Quasilinear Theory

Consider evolution of state variable $\underset{\sim}{q}(\underline{x},t)$

eg $\quad \underset{\sim}{q} = \begin{pmatrix} u \\ v \\ w \\ \vdots \end{pmatrix}$

such that

$$\partial_t \underset{\sim}{q} = \mathcal{L}[\underset{\sim}{q}] + \mathcal{N}[\underset{\sim}{q},\underset{\sim}{q}] \qquad (1)$$

could also have forcing etc.

$\mathcal{L}[\underset{\sim}{q}]$ - linear operator

$\mathcal{N}[\underset{\sim}{q},\underset{\sim}{q}]$ - nonlinear operator

Use 3 example Systems

1) Incompressible Navier Stokes

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right) &= -\nabla p + \nu\nabla^2\underline{u} \\ \underline{\nabla}\cdot\underline{u} &= 0 \end{aligned}\right\} \qquad (2)$$

wlog set $\rho = 1$

Note: in a magnetised fluid $\exists$ extra body force on fluid

Lorentz Force $= \underline{j} \times \underline{B}$ ($\underline{B}$ magnetic field)

$\underline{j} = \frac{1}{\mu_0} \underline{\nabla} \times \underline{B}$ (current)

2) <u>Induction Eqn</u>

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

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Derived by combining pre-Maxwell eqns with Ohm's Law for a moving conductor

$$\left(\frac{\partial \underline{B}}{\partial t} = -\underline{\nabla} \times \underline{E} \; ; \; \underline{\nabla} \cdot \underline{B} = 0 \; ; \; \underline{j} = \sigma (\underline{E} + \underline{u} \times \underline{B}) \right)$$

Exercise: Derive induction Eqn; what is η?

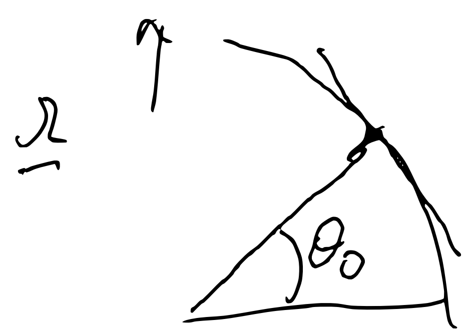
3) <u>Barotropic Vorticity Equation</u>

(see eg Vallis' Book).

Rotation often important! Sphericity not so...

Consider local Cartesian plane at latitude θ_0

$(x, y, z) = (a\lambda \cos\theta_0, a(\theta - \theta_0), z)$



Now in a rotating frame:

$$\frac{D\underline{u}}{Dt} + 2\underline{\Omega}\times\underline{u} = \frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u} \qquad \underline{u} = (u, v, w)$$

so

$$\frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})u + 2(\Omega^y w - \Omega^z v) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + (\underline{u}\cdot\underline{\nabla})v + 2(\Omega^z u - \Omega^x w) = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{\partial t} + (\underline{u}\cdot\underline{\nabla})w + 2(\Omega^x v - \Omega^y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - g$$

Traditional approximation: ignore components of Ω not in local vertical

$$\Rightarrow \quad \frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \qquad (5)$$

$$\frac{Dv}{Dt} + fv = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \qquad (6)$$

Also make hydrostatic approx

$$0 \sim -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

local approximation

$$f = 2\Omega\sin\theta \sim 2\Omega\sin\theta_0 + 2\Omega(\theta-\theta_0)\cos\theta_0$$

$$\Rightarrow \boxed{f \sim f_0 + \beta y}$$

$$\left(f_0 = 2\Omega\sin\theta_0 \; ; \; \beta = \frac{\partial f}{\partial y} = \frac{2\Omega\cos\theta_0}{a} \right)$$

<u>Barotropic Vorticity eqn</u>: set $v = v(x,y)$ $u = u(x,y)$

remove pressure by taking $\partial_x ⑥ - \partial_y ⑤$

+ setting $\zeta = \partial_x v - \partial_y u$

$\zeta = -\nabla_H^2 \psi$ where ψ is a streamfn.

to derive (Exercise)

$$\zeta_t + J(\psi, f+\zeta) = \nu\nabla^2\zeta + \text{forcing} + \text{dissipation}.$$

$$J = (A, B) = A_x B_y - A_y B_x$$

$$\Rightarrow \boxed{\zeta_t + J(\psi,\zeta) + \beta\psi_x = \nu\nabla^2\zeta + \text{forcing} + \text{dissipation}} \quad ⑦$$

Quasilinear Theories.

