are the excited states of the $n$ and $n+1$ spins, namely, the triplet states:

$$|s_{tot} = 1|m_z = \pm 1, 0\rangle_{n,n+1} \tag{19}$$

with the rest of the chain not playing a role. So $E^{(2)}$ can actually be thought of as an effective Hamiltonian for all spins *except* $n$ and $n+1$:

$$E^{(2)} \to H^{(2)}_{n-1,\,n+1} =_{n,n+1} \langle s_{tot} = 0| \frac{V \sum_{m=-1}^{1} |s_{tot} = 1;\rangle_{n,n+1} \langle s_{tot} = 1; m| V}{-J_n} |s_{tot} = 0\rangle_{n,n+1} \tag{20}$$

Eq. (20) has many terms in it, but they all follow a simple pattern. Let me demonstrate just with one term, which takes the z-z terms from the $n-1$ bond and from the $n+1$ bond:

$$H^{(2)}_{n-1,\,n+1} = \ldots -\frac{1}{J_{n\,n,n+1}} \langle s_{tot} = 0| J_{n-1} S^z_{n-1} S^z_n |s_{tot} = 1; m = 0\rangle_{n,n+1} \langle s_{tot} = 1; m = 0| S^z J_{n+1} S^z_{n+1} S^z_{n+2} |s_{tot} = 0\rangle_{n,n+1} + \ldots \tag{21}$$

Which we see becomes:

$$\to \ldots \frac{J_{n-1} J_{n+1}}{4\Omega} S^z_{n-1} S^z_{n+2} + \ldots \tag{22}$$

the x-x and y-y terms in V will eventually all add up to give us the rotationally symmetric effective Hamiltonian:

$$H^{(2)}_{n-1,\,n+1} = \frac{J_{n-1} J_{n+1}}{2\Omega} S_{n-1} \cdot S_{n+2} \tag{23}$$

This is what we wanted. We eliminated the two strongest interacting spins, so the chain is now shorter. We made a cut in the chain, but quantum fluctuations expressed through second order perturbation theory introduce an rather small effective coupling between the nearest neighbors of the singlets:

$$J^{eff}_{n-1,n+2} = \frac{J_{n-1} J_{n+1}}{2\Omega} \ll J_{n-1}, J_n, J_{n+1} \tag{24}$$

where the much-less-than is a consequence of our strong disorder assumption: $J_{n\pm 1}/J_n \ll 1$. So we reduced the overall strongest coupling of the Hamiltonian, $\Omega$, since now the strongest coupling will be determined by what used to be the second strongest bond in the chain. Most importantly, after the decimation step, we have exactly the same form of the Hamiltonian: a nearest-neighbor Heisenberg model.

### C. Qualitative ground state picture: the random singlet phase

The next step in our analysis must be the repeated application of the decimation step. If disorder increases upon the repeated use of the Ma-Dasgupta rule, as it is called, then we are safe. The strong disorder assumption only becomes better and better. This is indeed the case, as Daniel Fisher proved, and we will notice soon.

The decimation rule gives us a simple picture of the random Heisenberg model ground state. The strong bonds are going to localize a singlet, with many singlets forming between nearest neighbors. But as the largest bonds are
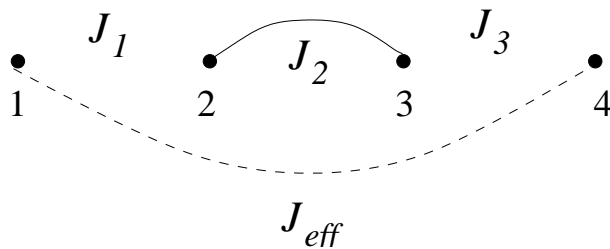
FIG. 3. The strong bond between sites 2 and 3 localizes a singlet. Quantum fluctuations induce a new bond between sites 1 and 4 which has the strength $J_{eff} = J_1 J_3 / 2 J_2$.
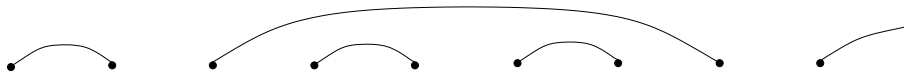


FIG. 4. The random singlet phase of a random Heisenberg model. Pairs of strongly interacting sites form non-overlapping singlets in a random fashion. These singlets mostly form between nearest neighbors, but also over an arbitrarily large distance. The long range singlets induce strong correlations between far away sites.

decimated, and the energy scale of the Hamiltonian, $\Omega$ is reduced, the largest bonds connect further nearest neighbors, and singlets may form between really far away sites (Fig. 4).

Already from this we can infer important information about our system: its *average* correlations. If we ask about the correlations in a particular chain (i.e., **in a particular realization of the random bonds**):

$$C_{nm} = \langle S_n \cdot S_m \rangle \tag{25}$$

most likely, they are really small - exponentially suppressed with the distance (as it turns out, with the square root of the distance):

$$C_{nm}^{typical} \sim e^{-c\sqrt{|m-n|}}. \tag{26}$$

But if you are very lucky in the choice of sites, $m$ and $n$ might be connected with a singlet, and then:

$$C_{nm}^{rare} \sim -1. \tag{27}$$

But how rare is rare? If the two sites at hand survive the violence of the decimation procedure until they are nearest neighbors, very likely they will form a singlet. The density of surviving sites at the stage of the decimation procedure where m and n could be nearest neighbor is $n \sim \frac{1}{|m-n|}$ (note that $n$ on the LHS is the density), so only one out of $|m-n|$ sites survives. What is the probability of the *two* sites surviving?

$$p_{m-n singlet} \sim \frac{1}{(m-n)^2}. \tag{28}$$

Now, when we **average over many realizations** of the random chain, or over many pairs of sites at the same distance, we obtain the **disorder averaged** correlation, marked with an overline:

$$\overline{C_{nm}} = (-1)p_{m-n singlet} + (1 - p_{m-n singlet})e^{-c\sqrt{|m-n|}} \approx -p_{m-n singlet} \sim \frac{1}{(m-n)^2}. \tag{29}$$

Power-law correlations!

So despite the localized nature of the ground state, the average correlations fall off only as a power law. This is a wonderful example of *Griffiths effects* - where the average correlations of a random system are dominated by rare instances with anomalously strong correlations.

## D. Distribution function flow

The behavior of the random Heisenberg model goes beyond the correlation properties of the ground state - it would be nice to also understand something about excitations. This requires us to understand the energy scales associated

with the decimated singlets. To find these, we must find the properties of the hamiltonian as we iterate the decimation procedure.

To formulate the problem at hand, consider what defines it. We begin our analysis with a chain of spins given to us where each bond has some interaction strength, $J_n$. One can construct a histogram of the bonds - this produces some distribution which characterizes the Hamiltonian:

$$\rho(J). \tag{30}$$

Let us now rethink our mission a bit. Instead of solving this particular hamiltonian, let us say that we are interested in the statistical properties of *all* Hamiltonians described by the same distributions $\rho(J)$, regardless of the arrangements of the various bonds. If we accept this, we can now ask the question: As we decimate the chains with distribution $\rho(J)$, how does the *distribution* evolve in the process? Namely, what is:

$$\rho_\Omega(J) \tag{31}$$

with $J < \Omega$. To obtain the *flow* of the distribution function with the repeated decimation procedure, we can do one of two things: (a) carry out the decimation procedure numerically on a set of random chains, and obtain the average distribution at each energy scale, (b)write some kind of a master equation, and pray that we can solve it. During your off time from Hiking, I'd expect you to carry out the former, and here we will pursue the latter.

Before we write a master equation, we need some prep work. The one thing we have to work with is the decimation rule, Eq. 24. It is a product. Plus, at each stage of the RG we have $0 < J < \Omega$ and $\omega$ keeps decreasing. Experience shows that it is much better to work with quantities, say, $\zeta$, where $\zeta$ is unbounded by any scale: $0 < \zeta < \infty$, and where the RG rule is a sum:

$$\zeta_{n-1,n+2}^{eff} = \zeta_{n-1} + \zeta_{n+1} \tag{32}$$

Turns out - we can do exactly that. Define:

$$\zeta_n = \ln \frac{\Omega}{J_n} \tag{33}$$

For the RG rule this reads:

$$-\ln \frac{J_{n-1,n+2}^{eff}}{\Omega} = -\ln \frac{J_{n-1}}{\Omega} \cdot \frac{J_{n+1}}{\Omega} \tag{34}$$

which is indeed right, if you are willing to overlook a $\ln 2$ which is negligile compared to $\ln \Omega/J$ due to the strong disorder assumption. Also, for all bonds, since $0 < J_m < \Omega$, $0 < \zeta_m < \infty$ indeed. Notice that the largest bonds have $\zeta_{largestbond} = 0$. And since logs seem to be so nice, let's continue with this defining inertia and define the RG flow parameter to be:

$$\Gamma = \ln \frac{\Omega_0}{\Omega} + \Gamma_0. \tag{35}$$

with $\Omega_0$ the initial largest energy in the chain, and $\Gamma_0$ some constant of order one (this is like an initial "RG time" from which we flow, and will be determined by the initial distribution of the chain we are solving) .

