# Stabilizer Codes

Codespace is +1 eigenspace of Abelian group of unitary operators = stabilizer

Measuring eigenvalues of these unitaries is the way to recovery ...

- n qudits, d = prime. (e.g. qubits d = 2)
- n oscillators, $d \to \infty$

d = 2 stabilizer group is subgroup of Pauli group.

d prime " " " " " generalized Pauli group

$d \to \infty$ " " " " " group of displacements

$\Pi_I$

Pauli group $P_n = \langle iI , Z_1 \dots , Z_n, X_1, \dots X_n \rangle$

$\langle A_1, A_2 \dots \rangle =$ group generated by $A_1, A_2 \dots$

Take Abelian subgroup $S \leq P_n$ such that $-I \notin S$

Codespace $= \{ |\psi\rangle \mid \forall s \in S \quad s|\psi\rangle = |\psi\rangle \}$

Other elements in $P_n$ play role of logical operators and errors!

Qudits $\quad X: |j\rangle \to |j \bmod d\rangle$

$\omega = e^{2\pi i/d} \quad Z: |j\rangle \to \omega^j |j\rangle$

$j = 0 \dots d-1$

Element $P \in \mathcal{P}_n$ characterized by a 2n-bit string $\vec{u} = (\vec{u}_1, \vec{u}_2)$

$$P(\vec{u}) = X^{u_1} Z^{u_{n+1}} \otimes X^{u_2} Z^{u_{n+2}} \; \dots \; X^{u_n} Z^{u_{2n}}$$

(up to a phase)

$$P(\vec{u})\, P(\vec{v}) = (-1)^{\vec{v}^T S \vec{u}}\, P(\vec{v})\, P(\vec{u})$$

symplectic innerproduct

$$(\vec{v}_1, \vec{v}_2) \underbrace{\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}}_{= S} \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} = \vec{v}^T S \vec{u}$$

$$= \vec{v}_2 \cdot \vec{u}_1 - \vec{v}_1 \cdot \vec{u}_2$$

mod 2 arithmetic

"Paulis commute or anticommute"

Now n oscillators .... bitstring $\vec{u} \rightarrow \vec{u} \in \mathbb{R}^{2n}$

$$\mathbb{Z}_2 \rightarrow \mathbb{R}$$

3 extra

For 1 qubit

$$X^{u_1} Z^{u_2} X^{v_1} Z^{v_2} =$$

$u_2 \cdot v_1 = 0$

$u_2 \cdot v_1 = 1$

$$\begin{cases} X^{u_1} X^{v_1} Z^{u_2} Z^{v_2} = X^{v_1} X^{u_1} Z^{v_2} Z^{u_2} \\ - X^{v_1} X^{u_1} Z^{v_2} Z^{u_2} \end{cases}$$

$u_1 \cdot v_2 = 0$ $\quad X^{v_1} Z^{v_2} X^{u_1} Z^{u_2}$

$u_1 v_2 = 1$ $\quad -X^{v_1} Z^{v_2} X^{u_1} Z^{u_2}$

$u_1 v_2 = 0$ $\quad -X^{v_1} Z^{v_2} X^{u_1} Z^{u_2}$

$u_1 \cdot v_2 = 1$ $\quad +X^{v_1} Z^{v_2} X^{u_1} Z^{u_2}$

$$P(\vec{u}) P(\vec{v}) = (-1)^{u_2 \cdot v_1} (-1)^{u_1 \cdot v_2} P(\vec{v}) P(\vec{u})$$

n qubits $\longrightarrow (-1)^{\vec{u}_2 \cdot \vec{v}_1} (-1)^{\vec{u}_1 \cdot \vec{v}_2}$

$$\left[ (-1)^{\vec{x} \cdot \vec{y}} = (-1)^{x_1 \cdot y_1} (-1)^{x_2 \cdot y_2} \cdots (-1)^{x_n \cdot y_n} \right]$$

III

Define $X(a) = \exp(i\sqrt{2\pi}\, a\, \hat{p}) = D(\sqrt{\pi}\, a)$

$Z(b) = \exp(i\sqrt{2\pi}\, b\, \hat{q}) = D(i\sqrt{\pi}\, b)$

$\boxed{a, b \in \mathbb{R}}$

$$Z(b)\, X(a) = \exp(i 2\pi a b)\, X(a)\, Z(b)$$

they commute when $ab$ is integer

Remember
$D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*} a)$

More generally, it relies on

$$\exp(A)\exp(B) = \exp(B)\exp(A)\; \exp([A,B])$$

when $[A,B] \propto I$

exercise IV : show this

III

Multiple oscillators : $\vec{u} = (\vec{u}_1, \vec{u}_2) \in \mathbb{R}^{2n}$

$$P(\vec{u}) = X(u_1) Z(u_{n+1}) \otimes X(u_2) Z(u_{n+2}) \otimes \dots\dots X(u_n) Z(u_{2n})$$