Historical Perspective

- Early roots in linear Rapid Distortion Theory (Batchelor + Proudman 1954)
- Malkus (1954) quasi-linear convection
  → Ledoux et al (1961), Spiegel (1962), Herring (1963)

Fluids

- Fried et al (1960), Vedenov et al (1961) Noerdlinger (1963)

Plasmas.

Diagrammatic (Feynman) Representations

Vedenov et al (1961)
Hasselmann (1966) wave-wave interactions

Herring "The discarding of the fluctuating self-interaction then corresponds to closing the system of moment eqns by discarding the third order cumulants"

Averaging Choices (see e.g. Tobias 2021)

- Derivation + solution for average properties of state variables.

- Decompose variables into mean (average) + fluctuating parts.

e.g. $q = \bar{q} + q'$

overbar linear averaging process that satisfies Reynolds Averaging Rules

$$\overline{q_1 + q_2} = \bar{q}_1 + \bar{q}_2$$

$$\bar{\bar{q}} = \bar{q}$$

$$\overline{\bar{q} q} = \bar{q}\bar{q}$$

Possible forms of averaging

Spatial Averaging

- Assume fluctuations on scale l_0
- $l_0 \ll L$ (scale of system)
- Can define intermediate scale $l_0 \ll a \ll L$

s.t.

$$\underline{q} \equiv \langle \underline{q} \rangle_a \equiv \frac{3}{4\pi a^3} \int_V \underline{q}(\underline{x}+\underline{\xi}, t)\, d^3\underline{\xi} \qquad (8)$$

· V sphere of Radius a.

Or averaging over one spatial coordinate

e.g. $\underline{q}(r, \theta, \phi)$

$$\overline{\underline{q}}(r, \theta) \equiv \frac{1}{2\pi} \int_0^{2\pi} \underline{q}(r, \theta, \phi)\, d\phi \quad . \qquad (9)$$

Temporal Averaging

If $\underline{q}$ varies on a rapid timescale t_0 + a long timescale T, then one can average over an intermediate timescale τ, $t_0 \ll \tau \ll T$

$$\overline{\underline{q}} \equiv \langle \underline{q} \rangle_\tau \equiv \frac{1}{2\tau} \int_{-\tau}^{\tau} \underline{q}(\underline{x}, t+\tau')\, d\tau' \qquad (10)$$

Ensemble Averaging

· If no separation of scales can take an <u>average over realisations</u> of the turbulence.

· If flow is ergodic (not usually case!) all of these averages should give same answer.

Averaging via Filtering

- e.g. spectral filtering (see later)
- Gaussian filtering (Germano 1992).
- Don't necessarily satisfy Reynolds rules
  - but can, in some cases, be made to.

The Quasilinear (QL) Approximation

Take equation (1) + average.

$$\Rightarrow \quad \partial_t \bar{\underline{q}} = \mathcal{L}[\bar{\underline{q}}] + \underbrace{\mathcal{N}[\bar{\underline{q}},\bar{\underline{q}}]}_{\text{mean/mean} \to \text{mean}} + \underbrace{\overline{\mathcal{N}[\underline{q}',\underline{q}']}}_{\text{fluc/fluc} \to \text{mean}} \qquad (11)$$

Take Eqn (11) from Eqn (1)

$$\Rightarrow \quad \partial_t \underline{q}' = \mathcal{L}[\underline{q}'] + \underbrace{\mathcal{N}[\bar{\underline{q}},\underline{q}'] + \mathcal{N}[\underline{q}',\bar{\underline{q}}]}_{\mathcal{L}_{\bar{q}}[\underline{q}']}$$

mean/fluc → fluc

$$+ \underbrace{\left(\mathcal{N}[\underline{q}',\underline{q}'] - \overline{\mathcal{N}[\underline{q}',\underline{q}']}\right)}_{G[\underline{q}',\underline{q}']}$$

