

July 7, 2010

I. REAL SPACE RG OF DISORDERED 1D SYSTEMS.

A. overview

The ultimate goal of these lectures is to first and foremost, teach you how to use real space renormalization group to solve strongly disordered systems, and second to explore the application of this method to understand the low energy and critical properties of the strongly random interacting bosonic chain near its superfluid insulator transition.

If you think about problems you have so far encountered as researchers and students, the vast majority of them had translational symmetry. Therefore the number one hit of mathematical methods you learned over and over again was Fourier analysis. Regardless of what the problem is - say diagonalizing a tight binding Hamiltonian in 1d:

$$\hat{\mathcal{H}} = -t \sum_i (c_i^\dagger c_{i+1} + H.C.) \quad (1)$$

we use Fourier transform, and diagonalize the Hamiltonian:

$$c_k = \sum_j c_j e^{ikx_j} \quad (2)$$

and as we all know, this gives:

$$\hat{\mathcal{H}} = -2t \sum_k \cos(ka) c_k^\dagger c_k \quad (3)$$

with a the lattice constant.

But what happens when we have disorder present? For instance, what happens if the hoppings are random?

$$\hat{\mathcal{H}} = - \sum_i J_i (c_i^\dagger c_{i+1} + H.C.) \quad (4)$$

Which means that there is a distribution of hopping strengths, and we can assume no correlation between hopping at different places. (Fig. 1)

Of course the number one answer is - plug it into a computer and see what you get! But in my two classes I will teach you a very powerful method which tackles this problem analytically. Today I will discuss a problem related to the random hopping of fermions, or, hard core bosons - the random 1d Heisenberg model:

$$\hat{\mathcal{H}} = \sum_i J_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j. \quad (5)$$

The solution for this problem was developed by Ma and Dasgupta^{1,2} and Daniel Fisher³. Using real-space Renormalization group, Fisher found the ground state and low lying excitations of this model. What I will show you applies equally well to the random hopping problem of hard core bosons, or, of non-interacting fermions. So part of your job for tonight or this weekend would be to plug in the random hopping problem to Mathematica or Matlab, and see if what I'm telling you is correct.

Tomorrow, we will apply the same methods to a problem of current interest: The superfluid insulator transition in 1d in the presence of strong randomness:

$$\hat{\mathcal{H}} = \sum_i [-J_i \cos(\phi_{i+1} - \phi_i) + U_i \hat{n}_i^2]. \quad (6)$$

Which is the Hamiltonian describing a random array of superconducting grains.

The method we will develop is the real-space renormalization group. For those of you who have nightmares about epsilon-expansions - don't worry, we will do everything from scratch, and by the end (hopefully...) you'll have a good intuition about the process of renormalization. Our work plan will be: Decimation:

- Identify the highest energy piece in the Hamiltonian.
- Eliminate the highest energies by solving the highest energy piece as if no other terms in the Hamiltonian exist.

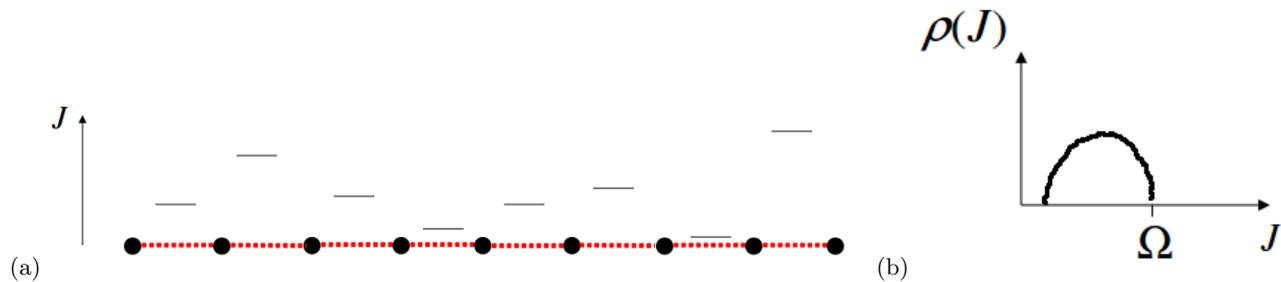


FIG. 1: (a) The bond strength J varies from cite to cite as if each J is an independent random variable. (b) Each Hamiltonian with random J 's is described by a distribution of J 's, say, the probability density $\rho(J)$.