Now we are ready to derive the master equation. Let's draw the initial distribution of $\zeta$'s - which we denote $P_\Gamma(\zeta)$. How do we decimate? We take all the bonds $m$ that are large, with $\Omega - d\Omega < J_m < \Omega$. It is really unlikely they are close to each other, so we consider them all in one fell swoop. We remove the strong bonds from the distribution, and introduce new bonds, deriving from the nearest neighbors.

Let's consider this in the language of the $\zeta$'s. We take all bonds with $0 < \zeta < d\Gamma$ with:

$$d\Gamma = d\ln \frac{\Omega_0}{\Omega} = \frac{d\Omega}{\Omega} \tag{36}$$

We eliminate these small $\zeta$'s, but this also implies a redefinition of all $\zeta_m$'s:

$$\zeta_m \to \tilde\zeta_m = \ln \frac{\Omega - d\Omega}{J_m} = \zeta_m - \frac{d\Omega}{\Omega} = \zeta_m - d\Gamma \tag{37}$$

So the entire distribution $P(\zeta)$ moves to the left. This can be expressed mathematically:

$$dP(\zeta) = \frac{\partial P(\zeta)}{\partial \zeta} d\Gamma. \tag{38}$$

But there is a second contribution: adding the renormalized bonds. First, what is the probability distribution of a renormalized bond? It is the convolution of the probabilities of the left bond $\zeta_\ell$ and right bond $\zeta_r$:

$$P^{RG}(\zeta) = \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell)P(\zeta_r)\delta(\zeta - \zeta_\ell - \zeta_r) \tag{39}$$

Now, how many of these do we produce? We look at each bond and ask - what is the probability that it will get decimated in the next step when $\Gamma \to \Gamma + d\Gamma$? It is $dp = d\Gamma P(\zeta = 0)$. And this contribution also gives a differential change to the full distribution:

$$dP(\zeta) = dp P^{RG} = d\Gamma P(\zeta = 0) \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell)P(\zeta_r)\delta(\zeta - \zeta_\ell - \zeta_r) \tag{40}$$

Together, Eqs. (38) and (40) give the full master equation:

$$\frac{dP(\zeta)}{d\Gamma} = \frac{\partial P(\zeta)}{\partial \zeta} + P(0) \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell)P(\zeta_r)\delta(\zeta - \zeta_\ell - \zeta_r) \tag{41}$$

where we dropped the $\Gamma$ subscript of $P(\zeta)$.

There is a point we glossed over. We removed some probability by getting rid of all the probability density at small $\zeta$. We added probability by adding all the new bonds. Do we need to adjust the normalization of our distribution function? Integrating both sides of Eq. (41) reveals that the normalization is unchanged. For each bond we lost, we added a renormalized bond.

That's it - we have the master equation. Can we solve it?

## E. Scaling of the flow equation

The flow equation (41) looks like the physicists worst nightmare: integro-differential, multivariable, and non-linear. To get some color back to our cheeks, let's do what we often do faced with a complicated non-linear equation (as well as an RG flow!): rescaling.

$P_\Gamma(\zeta)$ has two variables, $\Gamma$ and $\zeta$. We can ask whether there is a *scaling solution*. Namely, let's try to use a new variable $x = \zeta/\Gamma^\phi$ instead of $\zeta$. The distribution function for $x$ will be:

$$P_\Gamma(\zeta) = \frac{1}{\Gamma^\phi}Q(x) \tag{42}$$

(which we obtain by saying $Q(x)dx = P(\zeta)d\zeta$). This is our *scaling ansatz*. Can we find a $\phi$ which eliminates $\Gamma$ altogether? Let's see.

The master equation in terms of $Q$ and $x$ is:

$$\frac{dQ(\zeta/\Gamma^\phi)/\Gamma^\phi}{d\Gamma} = \frac{1}{\Gamma^{2\phi}}\frac{\partial Q(x)}{\partial x} + \frac{1}{\Gamma^{2\phi}}Q(0) \int dx_\ell \int dx_r Q(x_\ell)Q(x_r)\delta(x_r + x_\ell - x) \tag{43}$$

The LHS is simply:

$$-(Q(x) + xQ'(x))\phi/\Gamma^{\phi+1} \tag{44}$$

where we assumed that $Q(x)$ does not have a $\Gamma$ dependence. We see that

$$\phi = 1 \tag{45}$$

eliminates $\Gamma$ altogether, and yields:

$$0 = (x + 1)Q'(x) + Q(x) + Q(0) \int dx_\ell \int dx_r Q(x_\ell)Q(x_r)\delta(x_r + x_\ell - x) \tag{46}$$

Even before solving this equation, within the scaling ansatz we already managed to glean useful information:

$$\zeta_{typical} \sim \Gamma \tag{47}$$

which means:

$$\ln \frac{\Omega}{J_\Omega^{typical}} \sim \ln \frac{\Omega_0}{\Omega}$$

and:

$$J^{typical} \sim \frac{\Omega^2}{\Omega_0} \tag{48}$$

So as the highest energy scale in the Hamiltonian drops, the typical bond strength drops even more. The distribution becomes more and more weighted towards $J = 0$. This confirms the strong disorder assumption. In fact, it tells us that disorder is *relevant* in the RG sense. This means that disorder *grows* upon RG flow.

## F.   Solving the master equation

Let's us take the leap and find a solution of the flow equation. Roughly speaking we need a function that retains its form upon convolution. There is a simple answer: an exponent. Once we realize this, it is easy to see that the solution is simply:

$$Q(x) = e^{-x}. \tag{49}$$

All this pain for this simple answer! But what does it mean? First, it is a fixed point - in the sense that the solution for $Q(x)$ is stationary under the flow. The solution for $P(\zeta)$ is:

$$P_\Gamma(\zeta) = \frac{1}{\Gamma} e^{-\zeta/\Gamma} \tag{50}$$

The width of the $\zeta$ distribution increases indefinitely upon the RG flow. This has received the name "infinite randomness fixed point" to this phase of matter. The $\zeta$ distribution transformed to a J distribution (don't forget the Jacobian! $P(\zeta)d\zeta = \rho(J)dJ$) is:

$$\rho_\Omega(J) = \frac{1}{\Omega \cdot \Gamma} \left( \frac{\Omega}{J} \right)^{1-1/\Gamma} \tag{51}$$

As $\Gamma$ tends to $\infty$ this distribution becomes less and less integrable. Infinite randomness again.

What we found, though, is *a solution* of the flow equation for the distributions of bond strengths upon RG. How do we know it is in anyway *the solution*? Namely, there are at least three possibilities: (a) This solution is obtained only if the initial distribution of the couplings conforms with it at some $\Gamma_0$, (b) This is the low-energy distribution function for a range of coupling distributions, (c) this is the low-energy distribution to which *all* initial distributions flow to. In option (a) the solution we found is unstable, since any deviation will destroy it. Option (c) is the most tantalizing one: In this case, the scaling solution we found is a *global attractor*.

Option (c) obtains: the solution we found is *universal*! Every chain you consider, will, after enough sites have been eliminated, look like a chain with the distribution in Eq. (51).

## G.   Physical properties

Now that we have the distribution as a function of energy scale, we should be able to extract all physical properties. Let's start with the density of surviving spins as a function of energy.

Every time we decimate a strong bond, we remove two sites. The probability of eliminating a bond upon changing the RG scale by $d\Gamma$ is simply $dp = d\Gamma P(0)$ from the discussion above. This implies:

$$dn = -2 \cdot n \cdot d\Gamma P(0) = -2n \frac{d\Gamma}{\Gamma}. \tag{52}$$

where $dn$ is the change in density, and the 2 is due to the two spins that disappear. The solution of this equation is simply:

$$n = \frac{n_0}{\Gamma^2}. \tag{53}$$

The density decreases with energy scale as $1/\ln \Omega^2$.

### 1. Energy-length scaling

Eq. (53) also yields the energy-length scaling of the random-singlet phase. The length of a singlet is always $\ell = 1/n$, since singlets form between nearest-neighbors. So singlets that form at RG scale $\Gamma$ are of length:

$$\ell = \ell_0 \Gamma^2 \tag{54}$$

Which means that the *excitation energy* of singlets of length $\ell$ is:

$$J_\ell \sim e^{-\sqrt{\ell}} \tag{55}$$

which is contrary to usual quantum-critical point scaling where $E \sim 1/\ell^z$, where $z$ is the dynamical scaling exponent. This type of scaling is called infinite-randomness scaling.

### 2. Susceptibility.

Now is our chance to tie it all together. The motivation for our study was the organic salts showing susceptibility $\chi \sim 1/T^\alpha$. What is the susceptibility expected for the random-singlet theory as a function of temperature?