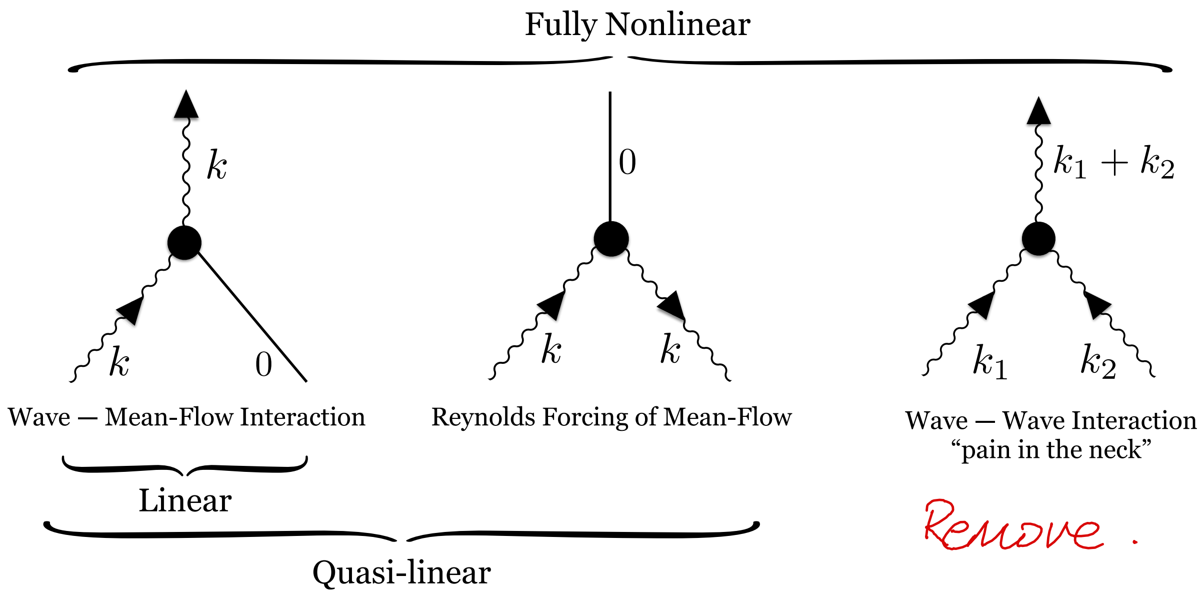
fluc/fluc → fluc.

(12)

For the QL approximation, set $G(q', q') = 0$

- EENL (Eddy/Eddy → Eddy Nonlinearity)
- "Pain in the Neck" term

- If thinking about modes with spatial wavenumber k.



- Example of constrained triad decimation (Kraichnan 1985).

Exercise

Derive the mean-field + fluctuating induction equations.

The heart of astrophysical dynamo theory is to try to understand the evolution of the mean field.

· Notes

(i) Ensemble averaging often better than zonal averaging in QL approx
- ensemble mean flow modes need not be purely zonally symmetric.
- more expensive

(ii) QL is a self-consistent mean-field theory.

(iii) QL is a conservative approximation
- in absence of driving / dissipation total energy + other linear and quadratic invariants are conserved.

QL via asymptotic theories

· Formally QL applicable when $G[q', q']$ is small compared with $\mathcal{L}_{\bar{q}}[q']$ or $\partial_t q'$

· This is a property of the nonlinear terms in the equation.

eg if nonlinearity is $(\underline{u} \cdot \underline{\nabla})\underline{u}$

might compare

$$(\underline{\bar{u}} \cdot \underline{\nabla})\underline{u}' \sim (\underline{u}' \cdot \underline{\nabla})\underline{u}' \qquad \underline{\bar{u}} \gg \underline{u}'$$

or

$$\partial_t \underline{u}' \sim (\underline{u}' \cdot \underline{\nabla})\underline{u}' \qquad \frac{1}{\tau_c} \gg \frac{u'}{l_c}$$

so if $\frac{u'}{\bar{u}} \ll 1$

or $\frac{u' \tau_c}{l_c} \ll 1$

$$\boxed{Ku = \frac{u' \tau_c}{l_c} \ll 1} \quad (13)$$

then QL approx valid

<u>But</u>: we only know Ku after the event!