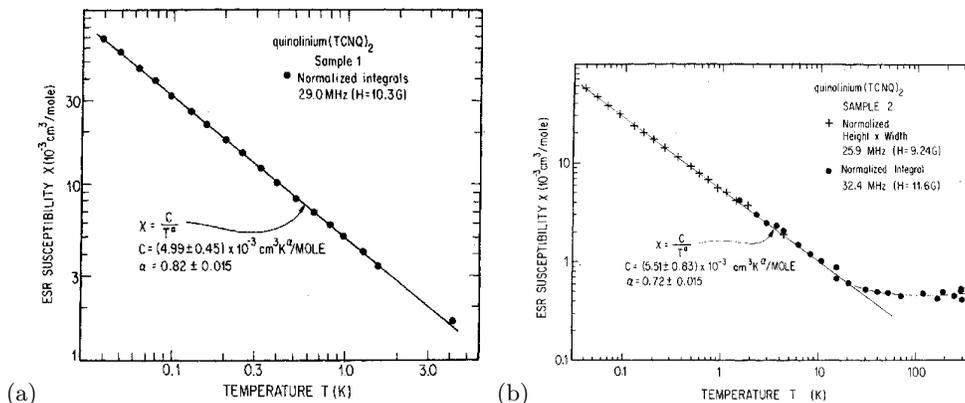


FIG. 2: Susceptibility of two different samples of the same material: the quasi-1d $Qn(TCNQ)_2$ ^{4,5}. The susceptibility of these spin-1/2 chains diverged as $T \rightarrow 0$ with a perceived $T^{-\alpha}$ power law, with α varying from sample to sample.

- Incorporate the rest of the Hamiltonian using perturbation theory, or other means. For the method to work, the resulting Hamiltonian will have a lower maximum energy scale. Flow:
- Iterate the decimation steps until the Hamiltonian is completely solved.

In the last step, we will take a conceptual leap and instead of following how one Hamiltonian evolves, we'll consider how an ensemble of Hamiltonians evolves. That means - we will look at distribution functions of couplings, and ask how they evolve during our RG treatment.

II. RANDOM HEISENBERG MODEL

So the model we would attack today is the random antiferromagnet in 1d:

$$\hat{\mathcal{H}} = \sum_i J_i \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_{i+1}. \quad (7)$$

The motivation to study this model came from experiments done on quasi-1d organic salts, mostly $Qn(TCNQ)_2$ ^{4,5}, which showed a remarkable susceptibility:

$$\chi \sim T^{-\alpha} \quad (8)$$

with $\alpha < 1$ and varying from sample to sample. These salts have chains of stacked double benzene rings, with each pair having one excess spin-1/2. (Fig. 2)

Recall that the Curie susceptibility of free spins is $\chi \sim T^{-1}$.

A. Pure Heisenberg chain?

What are the low energy properties of a uniform Heisenberg chain? Well, without going to much into details, a good rule of thumb is that the Heisenberg model is in the same phase as x-y chain, where the sz-sz coupling is missing:

$$\hat{\mathcal{H}} = \sum_i J \left(\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - \hat{S}_i^z \hat{S}_{i+1}^z \right) = \sum_i J \hat{S}_i^+ \hat{S}_{i+1}^- \quad (9)$$

Which is in turn, mappable to the fermionic chain model in Eq. (1), where a site with spin up maps to a full site, and a spin-down site maps to an empty site in the fermionic chain.

The fermionic chain is just a Fermi-surface at zero energy with a host of gapless excitations. No broken symmetries, no gap, such a state is called a spin liquid. Now, if fermion occupation is like spin, fermion chemical potential is like a z-magnetic field, and therefore the magnetic susceptibility of our Heisenberg chain is like the - density of states: a constant. Not a power law:

$$\chi_{\text{pure}} \sim \text{const.} \quad (10)$$

These results can be made more precise using Bosonization, or for the bold amongst you, but Bethe Ansatz.

B. Ma-Dasgupta decimation

When the disorder is strong, clearly there is no use for using Bosonization. Anything with a momentum-index is going to lead to a disappointment. So what do we do? We take the opposite limit to translational invariance - we assume a very strong disorder.