(upto a phase)

$$P(\vec{u}) P(\vec{v}) = e^{i 2\pi \vec{v}^T S \vec{u}} P(\vec{v}) P(\vec{u})$$

again $S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$P(\vec{u})$ and $P(\vec{v})$ commute when $\vec{v}^T S \vec{u} = 0 \mod 1$

Stabilizer group when $\vec{v}^T S \vec{u} = 0$ (no modular structure)

$\longrightarrow$ continuous-variable stabilizer codes : oscillator-versions of stabilizer codes, e.g q-qubit & toric code

NOT "INTERESTING"

///

5 extra

by "not interesting" we mean that if the noise model is small errors for each oscillator, i.e. small displacements, one can show that logical error model is only squeezed as compared to basic error model.

CV version of these codes was introduced to deal with different error model: large shifts on at most 1 oscillator or erase 1 oscillator, but what is physical motivation behind such model .....

5 extra extra

From

$$Z(b)\,X(a) = \exp(i2\pi ab)\,X(a)\,Z(b)$$

we have

$$X^\dagger(a)\,Z^\dagger(b) = \exp(-i2\pi ab)\,Z^\dagger(b)\,X^\dagger(a)$$

$X^\dagger(a) = X(-a)$
$Z^\dagger(b) = Z(-b)$

$$X(a)\,Z(b) = \exp(-i2\pi ab)\,Z(b)\,X(a)$$

Single oscillator:

$$P(\vec{u})P(\vec{v}) = X(u_1)\,Z(u_2)\,X(v_1)\,Z(v_2) = e^{i2\pi u_2 v_1}\,X(v_1)\,X(u_1)\,Z(v_2)\,Z(u_2)$$

$$e^{i2\pi[u_2 v_1 - u_1 v_2]}\,P(\vec{v})\,P(\vec{u})$$

Product of phases for $n$ oscillators $\rightarrow$ symplectic product.

Example : single-oscillator GKP code

$S_p = \exp(i\, 2\sqrt{\pi}\, \hat{p}) = X(\sqrt{2})$

$S_p S_q = S_p S_q$

$S_q = \exp(i\, 2\sqrt{\pi}\, \hat{q}) = Z(\sqrt{2})$

Stabilizer group generated by $S_p^\dagger, S_p, S_q, S_q^\dagger$.

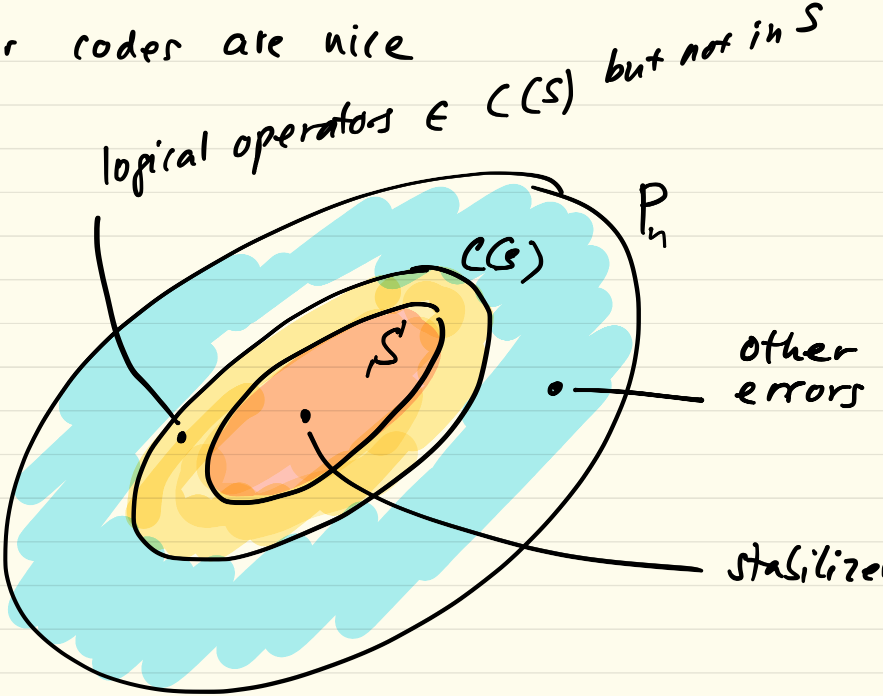
How do we know dimension of code space?