<u>Examples where you know a priori</u>

- Low amplitude wave-turbulence interacting with mean flow. (QBO)
- Separation of timescales of mean + fluctuations
  - Planetary atmospheres at high zonostrophy parameter (Bouchet et al 2013; Scott + Dritschel 2012)
  - Dynamo theory (First Order Smoothing)
    - Earth (Plumley et al 2018)
    - Disks (Squire + Bhattacharjee 2015)

Infinite $U(1)$ Symmetry

For co-ordinate averaging QL eqns exhibit $U(1)$ symmetry.
(replaces particle relabelling)

- In QL phase of each fluctuation can be changed by an arbitrary phase

$$q_m(t) \longrightarrow e^{i\theta_m} q_m(t)$$

- Does invariance of QL eqns under phase rotations correspond to infinite family of conserved quantities?
- For linear waves / QL steady states Hamiltonian/Lagrangian formulation permit application of Noether's thm

e.g. for barotropic flow on sphere with $f = f(\theta)$

$$M_m = \int \frac{|\zeta_m(\theta,t)|^2 \sin^2\theta \, d\theta}{\partial_\theta(\bar{\zeta}(\theta) + f(\theta))}$$

is conserved for each mode $\zeta_m(\theta,t) = \int_0^{2\pi} \zeta(\theta,\phi,t) e^{im\phi} d\phi$

PSEUDOMOMENTUM.

Choice of averaging Direction

- In local Cartesian domains, periodic in 2 directions $\exists$ choice of averaging direction.
- With imposed shear flow
  - streamwise averaging
  - spanwise "

$\Rightarrow$ Reduced Nonlinear Models (RNL)

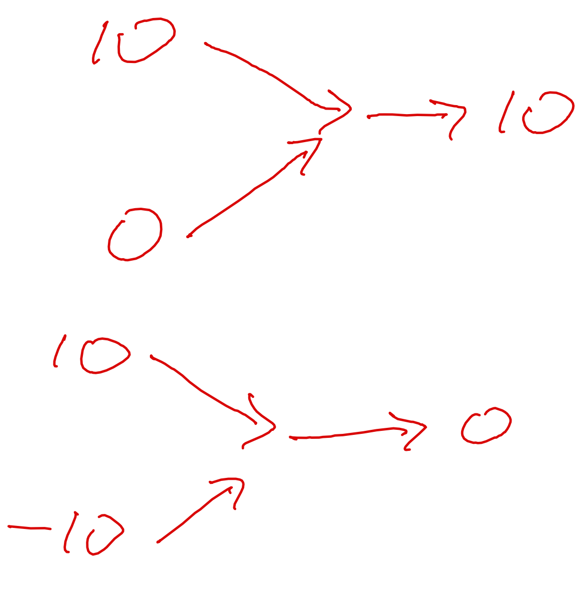
(Thomas et al 2014, 2015, Farrell et al 2016)

- Averaging over streamwise
  $\Rightarrow$ 2D average
  - fully nonlinear in spanwise/wall normal
  - <u>QL</u> in streamwise.

GENERALISED QUASILINEAR APPROXIMATION

- The QL approx with co-ordinate (zonal) averaging has very limited rules for energy exchanges.

i.e



No way of scattering energy between fluctuations!

- Extend to
  a) remove this deficiency
  b) allow more interactions between "mean" modes.
- But wish to allow for possibility of closure for small scales.

Set $$q = q_l + q_h \qquad (14)$$

low modes, high modes.