If disorder is very strong, if we find the largest coupling in the chain, say $J_n = \Omega$ (we use Ω henceforth to denote the highest coupling in the chain), its nearest neighbors are really unlikely to be as strong as Ω :

$$J_{n\pm 1} \ll \Omega \quad (11)$$

This means that we can definitely diagonalize the J_n piece in the Hamiltonian as if it is the only thing around. The spectrum of this piece is:

$$\hat{\mathcal{H}}_n = J_n S_n S_{n+1} = \frac{J}{2} \left[(S_n + S_{n+1})^2 - 2 \cdot \frac{3}{4} \right] = \frac{J_n}{2} \left[s_{\text{tot}}(s_{\text{tot}} + 1) - 2 \cdot \frac{3}{4} \right] \quad (12)$$

which means:

$$\begin{aligned} |s_{\text{tot}} = 0\rangle_{n,n+1} &= \frac{1}{\sqrt{2}} (|\uparrow\rangle_n |\downarrow\rangle_{n+1} - |\downarrow\rangle_n |\uparrow\rangle_{n+1}) & E &= -3J_n/4, \\ |s_{\text{tot}} = 1\rangle &= \frac{1}{\sqrt{2}} (|\uparrow\rangle_n |\downarrow\rangle_{n+1} + |\downarrow\rangle_n |\uparrow\rangle_{n+1}) & E &= J_n/4. \end{aligned} \quad (13)$$

Now - we are after the ground state, so we choose the singlet state.

But what about the other pieces in the Hamiltonian? certainly the neighboring bonds are not happy with our executive decision to put spins n and $n + 1$ in a singlet. They want a piece of the action. They get it through perturbation theory. The perturbation is simply the rest of the Hamiltonian, but more specifically, the nearest neighbors:

$$V = J_{n-1} S_{n-1} S_n + J_{n+1} S_{n+1} S_{n+2} \quad (14)$$

It is easy to verify that:

$${}_{n,n+1} \langle s_{\text{tot}} = 0 | V | s_{\text{tot}} = 0 \rangle_{n,n+1} = 0 \quad (15)$$

So no first order contribution. The next contribution we can get from a neat way of writing the second order perturbation:

$$E^{(2)} = \langle \psi | \frac{V \sum_m |m\rangle \langle m| V}{E_0 - E_m} | \psi \rangle \quad (16)$$

where the energy denominators are calculated according to $\hat{\mathcal{H}}_n$ only. Also:

$$|\psi\rangle = |\{s_i\}_{i < n}\rangle |s_{\text{tot}} = 0\rangle_{n,n+1} |\{s_i\}_{i > n+1}\rangle \quad (17)$$

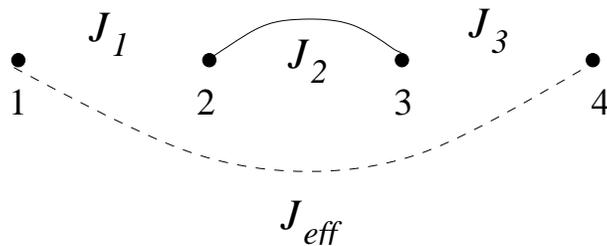


FIG. 3: The strong bond between sites 2 and 3 localizes a singlet. Quantum fluctuations induce a new bond between sites 1 and 4 which has the strength $J_{eff} = J_1 J_3 / 2J_2$.

so and $|m\rangle$ are the excited states of the n and $n + 1$ spins, namely, the triplet states:

$$|s_{tot} = 1; m = \pm 1, 0\rangle_{n,n+1} \quad (18)$$

with the rest of the chain not playing a role. So $E^{(2)}$ can actually be thought of as an effective Hamiltonian for all spins *except* n and $n + 1$:

$$E^{(2)} \rightarrow H_{n-1, n+1}^{(2)} = {}_{n,n+1} \langle s_{tot} = 0 | \frac{\sum_{m=-1}^1 |s_{tot} = 1; m\rangle_{n,n+1} \langle s_{tot} = 1; m| V}{-J_n} |s_{tot} = 0\rangle_{n,n+1} \quad (19)$$

Eq. (19) has many terms in it, but they all follow a simple pattern. Let me demonstrate just with one term, which takes the z-z terms from the $n - 1$ bond and from the $n + 1$ bond:

$$H_{n-1, n+1}^{(2)} = \dots - \frac{1}{J_n} {}_{n,n+1} \langle s_{tot} = 0 | J_{n-1} S_{n-1}^z S_n^z |s_{tot} = 1; m = 0\rangle_{n,n+1} \langle s_{tot} = 1; m = 0 | S_{n+1}^z J_{n+1} S_{n+1}^z S_{n+2}^z |s_{tot} = 0\rangle_{n,n+1} + \dots \quad (20)$$