If we would like to consider the properties of our system as a function of temperature, all we need to do is just run the RG until we hit:

$$\Omega = T. \tag{56}$$

At that point, all the singlets we formed are at energies much larger than T, so they are frozen. All the remaining sites are coupled to their neighbors with $J_{eff} \ll T$, so they are essentially free. Therefore the density of free spins at temperature T is:

$$n(T) = \frac{n_0}{\left(\Gamma_0 + \ln \frac{\Omega_0}{T}\right)^2}. \tag{57}$$

The susceptibility of free spins is the Currie susceptibility, $\sim 1/T$. Therefore:

$$\chi(T) = \frac{n_0}{T \ln^2 \frac{\Omega_1}{T}} \tag{58}$$

with $\Omega_1 = \Omega_0 e^{\Gamma_0}$. Not quite $T^{-\alpha}$; but when you take a log of the above function, you encounter the log of a log. There nearly no line straighter than a double log! If you look at Fig 5, you'll see that the two functions indeed compare perfectly well in the given range. One needs a big variation of $\Gamma_0$ (which in the figure is $\Gamma_I$) to accomodate for the entire experiment, which may raise some doubts in our mind. Since $\Gamma_0$ is determined by the large energy tail of the bare disorder distribution, it is not out of the question.
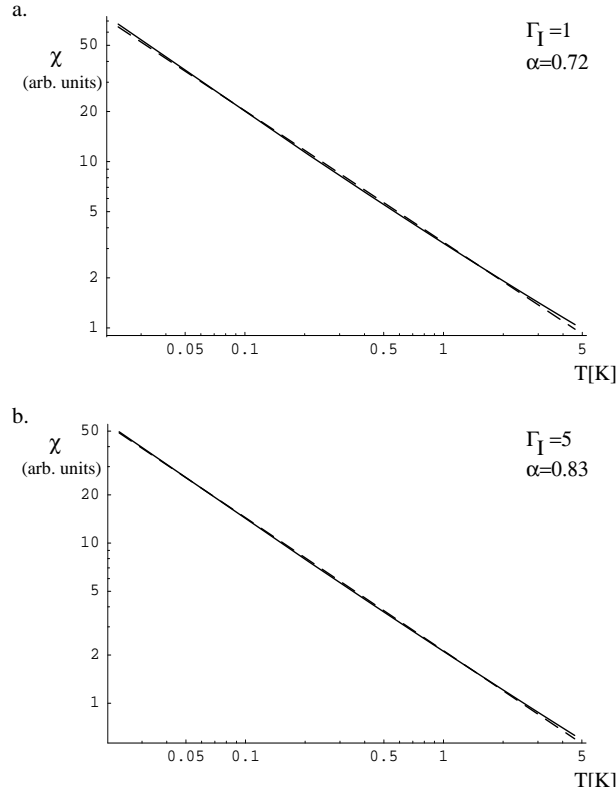
FIG. 5. A comparison between measured and predicted susceptibility for the random Heisenberg model. Experimental measurements[4,5] suggest $\lambda \sim 1/T^{\alpha}$, whereas the random-singlet theory predicts $\lambda \sim 1/T(\ln^2(T_0/T) + \Gamma_I)$, where $\Gamma_I$ is a non-universal number. (a) $\alpha = 0.72$ was measured in[4,5] (sample 2). The power law is the straight dashed line. This case is described very neatly by the random-singlet susceptibility with $\Gamma_I = 1$. (b) $\alpha = 0.83$ which is close to sample 1 and 3 fits very well the theory with $\Gamma_I = 5$.
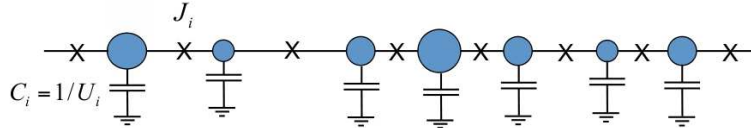


FIG. 6. The model we analyze is a chain of sites, each has a large filling of bosons (which are either atoms or Cooper pairs), and the bosons can hop between nearest neighbor sites. Sticking with the language of superconductors, each grain has a capacitance which is proportional to its size. Nearest neighbors are connected by Josephson junctions, whose strength depends exponentially on the distance between the grains. the degrees of freedom of each grain are the number of *excess* Cooper pairs $N_i$ and the superfluid phase $\theta_i$.

## III. RANDOM BOSONS

Now that we are experts in solving the random Heisenebrg model, let us try to apply the thery to the problem that Thierry Giamarchi discussed from the point of view of weak disorder using the bosonization approach. The method that we have at our disposal, is clearly only adequate at strong disorder. This chapter is worked out in a series of papers with Ehud Altman, Anatoli Polkovnikov, and Ehud Altman[6–8].

At strong disorder, we are better off starting the problem with a discretized picture, where sites that are either puddles of a superfluid, or in the superconducting language, superconducting grains, that are connected to each other by bosonic opping, or by a Josephson junction. The model in Fig. 6, consists of a chain of sites, each of which has charging energy, which is proportional to the inverse of capacitance, as well as hopping. Let's look at both these elements.

## A. Parametrization and Model

Before writing the Hamiltonian, let's decide on the operators we will use. I would like to think of microscopic superconducting grains that have many Cooper pairs. Together with the protons on the grain make it neutral. So the natural thing to do, is use number-phase variables, with $N_i$ counting the number of excess Cooper-pairs away from eutrality, and the superfluid phase $\theta_i$ which have canonical commutation relations:

$$[N_i, \theta_i] = i \tag{59}$$

Note that since $N_i$ measure relative to neutrality, it could also be negative and large.

You may wonder how these connect to the usual bosonic creation and annihilation operators. Here it is. With:

$$[b^\dagger, b] = -1 \tag{60}$$

we have:

$$b = e^{i\theta}\sqrt{N + N_0}, \; b^\dagger = \sqrt{N + N_0}e^{-i\theta} \tag{61}$$

with $N_0$ being the occupation of the grain at neutrality, relative to which $N$ measures. A fun exercise is to show that both are equivalent.

Now you may think that so far everything was trivial. It turns out that we have already chosen camps. Using b's instead of a's put us firmly in the solid state community, since atomic people tend to use a for their boson. That's not so bed. But then there is the choice of greek letter for the phase of the superfluid on each grain. I am a proud product of the Santa Barbara school, and therefore you'd find me use $\phi$ for phase and $\theta$ for the conjugate variable associated with charge $\nabla\theta = q$. But in your notes from Thierry's lecture, you will find the completely opposite notation. This has been a strong source of cross atlantic hostility for many years. So I choose to wave an olive branch, and use Thierry's notation. All Matthew Fisher disciples in the audience please forgive me.

Now that we have the variables, let's write the Hamiltonian. The hopping part is really easy:

$$\hat{\mathcal{H}}_{hopping} = -J\cos(\theta_{i+1} - \theta_i) \tag{62}$$

For some of you this is familiar as the Josephson coupling, and for some of you, you can recognize the $e^{i\theta_i}$ as the raising operator, except for the square-root N, in the definition of $b$ and $b^\dagger$. Because we are thinking of big puddles of bosons, or grains with a lot of cooper pairs, we absorb a factor of $\sqrt{N_i N_{i+1}}$ in the Josephson coupling strength $J$.

Charging is even more fun. Each grain is like a capacitor. Therefore, we should have a term like $N^2/2C$, which $C \propto L_i$ with $L_i$ being a size of the grain. Let's define $U = 1/2C$, to measure the interaction strength. In addition, dirt in the area may produce a bias voltage, which will shift the neutrality point of the grains. Let's define $\bar{n}_i$ as the number of Cooper pairs needed to make a grain neutral. Together we write the charging energy as:

$$\hat{\mathcal{H}} = U(N_i - \bar{n}_i)^2. \tag{63}$$

Let's leave the offset term alone for simplicity, and get back to it later.

For now, let us solve the particle-hole symmetric, offset-free problem:

$$\hat{\mathcal{H}} = U_i N_i^2 - J_i\cos(\theta_{i+1} - \theta_i) \tag{64}$$

This is the rotor representation of the Bose-Hubbard model at large filling.

## B. Physical properties of the model

How are the superfluid and insulating phase signified? amongst other things, through their stiffness and compressibility. The stiffness is defined as

$$\rho_S = \frac{1}{2}L\frac{\partial^2 E_{\Delta\theta}}{\partial\Delta\theta^2}. \tag{65}$$

Where $\Delta\theta = \theta_L - \theta_1$, the total phase difference on the chain, and $E_{\Delta\theta}$ is the energy of hte twisted chain. $L$ is the length of the chain. The compressibility is an additional important observable - how easy is it to charge the chain? A strong insulator cannot be charged, and we expect its compressibility to vanish. We define compressibility as:

$$\kappa = \left(\frac{\partial n}{\partial\mu}\right)_T \tag{66}$$

with $n$ the particle *density*, and $\mu$ the chemical potential. The reason for the name is that using the Gibbs-Duhem relation you can show that $\kappa = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)$, with $p$ pressure and $V$ volume. For a strong insulator we would have $\kappa = 0$, and then it would just be incompressible.