$$\left.\begin{matrix} X(\alpha\sqrt{2}) \\ Z(\sqrt{2}/\alpha) \end{matrix}\right\}$$ also commute, e.g. $\alpha = \sqrt{\pi} \rightarrow \hat{p}$ has to be integer for $S_p = +1$

$\hat{q}$ is $0 \bmod \pi$ for $S_q = +1$

III

# Stabilizer codes are nice




example:

$S = \langle XXXX, ZZZZ \rangle$

$C(S) = \langle S, Z_1 Z_2, Z_2 Z_3, X_1 X_2, X_2 X_3 \rangle$

code $[[4,2,2]]$

Centralizer of $S$ in $P_n = C(S) = \{ P \mid \forall s \in S \; [P, s] = 0 \}$

other errors: do not commute with some $s \in S$

GKP code: what is in $C(S)$?

$X(\frac{1}{\sqrt{2}}) = \exp(i\sqrt{\pi}\,\hat{p}) = \sqrt{S_p} = \bar{X}$ $\qquad$ $\bar{X}\bar{Z} = \bar{Z}\bar{X} \times$

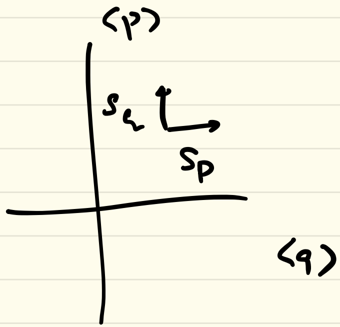
$Z(\frac{1}{\sqrt{2}}) = \exp(i\sqrt{\pi}\,\hat{q}) = \sqrt{S_q} = \bar{Z}$ $\qquad$ $\exp(-\pi[\hat{p},\hat{q}])$

$= -1.$

By definition $|\bar{0}\rangle$ is $\bar{Z}|\bar{0}\rangle = |\bar{1}\rangle$

$X|\bar{0}\rangle = |\bar{1}\rangle \qquad \bar{Z}|\bar{1}\rangle = -|\bar{1}\rangle$

1 qubit encoded




Code states are invariant under phase space translations $S_p$ and $S_q$ $\rightarrow$ "grid states"

What errors can stabilizer codes correct?

Qubit stabilizer codes: what Pauli errors meet the QEC conditions?

distance of code $d = \min_{P \in C(S)/S} |P|$

$|P|$ weight of Pauli = # qubits on which it acts nontrivially

Code can correct all Pauli errors of weight $t = \lfloor \frac{d-1}{2} \rfloor$, i.e error set $\mathcal{E} = \{E_1, E_2 \ldots E_k\}$ where each $E_i$ has $|E_i| \leq t$

Why? $|E_i^\dagger E_j| < d$ for errors in $\mathcal{E}$.

Hence $E_i^\dagger E_j$ — $\in S$

— anti-commute with at least 1 element in $S$

$\in P_n$

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$E_i^+ E_j \in S \qquad E_i^+ E_j P_c = P_c$

$P_c E_i^+ E_j P_c = P_c$ ✓

$\exists\, s \in S \qquad E_i^+ E_j s = -s E_i^+ E_j$

Consider $\langle \bar{\ell} | E_i^+ E_j | \bar{k} \rangle \xrightarrow[\text{be}]{\text{should}} \alpha_{ij} \delta_{\ell k}$

$= \langle \bar{\ell} | E_i^+ E_j\, s | \bar{k} \rangle$ ✓

$= - \langle \bar{\ell} | E_i^+ E_j | \bar{k} \rangle = 0$

"Correct up to half the distance" for any $\ell, k$

( Basically $\mathcal{E}$ needs to be such that no $E_i^+ E_j$ is a logical )

$\overline{\underline{III}}$

How to do reversal/correction?

$\mathcal{E} = \{E_1, E_2 \ldots\}$

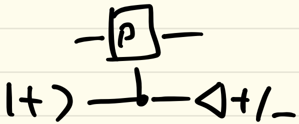
Project onto error-spaces $C_i$ with projectors $P_i \propto E_i P_c E_i^\dagger$

$$\exists s: \; s \, E_i |\bar{\ell}\rangle = -E_i s |\bar{\ell}\rangle = -E_i |\bar{\ell}\rangle$$

Error-spaces $C_i$ characterized by eigenvalues $\pm 1$ of generators of stabilizer.