Which we see becomes:

$$\rightarrow \dots \frac{J_{n-1} J_{n+1}}{4\Omega} S_{n-1}^z S_{n+2}^z + \dots \quad (21)$$

the x-x and y-y terms in V will eventually all add up to give us the rotationally symmetric effective Hamiltonian:

$$H_{n-1, n+1}^{(2)} = \frac{J_{n-1} J_{n+1}}{2\Omega} S_{n-1} \cdot S_{n+2} \quad (22)$$

This is what we wanted. We eliminated the two strongest interacting spins, so the chain is now shorter. We made a cut in the chain, but quantum fluctuations expressed through second order perturbation theory introduce a rather small effective coupling between the nearest neighbors of the singlets:

$$J_{n-1, n+2}^{eff} = \frac{J_{n-1} J_{n+1}}{2\Omega} \ll J_{n-1}, J_n, J_{n+1} \quad (23)$$

where the much-less-than is a consequence of our strong disorder assumption: $J_{n\pm 1}/J_n \ll 1$. So we reduced the overall strongest coupling of the Hamiltonian, Ω , since now the strongest coupling will be determined by what used to be the second strongest bond in the chain. Most importantly, after the decimation step, we have exactly the same form of the Hamiltonian: a nearest-neighbor Heisenberg model.

C. Qualitative ground state picture: the random singlet phase

The next step in our analysis must be the repeated application of the decimation step. If disorder increases upon the repeated use of the Ma-Dasgupta rule, as it is called, then we are safe. The strong disorder assumption only becomes better and better. This is indeed the case, as Daniel Fisher proved, and we will notice soon.

The decimation rule gives us a simple picture of the random Heisenberg model ground state. The strong bonds are going to localize a singlet, with many singlets forming between nearest neighbors. But as the largest bonds are



FIG. 4: The random singlet phase of a random Heisenberg model. Pairs of strongly interacting sites form non-overlapping singlets in a random fashion. These singlets mostly form between nearest neighbors, but also over an arbitrarily large distance. The long range singlets induce strong correlations between far away sites.

decimated, and the energy scale of the Hamiltonian, Ω is reduced, the largest bonds connect further nearest neighbors, and singlets may form between really far away sites (Fig. 4).

Already from this we can infer important information about our system: its *average* correlations. If we ask about the correlations in a particular chain (i.e., **in a particular realization of the random bonds**):

$$C_{nm} = \langle S_n \cdot S_m \rangle \quad (24)$$

most likely, they are really small - exponentially suppressed with the distance (as it turns out, with the square root of the distance):

$$C_{nm}^{typical} \sim e^{-c\sqrt{|m-n|}}. \quad (25)$$

But if you are very lucky in the choice of sites, m and n might be connected with a singlet, and then:

$$C_{nm}^{rare} \sim -1. \quad (26)$$

But how rare is rare? If the two sites at hand survive the violence of the decimation procedure until they are nearest neighbors, very likely they will form a singlet. The density of surviving sites at the stage of the decimation procedure where m and n could be nearest neighbor is $n \sim \frac{1}{|m-n|}$ (note that n on the LHS is the density), so only one out of $|m-n|$ sites survives. What is the probability of the *two* sites surviving?

$$p_{m-n\text{singlet}} \sim \frac{1}{(m-n)^2}. \quad (27)$$

Now, when we **average over many realizations** of the random chain, or over many pairs of sites at the same distance, we obtain the **disorder averaged** correlation, marked with an overline:

$$\overline{C_{nm}} = (-1)p_{m-n\text{singlet}} + (1 - p_{m-n\text{singlet}})e^{-c\sqrt{|m-n|}} \approx -p_{m-n\text{singlet}} \sim \frac{1}{(m-n)^2}. \quad (28)$$

Power-law correlations!

So despite the localized nature of the ground state, the average correlations fall off only as a power law. This is a wonderful example of *Griffiths effects* - where the average correlations of a random system are dominated by rare instances with anomalously strong correlations.

D. Distribution function flow

The behavior of the random Heisenberg model goes beyond the properties of the ground state, and requires us to understand the energy scales associated with the decimated singlets. To find these, we must find the properties of the hamiltonian as we iterate the decimation procedure.