### 1. Superfluid stiffness

In a superfluid phase, all phase differences must be small, and the charging effects should be secondary. Let's get a feeling for the stiffness neglecting the effects of charging energy, and expanding the cosine in the Hamiltonian we get:

$$E = \sum_i \frac{1}{2} J_i (\theta_{i+1} - \theta_i)^2 \tag{67}$$

How do we find the individual differences from the total difference? It's kind of like resistors in series. What is conserved along the chain? The current is constant, which on each Josephson junction is:

$$I_n = J_n \sin(\theta_{n+1} - \theta_n) \approx J_n (\theta_{n+1} - \theta_n) \tag{68}$$

If $I = I_n$ for all $n$, then:

$$(\theta_{n+1} - \theta_n) = \frac{I}{J_n} \tag{69}$$

and

$$E = \sum_i \frac{1}{2} \frac{1}{J_i} I^2 \tag{70}$$

On the other hand

$$\Delta\theta = \sum_{i=2}^{L} (\theta_i - \theta_{i-1}) = \sum_{i=2}^{L} \frac{I}{J_n} \tag{71}$$

Putting this all together, we have:

$$E = \frac{1}{2}\left(\sum_i \frac{1}{J_i}\right)^{-1} \Delta\theta^2 \tag{72}$$

reading off the stiffness we have:

$$\frac{L}{\rho_S} = \left(\sum_i \frac{1}{J_i}\right)^{-1} = L\overline{(1J)}. \tag{73}$$

### 2. Compressibility

For a single grain, $\kappa = C$, is just the capacitance. This comes from the capacitance law: $S \sim Q = CV$, and $V$ is the chemical potential over the charge. If the system is superconducting, its total capacitance to ground is just the sum of all capacitance, and then:

$$N_{total} = \sum_i C_i \mu \tag{74}$$

with $\mu$ playing the role of a voltage. Then:

$$\frac{\partial n}{\partial \mu} = \frac{\partial N/L}{\partial \mu} = \frac{1}{L}\sum_i C_i = \overline{(1U)} \tag{75}$$

### 3. Luttinger parameter

The Luttinger parameter, which Thierry defined is the product of the stiffness and the compressibility:

$$K = \pi\sqrt{\kappa\rho_S} \tag{76}$$

For our model, we immediately get an approximation for the Luttinger parameter:

$$K = \pi\sqrt{\overline{C/J^{-1}}} \sim \sqrt{J/U}. \tag{77}$$

Clearly, $J/U$ large gives a superfluid phase, and $J/U$ small, insulator.

## C.  Real space RG

Our goal is to solve the random boson chain in Eq. (64). Our analysis is going to follow pretty closely along the random-singlet line. We are going to look at the Hamiltonian, and find the largest coupling in the chain:

$$\Omega = max\{J_i, U_i\} \tag{78}$$

and eliminate these large couplings, while renormalizing the Hamiltonian.

### 1.  large U

Suppose that the largest term in the H is $U_n$ (Fig. 7a). This means that grain $n$ is the tiniest grain, and it is not going to help our superfluidity cause. The we immediately put $N_n = 0$. But the hopping terms to the left and to the right are not so happy with this deision, and try ot excite the site to $N_n = \pm1$. A quick look back at the omnipotent second order perturbation theroy, Eq. (17) allows us to calculate the effect of these hopping terms. Qualitatively, what do we expect? A weakend hopping term.

Applying this here we get:

$$H_{eff} = \tfrac{1}{4}\,_n\langle 0| J_{n-1}(e^{i(\theta_{n-1}-\theta_n)}\frac{\sum\limits_{q=\pm1}|q\rangle_n\,_n\langle q|}{-U_n}J_n(e^{i(\theta_n-\theta_{n+1})}|0\rangle_n \tag{79}$$
$$+\tfrac{1}{4}\,_n\langle 0| J_n(e^{i(\theta_n-\theta_{n+1})}\frac{\sum\limits_{q=\pm1}|q\rangle_n\,_n\langle q|}{-U_n}J_{n-1}e^{i(\theta_{n-1}-\theta_n)}|0\rangle_n + H.C.$$

The denominator tells us the energy cost to go away from the neutrality of grain $n$. and I picked out the term that hops a boson from $n-1$ to $n$ and then to $n+1$.Exchanging the order of the terms gives the opposite process, and adding the hermitian conjugate jsut doubles the terms. All in all we get what we expect:

$$H_{eff} = -\frac{J_{n-1}J_n}{U_n}\cos(\theta_{n-1}-\theta_{n+1}) \tag{80}$$

Thus the effective hopping is the hopping to the right, times the hopping to the left, over the energy denominator:

$$J^{eff}_{n-1,n+1} = \frac{J_{n-1}J_n}{U_n} \tag{81}$$

This is just cotunneling, and it leaves the hamiltonian with the same form, but with the smallest grain eliminated.

### 2.  large J

If two grains are strongly connected, i.e., $J_n$ is the largest term in the H, we might as well fuse them into a single site - make a cluster out of them (Fig. 7b). This is a bit subtle, since the cosine term is not associated with any gap in the problem. I'll get back to that later, if there is interest, but meanwhile, for the experts, I'll just say that this amounts to saying that there are no phase slips allowed on the bond. In the analysis, let's even make a bolder move. Set:

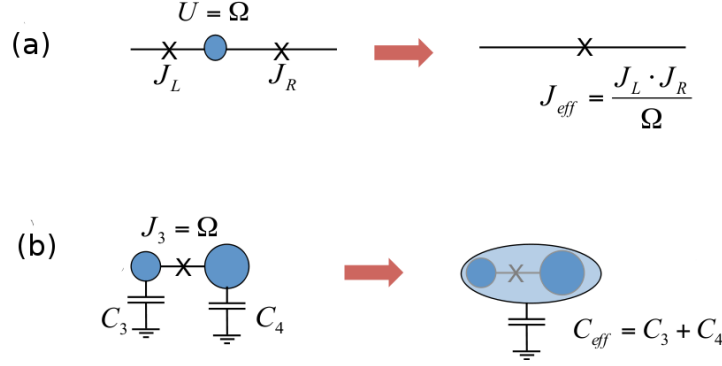$$\theta_n = \theta_{n+1}. \tag{82}$$

FIG. 7. The decimation steps of the random boson chain with no offset charges. (a) Particularly small grains are eliminated due to their large charging energy. Cotunneling of bosons through the eliminated grain gives renormalized Josephson coupling between the the grain's neighbors. (b) If two grains are particularly close, and the Josephson coupling between them is strong, they form a superfluid cluster, with the total grain capacitance being the sum of capacitances of the individual grains

This approximation makes us neglect intra-cluster fluctuations. These fluctuations suppress the Josephson couplings to the grain's neighbors, but this (Debye-Waller) suppression is $\propto \sqrt{U_{n/n+1}/J_n}$, which should be small since we chose a $J_n > U_n, U_{n+1}$.

Now, if there are two sites that fuse to form a cluster with a single phase $\theta_{n,n+1}$, what would be their joint capacitance? Easy: the sum of capacitances. Thierefore, the effective charging energy would have the form of resistances in parallel:

$$C_{n,n+1} = C_n + C_{n+1}, \ \ and \ \ U_{n,n+1}^{-1} = U_n^{-1} + U_{n+1}^{-1}. \tag{83}$$

And the Josephson coupling to the neighbors remains $J_{n-1}$ and $J_{n+1}$. We got rid of the strongest bond, and we reduced the interaction energy in the process

### 3. Qualitative picutre of the random superfluid phase insulator phases

Now it is easy to see that we have two possiblities. if the charging energies are large, then we are mostly going to eliminate one small grain after the other, and even if we form some clusters, they are not going to be large enough to pin down the phase in the entire chain. The end result is a Mott insulator, with a carging gap that varies along the chain, but absolutely no stiffness.

## D. Flow equations and phase diagram

### 1. Scaling variables

The next step is understanding how the iteration scheme above translates to a flow of distribution functions for the Josephson couplings and interaction strengths. As in the Heisenberg case, you can save yourself A LOT of work by choosing the right variables for the job. From previous experience, you already know that we want variables that make the decimation steps become approximate sum rules.

For the Josephson couplings this is achieved by using a logarithmic parametrization:

$$\beta_i = \log \Omega/J_i. \tag{84}$$

The charging energies are best expressed through the capacitance, which is additive in the cluster formation step. We define:

$$\zeta_i = \Omega/U_i. \tag{85}$$

Note that $\zeta = 2\Omega C$ with $C$ the capacitance. $\zeta$ obeys $1 \le \zeta < \infty$. Offset charges are added up as well upon decimation, and therefore are good variables according to the above criterion. As the dimensionless flow parameter, we define as

before:

$$\Gamma = \ln \frac{\Omega_0}{\Omega}, \tag{86}$$

where $\Omega_0$ is of the order of the largest bare energy scale in the Hamiltonian.

### 2. Flow equations

Now is the time to speak about distributions of $J$ and $U$. The Josephson couplings and the charging energies in nearby sites remain uncorrelated if they started so. Thus we can parametrize the coupling distribution functions in terms of two functions: $g(\beta)$ the bond log-coupling distribution, and $f(\zeta)$ the distribution function of the inverse charging energy where $\zeta_i = \Omega/U_i$.