= error syndrome

Thus: measure eigenvalues of generators of stabilizer.




Pauli $P$, e.g.

$P = XXXX$ etc.

$\overline{\underline{III}}$

# Oscillator codes ...

What displacement errors meet the QEC conditions?

Any set $\mathcal{E} = \{ P(\vec{u}_1), P(\vec{u}_2) \dots \}$

s.t $\forall\, i,j \quad P(-\vec{u}_i)\, P(\vec{u}_j) \notin C(S)/S$. (up to phases)

Only difference with qubit case: $P(\vec{u})\,P(\vec{v}) = e^{i 2\pi \vec{v}^T S \vec{u}}\, P(\vec{v})\, P(\vec{u})$

phase instead of +1 or −1

$P(\vec{u}_1) \in S \rightarrow$ does not change syndrome

$\exists\, s \in S : s\, P(\vec{u}_1) \neq P(\vec{u}_1)\, s \rightarrow$ will change eigenvalue of $s$.

Measure eigenvalues of generators of $S$

III

GKP code: generators are $S_p$, $S_p^\dagger$, $S_q$ and $S_q^\dagger$

$X(\frac{1}{\sqrt{2}}) = \exp(i\sqrt{\pi}\,\hat{p}) = \sqrt{S_p} = \bar{X}$

$Z(\frac{1}{\sqrt{2}}) = \exp(i\sqrt{\pi}\,\hat{q}) = \sqrt{S_q} = \bar{Z}$

any $X(u)$ with $u < \frac{1}{2\sqrt{2}}$

and

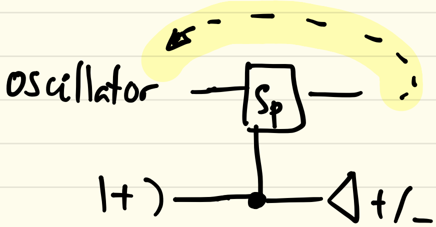
$Z(v)$ with $v < \frac{1}{2\sqrt{2}}$

is correctable

"At most half a logical shift".

Measure eigenvalue of some $U$

= Phase estimation




If oscillator is in eigenstate $|\psi_\theta\rangle$

i.e. $S_p|\psi_\theta\rangle = e^{i\theta}|\psi_\theta\rangle$

then $P_\pm = \frac{1}{2}(1 \pm \cos\theta)$

does not yet determine $\theta$ ... repeat

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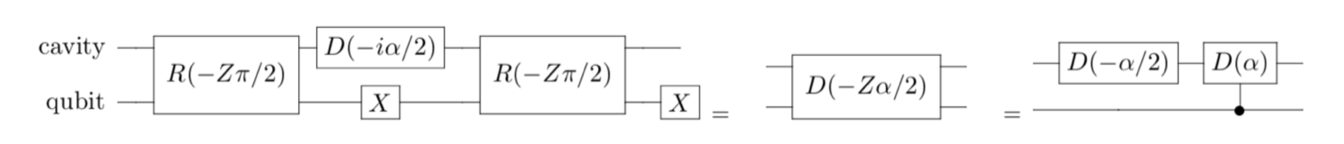
Requires controlled-displacement between qubit and oscillator → displacement + controlled-rotation




$$R(-Z\pi/2) = \exp(i\,\pi/2\; a^\dagger a\, Z)$$

Derive this ... $Z = \pm 1$

rotate the direction of displacement

$Z=1$

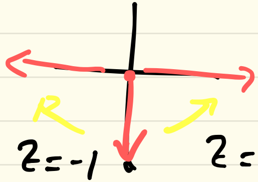
$$R(-\pi/2)\, D(-i\alpha/2)\, R(+\pi/2)\, R(-\pi/2)$$