To formulate the problem at hand, consider what defines the problem. We begin our analysis with a chain of spins given to us where each bond has some interaction strength, J_n . One can construct a histogram of the bonds - this produces some distribution which characterizes the Hamiltonian:

$$\rho(J). \quad (29)$$

Let us now rethink our mission a bit. Instead of solving this particular hamiltonian, let us say that we are interested in the statistical properties of *all* Hamiltonians described by the same distributions $\rho(J)$, regardless of the arrangements of the various bonds. If we accept this, we can now ask the question: As we decimate the chains with distribution $\rho(J)$, how does the *distribution* evolve in the process? Namely, what is:

$$\rho_{\Omega}(J) \quad (30)$$

with $J < \Omega$. To obtain the *flow* of the distribution function with the repeated decimation procedure, we can do one of two things: (a) carry out the decimation procedure numerically on a set of random chains, and obtain the average distribution at each energy scale, (b) write some kind of a master equation, and pray that we can solve it. During your off time from Hiking, I'd expect you to carry out the former, and here we will pursue the latter.

Before we write a master equation, we need some prep work. The one thing we have to work with is the decimation rule, Eq. 23. It is a product. Plus, at each stage of the RG we have $0 < J < \Omega$ and ω keeps decreasing. Experience shows that it is much better to work with quantities, say, ζ , where ζ is unbounded by any scale: $0 < \zeta < \infty$, and where the RG rule is a sum:

$$\zeta_{n-1,n+2}^{eff} = \zeta_{n-1} + \zeta_{n+1} \quad (31)$$

Turns out - we can do exactly that. Define:

$$\zeta_n = \ln \frac{\Omega}{J_n} \quad (32)$$

For the RG rule this reads:

$$-\ln \frac{J_{n-1,n+2}^{eff}}{\Omega} = -\ln \frac{J_{n-1}}{\Omega} - \ln \frac{J_{n+1}}{\Omega} \quad (33)$$

which is indeed right, if you are willing to overlook a $\ln 2$ which is negligible compared to $\ln \Omega/J$ due to the strong disorder assumption. Also, for all bonds, since $0 < J_m < \Omega$, $0 < \zeta_m < \infty$ indeed. Notice that the largest bonds have $\zeta_{largestbond} = 0$. And since logs seem to be so nice, let's continue with this defining inertia and define the RG flow parameter to be:

$$\Gamma = \ln \frac{\Omega_0}{\Omega} + \Gamma_0. \quad (34)$$

with Ω_0 the initial largest energy in the chain, and Γ_0 some constant of order one (this is like an initial "RG time" from which we flow, and will be determined by the initial distribution of the chain we are solving).

Now we are ready to derive the master equation. Let's draw the initial distribution of ζ 's - which we denote $P_\Gamma(\zeta)$. How do we decimate? We take all the bonds m that are large, with $\Omega - d\Omega < J_m < \Omega$. It is really unlikely they are close to each other, so we consider them all in one fell swoop. We remove the strong bonds from the distribution, and introduce new bonds, deriving from the nearest neighbors.

Let's consider this in the language of the ζ 's. We take all bonds with $0 < \zeta < d\Gamma$ with:

$$d\Gamma = d \ln \frac{\Omega_0}{\Omega} = \frac{d\Omega}{\Omega} \quad (35)$$

We eliminate these small ζ 's, but this also implies a redefinition of all ζ_m 's:

$$\zeta_m \rightarrow \tilde{\zeta}_m = \ln \frac{\Omega - d\Omega}{J_m} = \zeta_m - \frac{d\Omega}{\Omega} = \zeta_m - d\Gamma \quad (36)$$

So the entire distribution $P(\zeta)$ moves to the left. This can be expressed mathematically:

$$dP(\zeta) = \frac{\partial P(\zeta)}{\partial \zeta} d\Gamma. \quad (37)$$

But there is a second contribution: adding the renormalized bonds. First, what is the probability distribution of a renormalized bond? It is the convolution of the probabilities of the left bond ζ_ℓ and right bond ζ_r :

$$P^{RG}(\zeta) = \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r) \quad (38)$$