We can now write the master flow equations for the distributions of coupling constants implied by the decimation rules discussed above:

$$\frac{\partial g}{\partial \Gamma} = \frac{\partial g}{\partial \beta} + f(1) \int d\beta_1 d\beta_2 \, g(\beta_1) g(\beta_2) \delta(\beta - \beta_1 + \beta_2) + g(\beta)(g(0) - f(1)), \tag{87}$$

$$\frac{\partial f}{\partial \Gamma} = \zeta \frac{\partial f}{\partial \zeta} + g(0) \int d\zeta_1 d\zeta_2 f(\zeta_1) f(\zeta_2) \delta(\zeta - \zeta_1 + \zeta_2) + f(1 + f(1) - g(0)). \tag{88}$$

Let us recount the origin of all terms contributing to the derivatives $dg(\beta)/d\Gamma$ and $df(\zeta)/d\Gamma$. First, a trivial, yet important, effect on the distributions stems from changing the cutoff $\Omega$. Since $\beta_i = \ln \Omega/J_i$, the change of $\Omega \to \Omega - d\Omega$ causes a shift $\beta_i \to \beta_i + d\beta$ with $d\beta = d\Omega/\Omega = -d\Gamma$. By the same token, this also shifts $\zeta \to \zeta + d\zeta$ with $d\zeta = -\zeta d\Omega/\Omega = -\zeta d\Gamma$. These simple shifts are captured through the chain rule by the first term in both equations, $\frac{\partial g}{\partial \beta}$ and $\zeta \frac{\partial f}{\partial \zeta}$.

The next terms are due to the formation of new bonds and clusters. As

$$\Omega \to \Omega - d\Omega,$$

a bond is decimated if its strength is $\Omega - d\Omega < J < \Omega$ is within this range, i.e., if

$$\beta < d\Gamma = d\Omega/\Omega.$$

The probability that the bond is decimated is therefore $g(\beta = 0)d\Gamma$ for each bond. Out of two clusters with paramters $\zeta_{1,2}$, a bond decimation produces a new cluster with parameters $\zeta = \zeta_1 + \zeta_2$. This is the essence of the convolution term in Eq. (88).

The convolution term in Eq. (87) describes the formation of a new bond upon site decimation with the new bond variable $\beta = \beta_1 + \beta_2$. The probability for such a site decimation event is $f(1) \cdot (\Omega/\Omega - \Omega/(\Omega - d\Omega)) = f(1)d\Gamma$.

The remaining terms in Eqs. (87) and (88) are needed to maintain the normalization of the distributions. Normalization can be verified by integrating over the entire range of $\zeta$ in Eq. (88) and $\beta$ in Eq. (87), and making sure that the derivative of the total probability integral on the LHS's is zero.

### 3. Scaling solution and reduced flow equations

And now comes the crucial moment - can we find solutions of these nasty looking integro-differential equations. But now we have intuition from the spin models analyzed in Sec **??**. When the decimation steps follow a sum rule for two positive couplings, the probability distributions for these couplings should consist of exponentials. This leads to the scaling ansazt:

$$g(\beta) = g_0 e^{-g_0 \beta}, \tag{89}$$

$$f(\zeta) = f_0 e^{-f_0(\zeta - 1)}. \tag{90}$$

This ansatz describes a family of functions that are parametrized by two variables, $f_0[= f(1)]$ and $g_0[= g(0)]$. An important result from numerics, is that this family of distributions is an attractor for essentially *all* initial distribiutions of sufficient width.
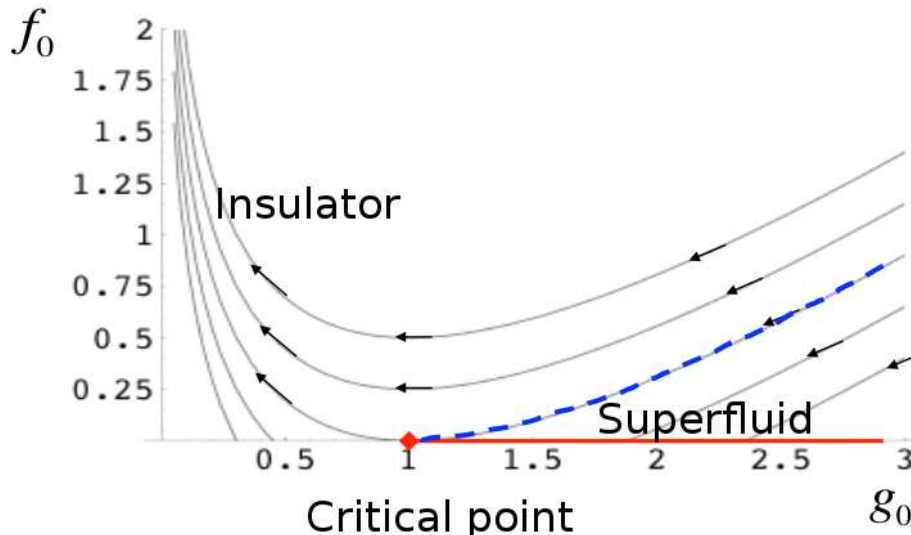
FIG. 8. RG flow for the reduced variables characterizing the distributions of Josephson and charging energies. Flows that are below the critical flow line (dashed) terminate on the superfluid fixed line (red). Flows above flow to the insulator phase

Plugging the scaling ansatz back to the flow equations, one hopes, will generate a flow equations in terms of $f_0$ and $g_0$ alone, and without any explicit functional dependence on $\zeta$ or $\beta$. Approximately, this is indeed the case. We assume that $f_0 \ll 1$, and the substitution of the scaling ansatz to the flow equations yields:

$$\frac{df_0}{d\Gamma} = f_0(1 - g_0) \tag{91}$$

$$\frac{dg_0}{d\Gamma} = -f_0\, g_0. \tag{92}$$

The flow lines in the space $f_0$ versus $g_0$, plotted in Fig. 8, are given by

$$f_0 = (g_0 - 1 - \ln g_0) + \epsilon. \tag{93}$$

The different flow lines are parameterized by $\epsilon$, which sets the detuning from the critical manifold at $\epsilon = 0$. Flows that lie below the critical manifold, $\epsilon < 0$, terminate on the line $g_0 > 1$, $f_0 = 0$ marking the superfluid phase. Values above the critical manifold, $\epsilon > 0$, flow to the region $g_0 < 1$ and where $f_0$ is relevant and flows to larger and larger values. This flow characterizes the insulating regime of the model.

The scaling equations (92) have the form of the Kosterlitz-Thouless flow equations, if we were to rewrite them in terms of $\sqrt{f_0}$ and $g_0$. These scaling variables, however, have a different physical meaning. In particular, the variable that gains a universal value at the transition is the exponent $g_0$ rather than the Luttinger parameter.

Below we elucidate the nature of the superfluid phase. We then explain the essence of the critical point in light of the anomalous properties of the superfluid leading up to it. This will also serve to clarify why a theory that is perturbative in the disorder strength, such as the replica treatment of Giamarchi Schultz can fail in this regime.

### E. Physical origin of the distributions

The universal distribution functions seemed to have appeared out of nowhere. It is interesting to note, as Ehud Altman did, that for continuous bosons near the no-interaction low disroder corner, these distributions appear quite naturally.

If we start with a Gross-Pitaevskii energy functional with:

$$F = \int dx[(\mu(x))n + Un^2] \tag{94}$$

where $n$ is the superfluid density, and $U$ the interaction strength. $\mu(x)$ is an effective chemical potential, which is varying in space. Puddles of superfluid form where $\mu(x) < 0$, as shown in Fig. (??). The distance between puddles

let's define as $r$, and their size, $\ell$. What would be the distributions of $r$ and $\ell$ if $\mu(x)$ is Gaussian with a finite a short correlation length? Probably Poissonian:

$$\rho(\ell) \sim e^{-\alpha\ell}, \;\; \rho(r) \sim e^{-\beta r} \tag{95}$$

Charging energy is inversely proportional to $\ell$, since:

$$E_{charging} \sim \ell n^2 U = \frac{U}{\ell}N^2 \tag{96}$$

so since:

$$U_{eff}^{-1} = \frac{\ell}{U} \tag{97}$$

we have:

$$\rho(U_{eff}^1) \sim e^{-\alpha U/U_{eff}} \tag{98}$$

Exactly the form we had from the RG! Is that also true for the Josephson distribution? you bet. In the most simplest version of this argument, we say that the Josephson coupling is exponentially proportional to the distance between puddles:

$$J \sim e^{-\gamma r} \tag{99}$$

but then, we can convert this directly to a distribution by using $e^{-\beta r} = J^{\beta/\gamma}$:

$$\rho(J) \sim J^{\beta/\gamma}\frac{\partial \ln J}{\partial J} = J^{\beta/\gamma-1}. \tag{100}$$

So the fact that every initial condition seems to flow to our universal distributions seems not even necessary - many systems are naturally described by these distributions. Of course, this is not too surprising, since the RG rules we used are consistent with a volume law for capacitance, and an exponential decay of wave functions

## IV.  PHYSICAL CONSEQUENCES

The RG analysis yielded a treasure trove of results. We know about the full universal distribution functions that describe the low energy physics of a random bosonic chain! This should allow us to calculate a variety of things. Some I will show you how to calculate here, and some I'll just mention, and give references for their calculation.