$$= D(\alpha/2)\, R(-\pi/2) \rightsquigarrow D(\alpha/2)$$

$Z=-1$

$$R(\pi/2)\, D(-i\alpha/2)\, R(-\pi/2)\, R(\pi/2)$$

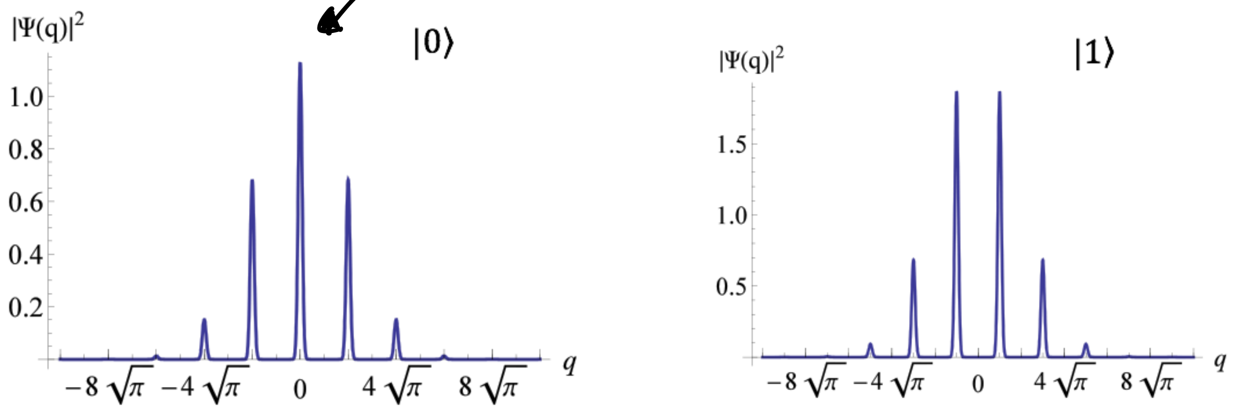
$$= D(-\alpha/2)\, R(\pi/2) \rightsquigarrow D(-\alpha/2)$$




III

# Approximate GKP code states

spacing $= S_p$




Squeezed peaks at even multiples of $\sqrt{\pi}$ Squeezed peaks at odd multiples of $\sqrt{\pi}$

How to make or describe ...

Eigenstate of $S_p = e^{i 2\sqrt{\pi}\,\vec{p}}$ and

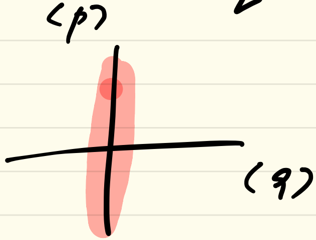
$S_q = e^{i 2\sqrt{\pi}\,\vec{q}}$ ?

Squeezed vacuum state is already approximate eigenstate of $S_q$.

squeezing parameter $\Delta = e^{-r}$

$$\Delta^2 = \frac{(\Delta q)^2}{(\Delta q_{vac})^2}$$

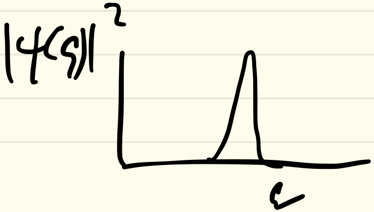
$$(\Delta q_{vac})^2 = 1/2$$




Squeezing in dB

$= 10 \log_{10}(\cosh^2 r)$

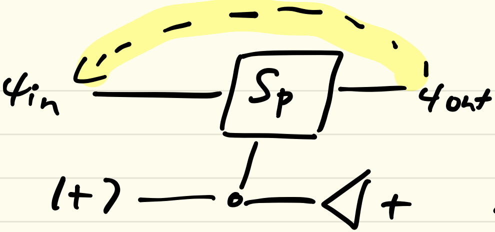
$$|sq.vac\rangle \propto \int dq\; e^{-q^2/2\Delta^2}\, |q\rangle$$




one peak ... how to create multiple peaks...

[1]

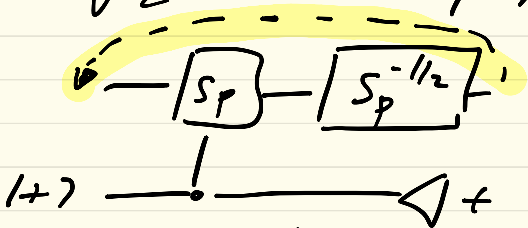
Apply




repeatedly

← post-select on +

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(I + S_p\right)|\psi_{in}\rangle$$