Now, how many of these do we produce? We look at each bond and ask - what is the probability that it will get decimated in the next step when $\Gamma \rightarrow \Gamma + d\Gamma$? It is $dp = d\Gamma P(\zeta = 0)$. And this contribution also gives a differential change to the full distribution:

$$dP(\zeta) = dp P^{RG} = d\Gamma P(\zeta = 0) \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r) \quad (39)$$

Together, Eqs. (37) and (39) give the full master equation:

$$\frac{dP(\zeta)}{d\Gamma} = \frac{\partial P(\zeta)}{\partial \zeta} + P(0) \int d\zeta_\ell \int d\zeta_r P(\zeta_\ell) P(\zeta_r) \delta(\zeta - \zeta_\ell - \zeta_r) \quad (40)$$

where we dropped the Γ subscript of $P(\zeta)$.

There is a point we glossed over. We removed some probability by getting rid of all the probability density at small ζ . We added probability by adding all the new bonds. Do we need to adjust the normalization of our distribution function? Integrating both sides of Eq. (40) reveals that the normalization is unchanged. For each bond we lost, we added a renormalized bond.

That's it - we have the master equation. Can we solve it?

E. Scaling of the flow equation

The flow equation (40) looks like the physicists worst nightmare: integro-differential, multivariable, and non-linear. To get some color back to our cheeks, let's do what we often do faced with a complicated non-linear equation (as well as an RG flow!): rescaling.

$P_\Gamma(\zeta)$ has two variables, Γ and ζ . We can ask whether there is a *scaling solution*. Namely, let's try to use a new variable $x = \zeta/\Gamma^\phi$ instead of ζ . The distribution function for x will be:

$$P_\Gamma(x) = \frac{1}{\Gamma^\phi} Q(\zeta) \quad (41)$$

(which we obtain by saying $Q(x)dx = P(\zeta)d\zeta$). This is our *scaling ansatz*. Can we find a ϕ which eliminates Γ altogether? Let's see.

The master equation in terms of Q and x is:

$$\frac{dQ_\Gamma(\zeta/\Gamma^\phi)/\Gamma^\phi}{\Gamma} = \frac{1}{\Gamma^{2\phi}} \frac{\partial Q(x)}{\partial x} + \frac{1}{\Gamma^{2\phi}} Q(0) \int dx_\ell \int dx_r Q(x_\ell) Q(x_r) \delta(x_r + x_\ell - x) \quad (42)$$

The LHS is simply:

$$-(Q_\Gamma(x) + xQ'_\Gamma(x))\phi/\Gamma^{\phi+1} \quad (43)$$

where we assumed that $Q(x)$ does not have a Γ dependence. We see that

$$\phi = 1 \quad (44)$$

eliminates Γ altogether, and yields:

$$0 = (x+1)Q'(x) + Q(x) + Q(0) \int dx_\ell \int dx_r Q(x_\ell) Q(x_r) \delta(x_r + x_\ell - x) \quad (45)$$

Even before solving this equation, within the scaling ansatz we already managed to glean useful information:

$$\zeta_{\text{typical}} \sim \Gamma \quad (46)$$

which means:

$$\ln \frac{\Omega}{J_{\Omega}^{\text{typical}}} \sim \ln \frac{\Omega_0}{\Omega}$$

and:

$$J^{\text{typical}} \sim \frac{\Omega^2}{\Omega_0} \quad (47)$$

So as the highest energy scale in the Hamiltonian drops, the typical bond strength drops even more. The distribution becomes more and more weighted towards $J = 0$. This confirms the strong disorder assumption. In fact, it tells us that disorder is *relevant* in the RG sense. This means that disorder *grows* upon RG flow.

F. Solving the master equation

Let's us take the leap and find a solution of the flow equation. Roughly speaking we need a function that retains its form upon convolution. There is a simple answer: an exponent. Once we realize this, it is easy to see that the solution is simply:

$$Q(x) = e^{-x}. \quad (48)$$

All this pain for this simple answer! But what does it mean? First, it is a fixed point - in the sense that the solution for $Q(x)$ is stationary under the flow. The solution for $P(\zeta)$ is:

$$P_\Gamma(\zeta) = \frac{1}{\Gamma} e^{-\zeta/\Gamma} \quad (49)$$

The width of the ζ distribution increases indefinitely upon the RG flow. This has received the name "infinite randomness fixed point" to this phase of matter. The ζ distribution transformed to a J distribution (don't forget the Jacobian! $P(\zeta)d\zeta = \rho(J)dJ$) is:

$$\rho_\Omega(J) = \frac{1}{\Omega \cdot \Gamma} \left(\frac{\Omega}{J} \right)^{1-1/\Gamma} \quad (50)$$

As Γ tends to ∞ this distribution becomes less and less integrable. Infinite randomness again.