### A.  Dynamical scaling exponent

The scaling exponent tells us about how time and space scale relative to each other. You can also think about it as determining the energy-length scaling of the fundamental excitations in the system:

$$E \sim \frac{1}{L^z} \tag{101}$$

To calculate z for the random boson transition, we should calculate the relationship between $\Gamma = \ln \Omega_0/\Omega$ and length, or density.

As we reduce energy scale, we decimate sites and bonds such that:

$$dn = -n(f_0 + g_0)d\Gamma \tag{102}$$

Near criticallity we know that $f_0 \ll 1$ and $g_0 \to 1$. Therefore, we have:

$$d\ln(n) = -d\Gamma, \; and \, n \sim \Omega \tag{103}$$

So we easily find:

$$z = 1 \tag{104}$$

As people have speculated anyway 25 years ago.

## B. Gaps in the insulating phase

In the inulating phase of this model, we expect $g_0 \to 0$. Therefore, the flow equation for $f_0$ becomes:

$$\frac{\partial f_0}{\partial \Gamma} \approx f_0, so, \ f_0 \sim e^{\Gamma}. \tag{105}$$

This result allows us to easily obtain the smallest charging gap for an insulator of size $L$. We simply decimate the system until it only has one site, and then ask - at what scale did this happen? Going back to Eq. (102), given all the above, we get:

$$\frac{\partial \ln n}{\partial \Gamma} \approx e^{\Gamma} \tag{106}$$

and therefore:

$$\ln n \sim -e^{\Gamma} = -\frac{\Omega_0}{\Omega}. \tag{107}$$

When only one site remains, we can set $n = 1/L$, and then $\Omega = \Delta$ is the charging gap of the system. We obtain:

$$\Delta \sim \frac{1}{\ln L}. \tag{108}$$

## C. Compressibility

Very crudely, the compressibility just asks how much will the density changes when the chemical potential changes by a bit. From the smallest gap, we know that to raise the density by $1/L$ we need to change the chemical potential by $\Delta \sim 1/\ln L$. This logic leads to the statement:

$$\kappa = \frac{\partial n}{\partial \mu} \sim \frac{1}{L}\frac{1}{\Delta} = \frac{\ln L}{L} \tag{109}$$

which vanishes as $L \to \infty$.

The insulating phase with no offset charges is thus a gaples phase at the thermodynamic limit, and is therefore loosely called a glass, and it is incompressible. This earned this phase the name 'Mott-Glass' by[7] and, earlier, by[9].

The Mott-glass is a classic example of a Griffiths phase. A Griffiths phase occurs in disordered systems where near a phase-transition, the system exhibits clear signatures of the phase on the other side of the transition. Particularly, this is seen in temporal correlations, which often decay much more slowly than expected due to a finite density of enclaves of the opposite phase, which have anomalously long-lived temporal correlations.

The Mott-Glass is incompressible, yet the average temporal auto-correlations of the system will be dominated by the essentially single large cluster of size $\ln L$ that determines the gap.

## D. Luttinger parameter

An advanced application (i.e., it was quite hard!) of the method is the calculation of the Luttinger parameter[8]. The difficulty arises from the fact that the stiffness of the RG'd chain is:

$$\rho_S^{-1} = \frac{1}{L}\sum_{n \in SFcluster}\frac{1}{\tilde{J}_n} \tag{110}$$

What do I mean by $\tilde{J}_n$ with $n \in clusters$? We need to add the inverse bonds that make up the SF cluster - the ones we decimated using the cluster formation step, which led to the formation of the cluster. We did this, and the result was non-universal. Without going into details, we obtain that when disorder is larger than some crossover value, the Lttinger parameter at the transition we calculate is *larger* than $3/2$ (in this case, since we have particle hole symmetry, the proper thing to say is that $K > 2 = K_c$ for the SF-Mott glass transition). This is summarized in Fig. 9. This has become controversial - how come weak disorder and large disorder are so different? This is still debated, but recent evidence from Thomas Vojta's group is supportive of this prediction[10].
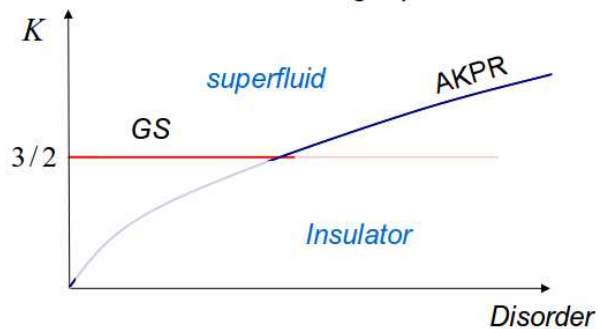
FIG. 9. The RSRG calculation yields that at large disorder, the superfluid phase is destroyed even at larger values of disorder than the Giamarchi Schulz (GS) value. At a finite value of disorder, there should be either a crossover or a transition between the GS weak disorder results, and the AKPR large disorder result. The graph is for the SF-Bose glass transition, where a random chemical potential is also presence, and therefore the comparison is to $K_{GS} = 3/2$. When the offset charges are all zero, we should have $K_{GS} = 2$.

## V.  THE TRANSVERSE FIELD ISING MODEL

### A.  Model and qualitative phases

The transverse field Ising model (TFIM), for good and for bad, forms many of our prejudices about quantum phase transitions. It pits an Ising interaction against a magnetic field that is normal to the Ising interaction axis, and induces quantum spin fluctuations:

$$\hat{\mathcal{H}} = -\sum_i (h\sigma_i^x + J\sigma_i^z \sigma_{i+1}^z). \tag{111}$$

Qualitatively, we can figure out what this model does in the extremes of $h/J$. When $h/J \gg 1$, all spins point in the x-direction, and the phase is paramagnetic. On the other hand, when $h/J \ll 1$, at $T = 0$, spins would like to order, and will form a cluster that either points up or down. This is a Ferromagnetic phase, and it breaks the $Z_2$ symmetry of the model.

The critical properties of this model have been worked out in many places, and they are pretty fun to calculate. Also, this model is quite magical in that it is self dual (moving to domain wall variables exchages h and J), it is also mappable to the Kitaev model, which means that it is a free Fermion model, and also, it is the quantum version of the 2d Ising model, and perhaps the simplest way of solving it. Here, let me quote just what we need. The correlation length exponent is $\nu = 1$:

$$\xi \sim \frac{1}{|h - J|}. \tag{112}$$

Given that Thomas already taught you what the Harris criterion is, let's use it to judge whether disorder is relevant or not. When we add disorder which is quenched in space:

$$\hat{\mathcal{H}} = -\sum_i (h\sigma_i^x + J\sigma_i^z \sigma_{i+1}^z), \tag{113}$$

we know that if

$$\nu < 2/d = 2$$

disorder must be relevant. This is indeed the case here, since $d = 1$ (the dimension of the disorder). Our expectation is that disorder will make the critical point completely different, and will in the very least change what $\nu$ is.

Our goal and program is by now really familiar to you. We want to find the ground state and low lying excitations of the model. We will do so by isolating the largest coupling of the chain:

$$\Omega = max\{J_i, h_i\} \tag{114}$$

and we will use perturbation theory to eliminate that bond or site. We will convert this into a flow distribution, and solve it. From there, we would be able to get anything.

To keep up the ante, let's add a twist to our analysis. Let's use unitary perturbation theory.

## B. Unitary perturbation theory

Something that I learned during grad school which I found really useful, is how to do perturbation theory using unitary transformation, rather than in terms of wave functions[11]. What's the idea? Split your Hamiltonian to:

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + V. \tag{115}$$

with $V$ small compared to $\hat{\mathcal{H}}_0$ in some sense. Now, we would like to find a transformation $U = e^{iS}$ for which:

$$e^{iS}\hat{\mathcal{H}}e^{-iS} = \hat{\mathcal{H}}_0 + \hat{\mathcal{O}}(V^2) + E_V. \tag{116}$$

with $E_V$ a shift in energy which commutes with everything. We can then also try to cancel higher powers of $V$.

Let's find an $S$ order by order that cancel the remnants of $V$. To first order:

$$e^{iS}\hat{\mathcal{H}}e^{-iS} = \hat{\mathcal{H}}_0 + V + i[S, \hat{\mathcal{H}}_0] + \dots \tag{117}$$

Now it is obvious what we would like to do. Look for:

$$V = -i[S, \hat{\mathcal{H}}_0] \tag{118}$$

If you manage to do this, you're in business! $S$ will be of order $V$, and then you could read off corrections to the Hamiltonian from the 2nd order of the transformed Hamiltonian. Continuing Eq. (117):

$$\dots i[S, V] - \frac{1}{2}S^2\hat{\mathcal{H}}_0 - \frac{1}{2}\hat{\mathcal{H}}_0 S^2 + S\hat{\mathcal{H}}_0 S. \tag{119}$$

In the end of the day, we hope to eliminate the non commuting pieces in the hamiltonian. We can then trivially diagonalize it, and obtain a wavefunction, $\left|\tilde{\Psi}\right\rangle$ to get to a true eigenstate of the original problem, we just need to undo the unitaries. Or:

$$e^{iS}\hat{\mathcal{H}}e^{-iS}\left|\tilde{\Psi}\right\rangle = E\left|\tilde{\Psi}\right\rangle \rightarrow \hat{\mathcal{H}}e^{-iS}\left|\tilde{\Psi}\right\rangle = Ee^{-iS}\left|\tilde{\Psi}\right\rangle \tag{120}$$

and we see that $e^{-iS}\left|\tilde{\Psi}\right\rangle$ is an eignestate.