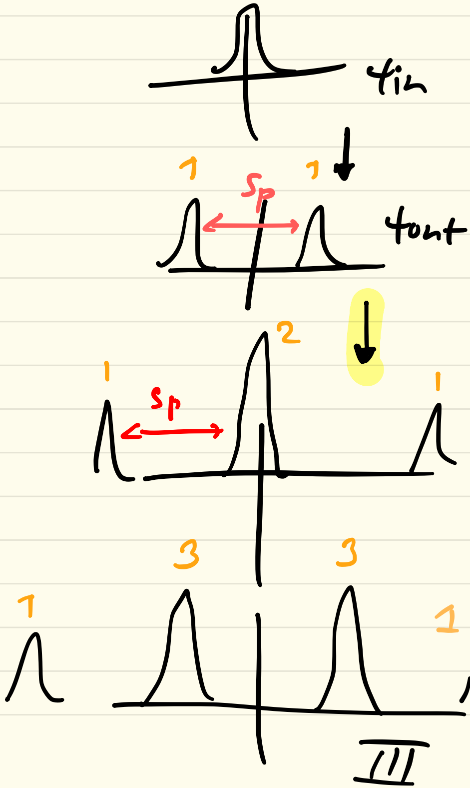



$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}\left(S_p^{-1/2} + S_p^{1/2}\right)|\psi_{in}\rangle$$

→ Binomial envelope

≈ Gaussian envelope (M rounds)

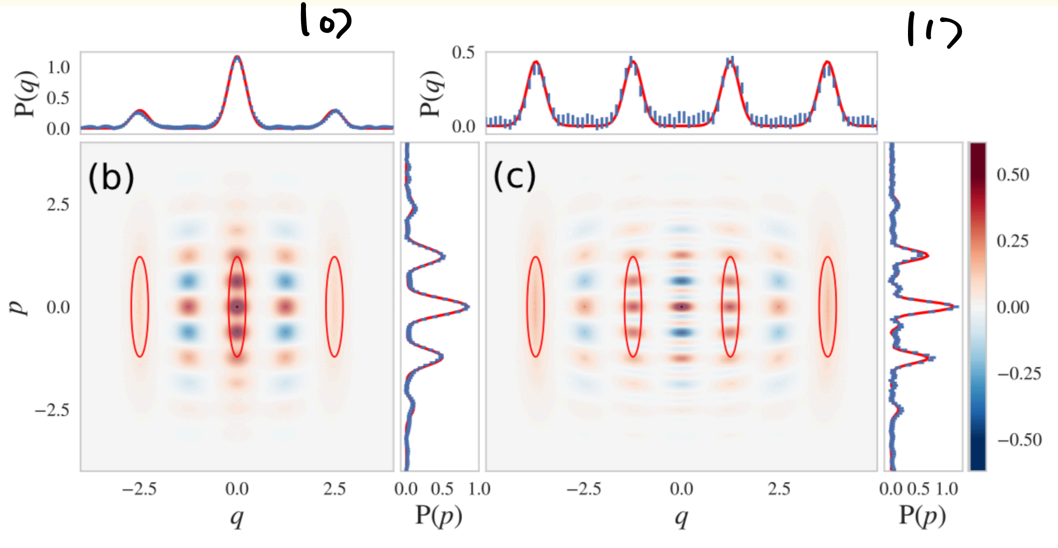
$$\approx \sum_{t=-M/2}^{M/2} e^{-2t^2/M}\, S_p^{t}$$



Flühmann et al. last week (arxiv.org 1807.01033)

: creation of states in motional state of $Ca^+$ ion

$|0\rangle$ $|1\rangle$

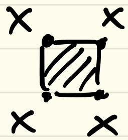


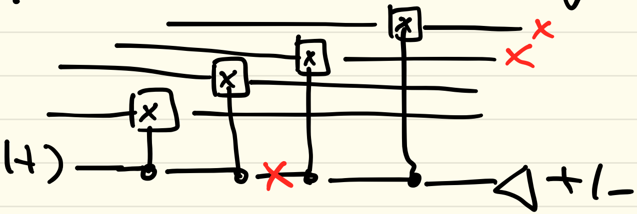

<u>Comments</u> :

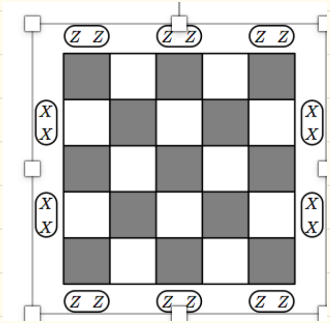
- no need for post-selection (Weigand / Terhal)
- procedure is not fault-tolerant