What we found, though, is *a solution* of the flow equation for the distributions of bond strengths upon RG. How do we know it is in anyway *the solution*? Namely, there are at least three possibilities: (a) This solution is obtained only if the initial distribution of the couplings conforms with it at some Γ_0 , (b) This is the low-energy distribution function for a range of coupling distributions, (c) this is the low-energy distribution to which *all* initial distributions flow to. In option (a) the solution we found is unstable, since any deviation will destroy it. Option (c) is the most tantalizing one: In this case, the scaling solution we found is a *global attractor*.

Option (c) obtains: the solution we found is *universal*! Every chain you consider, will, after enough sites have been eliminated, look like a chain with the distribution in Eq. (50).

G. Physical properties

Now that we have the distribution as a function of energy scale, we should be able to extract all physical properties. Let's start with the density of surviving spins as a function of energy.

Every time we decimate a strong bond, we remove two sites. The probability of eliminating a bond upon changing the RG scale by $d\Gamma$ is simply $dp = d\Gamma P(0)$ from the discussion above. This implies:

$$dn = -2 \cdot n \cdot d\Gamma P(0) = -2n \frac{d\Gamma}{\Gamma}. \quad (51)$$

where dn is the change in density, and the 2 is due to the two spins that disappear. The solution of this equation is simply:

$$n = \frac{n_0}{\Gamma^2}. \quad (52)$$

The density decreases with energy scale as $1/\ln \Omega^2$.

Energy-length scaling

Eq. (52) also yields the energy-length scaling of the random-singlet phase. The length of a singlet is always $\ell = 1/n$, since singlets form between nearest-neighbors. So singlets that form at RG scale Γ are of length:

$$\ell = \ell_0 \Gamma^2 \quad (53)$$

Which means that the *excitation energy* of singlets of length ℓ is:

$$J_\ell \sim e^{-\sqrt{\ell}} \quad (54)$$

which is contrary to usual quantum-critical point scaling where $E \sim 1/\ell^z$. this type of scaling is called infinite-randomness scaling.

Susceptibility.

Now is our chance to tie it all together. The motivation for our study was the organic salts showing susceptibility $\chi \sim 1/T^\alpha$. What is the susceptibility expected for the random-singlet theory as a function of temperature?

If we would like to consider the properties of our system as a function of temperature, all we need to do is just run the RG until we hit:

$$\Omega = T. \tag{55}$$

At that point, all the singlets we formed are at energies much larger than T , so they are frozen. All the remaining sites are coupled to their neighbors with $J_{eff} \ll T$, so they are essentially free. Therefore the density of free spins at temperature T is:

$$n(T) = \frac{n_0}{(\Gamma_0 + \ln \frac{\Omega_0}{T})^2}. \tag{56}$$

The susceptibility of free spins is the Currie susceptibility, $\sim 1/T$. Therefore:

$$\chi(T) = \frac{n_0}{T \ln^2 \frac{\Omega_1}{T}} \tag{57}$$

with $\Omega_1 = \Omega_0 e^{\Gamma_0}$. Not quite $T^{-\alpha}$; but when you take a log of the above function, you encounter the log of a log. There nearly no line straighter than a double log! If you look at the slide attached to the notes you'll see that the two functions indeed compare perfectly well in the given range.

Next: Bosons.

¹ S. K. Ma, C. Dasgupta, and C. K. Hu, Phys. Rev. Lett. **43**, 1434 (1979).

² C. Dasgupta and S. K. Ma, Phys. Rev. B **22**, 1305 (1980).

³ D. S. Fisher, Phys. Rev. B **50**, 3799 (1994).

⁴ L. C. Tippie and W. G. Clark, Phys. Rev. B **23**, 5846 (1981).

⁵ L. C. Tippie and W. G. Clark, **23**, 5854 (1981).

Susceptibility in the Random-Singlet phase

Currie:

$$\chi \sim \frac{n}{T} = \frac{n_0}{T \ln^2(T_0/T)}$$

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-How does this compare

with $\chi \sim \frac{1}{T^\alpha}$?

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