Turns out that one can use this method generically for diagonalizaing finite matrices, as well as it serves a basis for a field theoretic RG method coined 'Wegners flow equations', on which there is a nice book by Stefan Kehrein, who is a pioneer of the method. The upshot of my exposure to 'flow equations' is that $S$ is vary often $[\hat{\mathcal{H}}_0, V]$. This is indeed the case in the TFIM.

Let us now derive the unitaries for h and J decimations of the TFIM. Maybe it is simpler to start with a large h. So choose:

$$\hat{\mathcal{H}}_0 = -h_1\sigma_1^x \tag{121}$$

In this case we would like to freeze site 1 in the +x direction. We don't need to decide on that though, and let's just find th unitaries. the main perturbation we would like to avoid is connection to neighbors:

$$V = -J_{01}\sigma_0^z\sigma_1^z - J_{12}\sigma_1^z\sigma_2^z. \tag{122}$$

In order to eliminate the first order couplings to spin 0, $S$ must satisfy equation (118); thus we first choose

$$S_a = -\frac{J_{01}}{2h_1}\sigma_0^z\sigma_1^y - \frac{J_{12}}{2h_1}\sigma_1^y\sigma_2^z \tag{123}$$

which yields the following terms in the effective Hamiltonian:

$$\hat{\mathcal{H}}_{eff} =$$
$$\dots - \frac{J_{01}J_{12}}{h_1}\sigma_0^z\sigma_1^x\sigma_2^z - \frac{h_0 J_{01}}{h_1}\sigma_0^y\sigma_1^y - \frac{h_2 J_{12}}{h_1}\sigma_1^y\sigma_2^y + \dots \tag{124}$$

The $\sigma^y\sigma^y$ interaction appears as a correction to the Hamiltonian which does not commute with $\hat{\mathcal{H}}_0$; site 1 is still coupled to adjacent sites by a second order interaction. By going to the next order, the Hamiltonian is restored to its original form. To get rid of it, we perform another transformation using

$$S_b = -\frac{h_0 J_{01}}{2h_1^2}\sigma_0^y\sigma_1^z - \frac{h_2 J_{12}}{2h_1^2}\sigma_1^z\sigma_2^y. \tag{125}$$
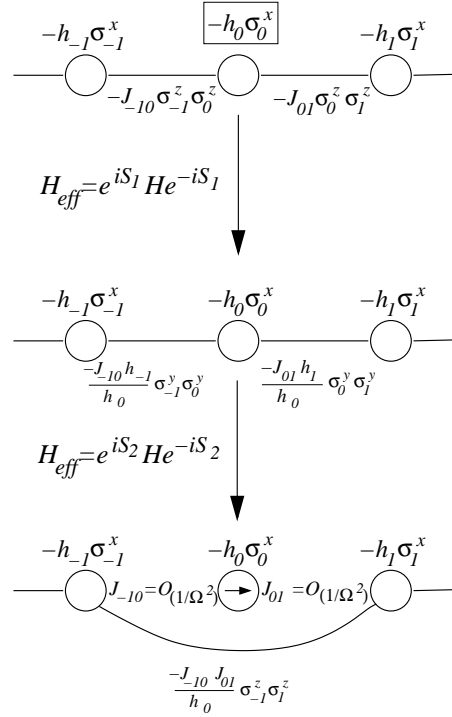
FIG. 10. Site decimation. Spin 0 is almost frozen in the $x$-direction due to the strong magnetic field $h_0$. Quantum fluctuations create a second nearest neighbor effective interaction between sites -1 and 1. This interaction is weaker than any of $J_{-10}$, $J_{01}$, $h_0$.

The effective Hamiltonian now includes

$$\hat{\mathcal{H}}_{eff} = \ldots - J_{-10}\sigma^z_{-1}\sigma^z_0 - h_0\sigma^x_0$$
$$-\tilde{J}_{02}\sigma^z_0\sigma^x_1\sigma^z_2 - h_2\sigma^x_2 - J_{23}\sigma^z_2\sigma^z_3 + \ldots, \tag{126}$$
$$-h_1\sigma^x_1,$$

from which we see that in the low-energy subspace of $\hat{\mathcal{H}}_0$, the effective exchange between spins 0 and 2 is given by

$$\tilde{J}_{02} = \frac{J_{01}J_{12}}{h_1}. \tag{127}$$

Since $h_1$ is the strongest coupling energy in the chain, the resulting effective bond obeys

$$J_{02} \ll h_0,\ J_{01},\ J_{12}, \tag{128}$$

where the sharpness of the inequality is because we assume strong randomness. We can now partially diagonalize $\hat{\mathcal{H}}_{eff}$ by writing

$$|H> = |\rightarrow>_1 |\tilde{G}^{(1)}>$$
$$|G> = e^{-iS_a}e^{-iS_b}|H>, \tag{129}$$

where $\sigma^x_1|\rightarrow_1> = |\rightarrow_1>$ and $|\tilde{G}^{(1)}>$ involves only the spins other than 1. We are left with a renormalized spin-chain with the spin at site 1 eliminated, and with an effective interaction $\tilde{J}_{02}\sigma^z_0\sigma^z_2$ between spin 0 and spin 2. [Note that we could also keep the high energy sector that involves $|\leftarrow_1>$; the effective Hamiltonian and state of the rest of the chain would differ from those of the low energy sector because of the presence of $\sigma^x_{(1)}$ in $\hat{\mathcal{H}}_{eff}$ of Eq. 126.]

The analog of the above results for the case where an exchange interaction, e.g. $\hat{\mathcal{H}}_0 = -J_{12}\sigma^z_1\sigma^z_2$, is eliminated is
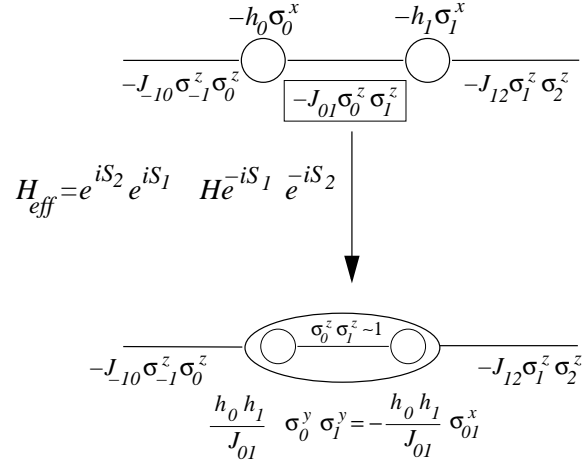
FIG. 11. Bond decimation. Sites 0 and 1 are frozen into one cluster by the strong Ising interaction, $J_{01}$. Quantum fluctuations produce an effective magnetic field, $\tilde{h}_{01} = \frac{h_0 h_1}{J_{01}}$, which flips the composite spin cluster. This field is weaker than any of $h_0$, $h_1$, $J_{01}$.

(Fig. 11)

$$\hat{\mathcal{H}}_0 = -J_{12}\sigma_1^z\sigma_2^z$$
$$V = -h_1\sigma_1^x - h_2\sigma_2^x$$
$$S_a = \frac{h_1}{2J_{12}}\sigma_1^y\sigma_2^z + \frac{h_2}{2J_{12}}\sigma_1^z\sigma_2^y \qquad (130)$$
$$S_b = -\frac{h_0 J_{01}}{2J_{12}^2}\sigma_1^y\sigma_0^z - \frac{h_3 J_{23}}{2J_{12}^2}\sigma_3^z\sigma_2^y.$$

This could be obtained by using the duality described in Ref. 12, or by direct computation. The ground state of $\hat{\mathcal{H}}_0 = -J_{12}\sigma_1^z\sigma_2^z$ is doubly degenerate with spins 1 and 2 either in the state $|\uparrow_{(12)}>= |\uparrow_1> |\uparrow_2>$ or in the state $|\downarrow_{(12)}>= |\downarrow_1> |\downarrow_2>$. Therefore in the ground state of $\hat{\mathcal{H}}_0 = \sigma_1^z\sigma_2^z$ spin 1 and 2 form a ferromagnetic cluster, which we denote as (12). We can define cluster operators, $\sigma_{(12)}^z$ and $\sigma_{(12)}^x$, that operate on the spin cluster (12) in the following way:

$$\sigma_1^z \Rightarrow \sigma_{(12)}^z$$
$$\sigma_2^z \Rightarrow \sigma_{(12)}^z \qquad (131)$$
$$-\sigma_1^y\sigma_2^y \Rightarrow \sigma_{(12)}^x$$

in terms of which

$$\hat{\mathcal{H}}_{eff} - \hat{\mathcal{H}}_0 =$$
$$\ldots - h_0\sigma_0^x - J_{0(12)}\sigma_0^z\sigma_{(12)}^z + \tilde{h}_{(12)}\sigma_{(12)}^x\sigma_1^z\sigma_2^z \qquad (132)$$
$$-J_{(12)3}\sigma_{(12)}^z\sigma_3^z - h_3\sigma_3^x + \ldots,$$

with

$$\tilde{h}_{(12)} = \frac{h_1 h_2}{J_{12}} \qquad (133)$$

being the effective transverse field on the new cluster (12) that has replaced the pair of spins 1 and 2 that now only appear separately in the high energy term in $\hat{\mathcal{H}}_0$. Again, since $J_{12}$ is the strongest energy, and strong randomness is assumed, the effective transverse field obeys:

$$h_{12} \ll h_1, \, h_2, \, J_{12}. \qquad (134)$$

The amazing thing in this process is that we didn't need to decide what the state we are intereseted in. We renormalized the *Hamiltonian* as a whole!