# Fault-tolerant error correction : some options

- stabilizer checks / generators act on few qubits, code is large : toric code



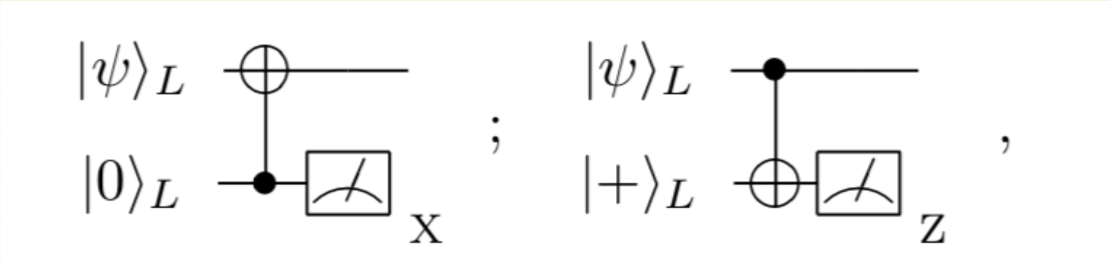




- for stabilizer codes where stabilizer

$$S = \langle \underbrace{S_1(X) \,..\, S_m(X)}_{\text{use only Pauli } X},\ \underbrace{S_{m+1}(Z) \,..\, S_\ell(Z)}_{\text{use only Pauli } Z.} \rangle$$

= CSS stabilizer codes

One can use so-called Steane error-correction



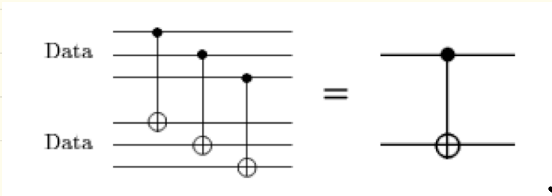

$\psi_L = \bar{\psi}$

$0_L = |\bar{0}\rangle$

$|\bar{0}\rangle$ and $|\bar{1}\rangle$ are logical ancillas.

CNOT is logical CNOT, for qubit CSS code such gate is

**transversal** :

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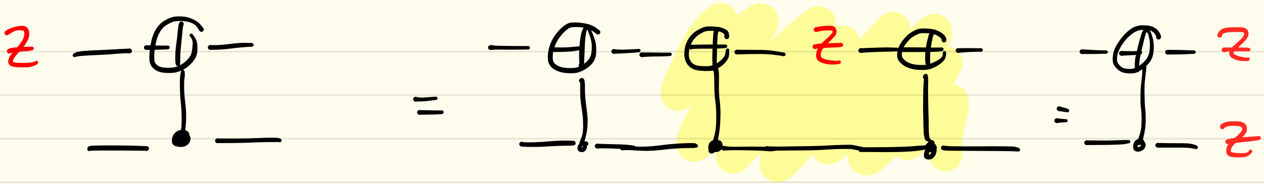
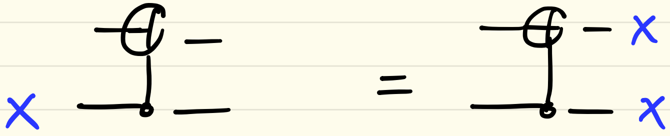
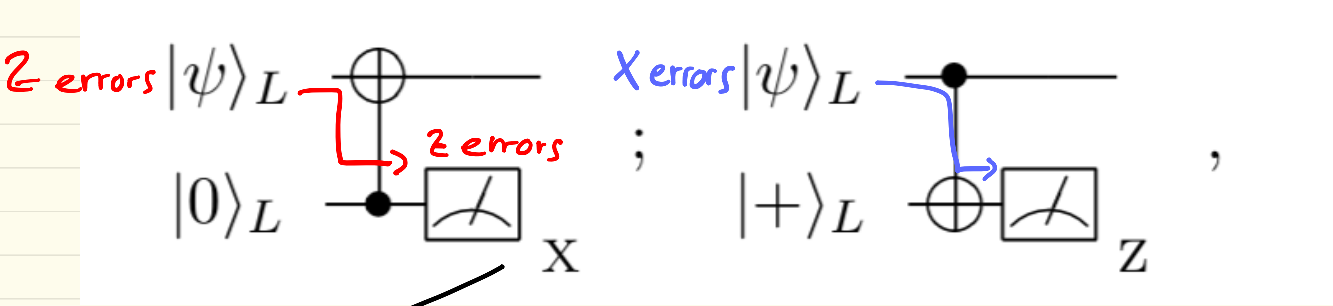