If performing approx in x-direction then

$$q_l = \sum_{k=-\Delta}^{\Delta} q_k(y,z)\, e^{ik'x} \qquad (15)$$

$$k' = \frac{2\pi k}{x_m}$$

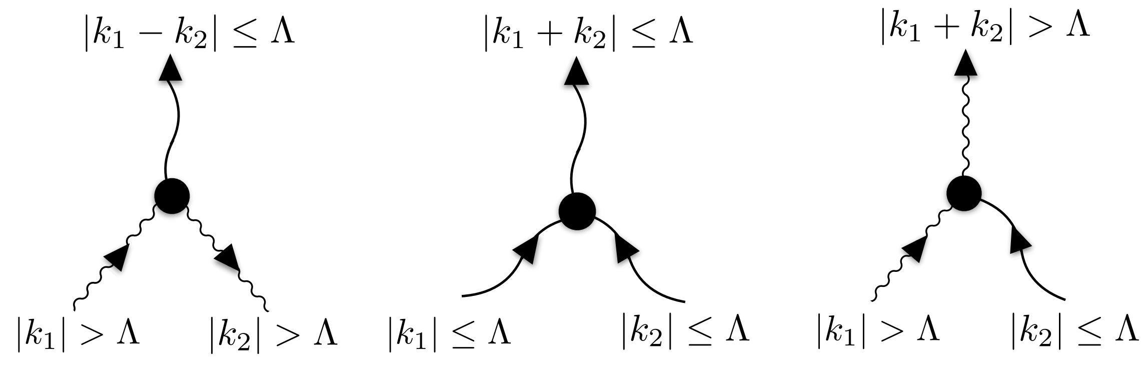
$$q_h = q - q_l$$

Note: as written (with no other rules) this partition does not satisfy Reynolds Averaging Rules:

i.e. $$(q_l q_l)_l \neq (q_l q)_l$$

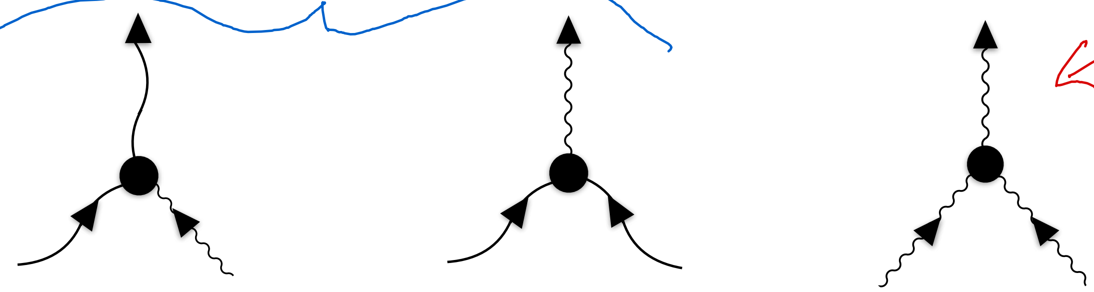
However we can remove some triad interactions in the spirit of QL.

THE GQL INTERACTION RULES



Retained

Ensures Reynolds Averaging Rules are respected



Ensures Eqns are linear in the high modes

Omitted

So eqns for modes can be written as

$$q_{l_t} = \mathcal{L}[q_l] + \left(\mathcal{N}[q_l, q_l]\right)_l + \left(\mathcal{N}[q_h, q_h]\right)_l$$

$$q_{h_t} = \mathcal{L}[q_h] + \mathcal{N}\left([q_l, q_h]\right)_h + \mathcal{N}[q_h, q_l])_h$$

· Note this is again an example of triad decimation in pairs
– linear / quadratic invariants are conserved.

- $\Lambda = 0$ QL
- $\Lambda = n_x$ DNS

} GQL systematically interpolates between the two.

$$10_h + 1_l \longrightarrow 11_h$$

$$10_h + (-11)_h \longrightarrow 1_l$$

∴ Mechanism for scattering energy into other high wavenumbers off the <u>non-zero low modes</u>.

Many numerical comparisons of QL + GQL as Λ is increased

- stochastic driving of jets (Marston et al 2016)
- 3D plane Poiseuille / Rotating Couette (Kellam 2019, Hernández et al 2022, T + Marston 2017)

- Busse Annulus (T et al 2018)
- Helical MRI (Child et al 2016)

- GQL always improves on QL
- Not monotonic improvement with $\Lambda \uparrow$.

RELATIONSHIP OF QL/GQL with RZIF/ RESOLVENT ANALYSIS/SCM

- If the mean-state is indep of time, separation makes sense in time (low freq vs high freq)

↳ see Marston + Tobias Ann Rev for details!