## C. Flow Equations and solution

Given the RG rules are products, it is clear we should define logarithmic variables:

$$\beta = \ln \frac{\Omega}{h} \ \ and \ \ \zeta = \ln \frac{\Omega}{J}. \tag{135}$$

With these, the RG rules are precise sum rules:

$$\beta_{eff} = \beta_L + \beta_R \tag{136}$$

for a $J$ decimation, and

$$\zeta_{eff} = \zeta_L + \zeta_R \tag{137}$$

for an $h$ decimation.

We don't need to derive the RG flow equations anew. They are essentially identical to the Random-Heisenberg flow, except that the probability in front of the convolution rule for J's belongs to the $h$ decimation, and vice versa. Denoting $P(\beta)$ and $R(\zeta)$ the distributions for field and bond variables, we get:

$$\begin{aligned} \frac{dP(\beta)}{d\Gamma} &= \frac{\partial P(\beta)}{\partial \beta} + R(0) \int d\beta_\ell \int d\beta_r P(\beta_\ell) P(\beta_r) \delta(\beta - \beta_\ell - \beta_r) + P(\beta) \cdot (P(0) - R(0)) \\ \frac{dR(\zeta)}{d\Gamma} &= \frac{\partial R(\zeta)}{\partial \zeta} + P(0) \int d\zeta_\ell \int d\zeta_r R(\zeta_\ell) R(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r) + R(\zeta) \cdot (R(0) - P(0)) \end{aligned} \tag{138}$$

Two things to note. First, unlike the Heisenberg model, we now do have terms that correct for normalization. The $\pm(R_0 - P_0)$ terms that are last in both equations just rescale the distribution which they multiply, to make sure that probability remains constant. Second, There is a subtle difference relative to the random boson equations - no $\zeta \frac{\partial f}{\partial \zeta}$, as opposed to Eq. (88). This subtle difference is what reflects the fact that the random bosons RG rules do not lead to infinite randomnes.

What are the fixed point distributions going to be? You can guess it - simple exponents. Again, let's plug in an exponenetial ansatz:

$$P(\beta) = be^{-b\beta}, \ \ and \ \ R(\zeta) = ge^{-g\zeta}. \tag{139}$$

where $b$ and $g$ are both functions of $\Gamma$. Plugging these into the flow equations (138) immediately reduces to the flow equation:

$$\frac{db}{d\Gamma} = \frac{dg}{d\Gamma} = -b \cdot g \tag{140}$$

Almost the same as random bosons, if we replace $f$ with $b$. Solving these is a bit easier, since the flow equations are self dual. We note that the difference of the two paramaters is constant, and therefore:

$$\frac{d}{d\Gamma}(b - g) = 0 \rightarrow b = g + 2\delta \tag{141}$$

where $\delta$ is going to serve as the tuning parameter of the FM-PM transition. Plugging this back in, we get:

$$\frac{dg}{d\Gamma} = -g(2\delta + g). \tag{142}$$

This equation can be directly integrated, and we obtain:

$$g = -\delta + \delta \cot \delta\Gamma, \ \ and \ \ b = +\delta + \delta \coth \delta\Gamma. \tag{143}$$

This is pretty much it! Next, consequences. All of this was worked out by Daniel Fisher in his 1995 killer paper, 12.

## D. Critical properties

We obtained the family of universal distributions as a function of a single parameter - $\delta$. Just as we had it for the random bosons, when $\delta = 0$ we have a critical point. We can see this since then $b = g \rightarrow \frac{1}{\Gamma}$, which results in the same

ditribution as the random singletfixed point. But, what is $\delta$ in terms of initial distributions? As it turns out, and as you can verify from the parameters $b$ and $g$,

$$\delta = \frac{\overline{\ln J} - \overline{\ln h}}{\sqrt{var(\ln J) + var(\ln h)}} \tag{144}$$

So $\delta > 0$ implies $J > h$, and a ferromagnetic phase, and $\delta < 0$ is the paramagnetic phase. Now we essentially have everything. What are the universal physical properties we can quickly get?

(1) First, as for the Heisenberg model, let's get the density as a function of energy scale. Every time we lower the energy scale, we have probability $P(0)d\Gamma$ to have an h-decimation, which removes a single site, and $R(0)d\Gamma$ probability to form a cluster out of two site, which also removes a single site. Together, their effect on the density is:

$$dn = -n \cdot (b+g)d\Gamma = -nd\Gamma 2\delta \coth(\delta\Gamma). \tag{145}$$

Integrated, this gives:

$$n = \frac{n_0 \delta^2}{\sinh^2 \delta\Gamma} \tag{146}$$

(2) How about correlation exponent? We said at first, that disorder must be important due to Harris criterion. In the clean case, $\nu = 1 < 2/d = 2$. Do we satisfy the Harris criterion now? Let's see. What determines the correlation length? It is the length scale where it becomes clear that the chain is going to one phase or the other. Looking at the distribution functions, it should be where the coth piece stops being important. This happens when $\Gamma \sim 1/|\delta|$.

Near criticality, the density-energy relationship is the same as that for the random singlet phase [as letting $\delta \to 0$ shows in Eq. (146)]. So the length in which this crossover occurs is:

$$\ell \sim \Gamma_{CO}^2 \sim \frac{1}{|\delta|^2} \tag{147}$$

But this is exactly

$$\nu = 2 \tag{148}$$

as the Harris criterion dictates.

(3) Here is a rather crucial fact that makes the TFIM near criticality into a Griffiths phase. What is the dynamical critical exponent $z$ $(E \sim \ell^{-z})$?

At criticallity we have infinite randomness, so $z \to \infty$, and we have the infinite randomness scaling. But off criticality, if we look back at eq. (146), we should be able to read off the relationship between the lengthscale of excitations, $\ell$, and their energy, by saying $\ell \sim 1/n$. We are looking for the scaling at large lengths, and therefore need to look at large $\Gamma$. we then have:

$$\ell \sim 1/n \sim e^{2|\delta|\Gamma} = e^{2|\delta| \ln \frac{\Omega_1}{E}} \sim E^{-2|\delta|} \tag{149}$$

This is looking more like regular energy-length scaling, and we read off $z$:

$$z = \frac{1}{2|\delta|} \tag{150}$$

This energy-length scaling is the consequence of having extended deffects of the opposite phase in both the paramagnetic phase, as well as the ferromagnetic phase. Otherwise, the phase would have been gapped. The probability of these defects diminishes as one moves away from the critical point. This is reflected in $z$ becoming more and more normal as $|\delta|$ grows.

(4) An interesting thing I won't derive is the correlations. Daniel Fisher calculated in his 95 paper that:

$$\overline{\langle \sigma_n^z \sigma_{n+x}^z \rangle} \sim \frac{1}{x^{2-\phi}} \tag{151}$$

with $\phi = \frac{1+\sqrt{5}}{2}$, which is the golden ratio. So the golden ratio pops up here as well.

## E.   Hilbert-glass transition in the TFIM

In the last few lines of this note I can add that the TFIM has played a strong role in our thinking of many-body localization (MBL) like transitions. The MBL transition is a finite temperature transition from a disordered electronic system having no conductance at all, to having a finite conductance. It has no thermodynamic signatures otherwise. This transition is, therefore, between two classes of eigenstates of the system - localized, and delocalized.

The fact that we can carry out the RSRG in the TFIM without determining the state until the very end, and not necessarily choose the ground state, means that the entire spectrum exhibits the FM-PM transition when $\delta = 0$ - not just the ground state. At finite temperatures, there are no signatures in the thermodynamics for this fact. Phase transitions in any case are not allowed at 1d, finite temperature. The dynamics, though, will show that transition.

What are the dynamical signatures? On the FM side, each eigenstate has an inifnte cluster where all spins point in the z-direction. Since most eigenstates are not ground states, there are going to be a lot of misalligned bonds in the typical state. But the z-magnetization pattern is not going to fluctuate. So:

$$C_t = \overline{\sum_n \langle n | \sigma_i^z(0)\sigma_i^z(t) | n \rangle} \tag{152}$$

(with $n$ being a state index, and the sum runs on the entire spectrum) is going to have a finite value in the $\delta > 0$ FM side, and will average to 0 at the paramagnetic side.

We speculate that this is the generic picture for dynamical phase transitions of the kind MBL belongs to. Becuase it is the eigenstates at arbitrary energy responsible for this, it is a transition in the Hilbert space, and we named it the Hilbert glass transition[13].

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