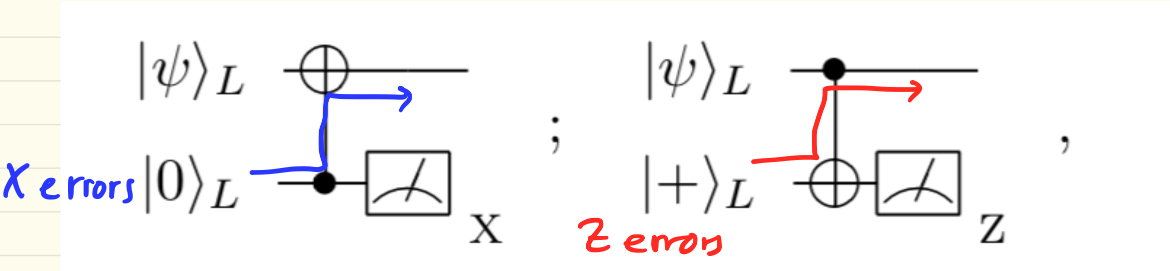



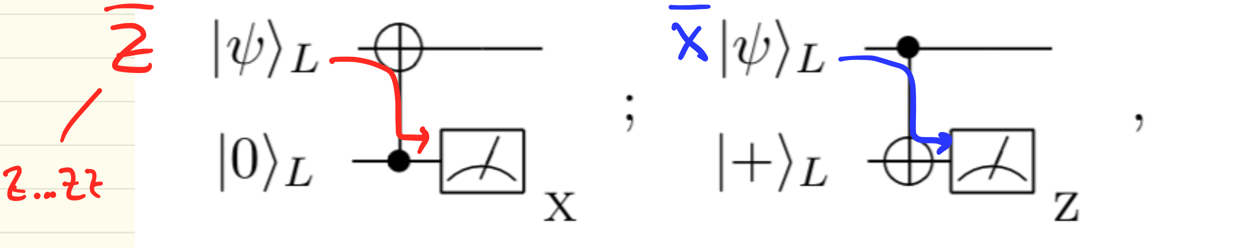

measure all qubits in X-basis
and construct eigenvalues
of
X-checks (like XXXX)

→ correct Z-errors

# But caution ...




Need good ancillas!  "the QEC backaction problem"




no revealing of logical information

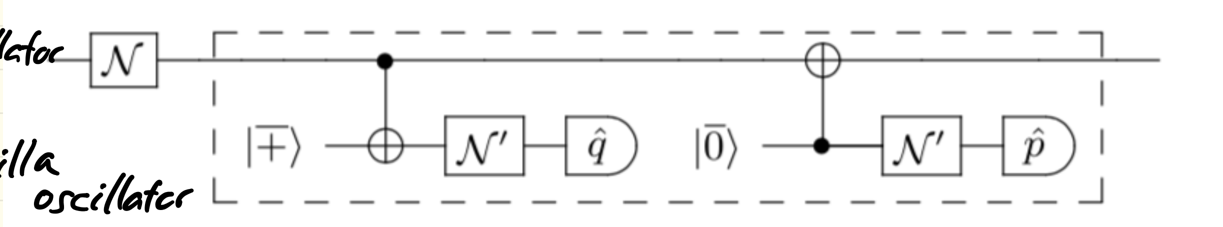
$\bar{Z}|\bar{0}\rangle = |\bar{0}\rangle$ ✓ $\qquad$ $\bar{X}|\bar{+}\rangle = |\bar{+}\rangle$ ✓

III

Same idea can be applied to GKP code

data oscillator

ancilla oscillator

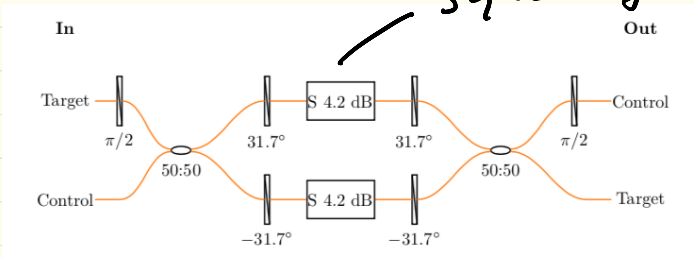



( Ignore noisy channels $\mathcal{N}$ and $\mathcal{N}'$ )

$-\boxed{q}$ homodyne measurement of $\hat{q}$

squeezing

CNOT (in optics language)




CNOT : $q_{target} \to q_{control} + q_{target}$ ; $p_{control} \to p_{control} - p_{target}$

Conclusion :

- QEC still has to prove its worth in practice
- Bosonic codes (combined with qubit codes) an interesting avenue for exploration (superconducting qubits, ion-trap qubits)

$\overline{III}$