Stabilizer Codes

Codespace is eigenspace of Abelian group of unitary operators = Stabilizer

Measuring eigenvalues of these unitaries is the way to recovery...

- n qudits, $d = \text{prime.}$ (e.g. qubits $d=2$)
- n oscillators, $d \to \infty$

d=2 stabilizer group is subgroup of generalized Pauli group
d prime " " " " " " " " " group of displacements
Pauli group $P_n = \langle iI, Z_1, ..., Z_n, X_1, ..., X_n \rangle$

Take Abelian subgroup $S \leq P_n$

such that $-I \notin S$

(Codespace = \{ $|\psi\rangle$ | $\forall S \in S$ $S |\psi\rangle = |\psi\rangle$ \})

Other elements in $P_n$ play role of logical operators and errors!

Qudits $X : |ij\rangle \rightarrow |ij\rangle_{\text{mod} d}$
$Z : |ij\rangle \rightarrow e^{2\pi i/|d|} |ij\rangle$
$j = 0, ..., d-1$
Element $P \in P_n$ characterized by a $2n$-bit string $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$

$$P(\tilde{u}) = X^{u_1} Z^{u_{n+1}} \otimes X^{u_2} Z^{u_{n+2}} \ldots X^{u_n} Z^{u_{2n}} \quad \text{(up to a phase)}$$

$$P(\tilde{u}) P(\tilde{\nu}) = (-1)^{\tilde{u}^T S \tilde{\nu}} P(\tilde{\nu}) P(\tilde{u})$$

Symplectic inner product

$$(\tilde{v}, \tilde{v}_2) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \tilde{v}_1 \\ -\tilde{v}_2 \end{pmatrix}$$

Now $n$ oscillators ... bitstring $\tilde{u} \rightarrow \tilde{u} \in \mathbb{R}^{2n}$

$\mathbb{Z}_2 \rightarrow \mathbb{R}$
For 1 qubit

\[ X^{u_1} z \cdot X^{v_1} z \cdot = \]

\[ u_2 \cdot v_1 = 0 \]

\[ u_2 \cdot v_1 = 1 \]

\[ P(\bar{u}) P(\bar{v}) = (-1)^{u_2 \cdot v_1} (-1)^{u_1 \cdot v_2} P(\bar{v}) P(\bar{u}) \]

n qubits → \((-1)^{u_2 \cdot v_1} (-1)^{u_1 \cdot v_2} \left[ (-1)^{x_1 \cdot y_1} \cdots (-1)^{x_n \cdot y_n} \right] \]
Define \( X(a) = \exp(i\sqrt{2\pi}a) = D(\sqrt{2\pi}a) \)
\[ Z(b) = \exp(i\sqrt{\pi}b) = D(i\sqrt{\pi}b) \]

\[ Z(b) X(a) = \exp(i2\pi ab) X(a) Z(b) \]

They commute when \( ab \) is integer

Remember
\[ D(a) = \exp(da^+a) \]

More generally, it relies on
\[ \exp(A) \exp(B) = \exp(B) \exp(A) \]
when \([A,B] = I\)  
\[ \exp([A,B]) \]

Exercise IV: Show this
Multiple oscillators: \( \vec{u} = (\vec{u}_1, \vec{u}_2) \in \mathbb{R}^{2n} \)

\[
P(\vec{u}) = X(u_1) Z(u_{n+1}) \otimes X(u_2) Z(u_{n+2}) \otimes \ldots \otimes X(u_n) Z(u_{2n})
\]

(up to a phase)

\[i 2\pi \vec{u}^T S \vec{u}\]

\[P(\vec{u}) P(\vec{v}) = e^{i 2\pi \vec{u}^T S \vec{v}}\]

again \( S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \)

\[P(\vec{u}) \text{ and } P(\vec{v}) \text{ commute} \]

when \( \vec{u}^T S \vec{v} = 0 \mod 1 \)

Stabilizer group when \( \vec{u}^T S \vec{u} = 0 \) (no modular structure)

\( \rightarrow \) continuous-variable stabilizer codes: oscillator-versions of stabilizer codes, e.g., q-qudit & toric code

NOT "INTERESTING"
by "not interesting" we mean that if the noise model is small errors for each oscillator, i.e., small displacements, one can show that logical error model is only squeezed as compared to basic error model.

CV version of these codes was introduced to deal with different error model: large shifts on at most 1 oscillator or erase 1 oscillator, but what is physical motivation behind such model...
From

\[ Z(b) X(a) = \exp \left( i 2\pi a b \right) X(a) Z(b) \]

we have

\[ X^+(a) Z^+(b) = \exp (-i 2\pi a b) Z^+(b) X^+(a) \quad X^+(a) = X(-a) \]
\[ Z^+(b) = Z(-b) \]

\[ X(a) Z(b) = \exp (-i 2\pi a b) Z(b) X(a) \]

Singly oscillator:

\[ P(\bar{v}) P(\bar{u}) = X(u_1) Z(u_2) X(u) Z(u) \]
\[ = e^{i 2\pi u_2 v_1} X(u_1) X(u_1) Z(u_2) Z(u_2) \]
\[ e^{i 2\pi [u_2 v_1 - u_1 v_2]} P(\bar{v}) P(\bar{u}) \]

Product of phases for \( n \) oscillators → symplectic product.
Example: single-oscillator GKP code

\[ S_p = \exp(i 2\sqrt{\pi} \hat{p}) = X(U_2) \]
\[ S_q = \exp(i 2\sqrt{\pi} \hat{q}) = Z(U_2) \]

Stabilizer group generated by \( S_p, S_p, S_q, S_q^+ \).

How do we know dimension of code space?

\[ X(\alpha U_2) \]
\[ Z(\sqrt{2}/\alpha) \]

also commute, e.g. \( \alpha = \sqrt{\pi} \rightarrow \hat{p} \) has to be integer for \( S_p = +1 \)

\( \hat{q} \) is \( 0 \mod \pi \) for \( S_q = +1 \)
Stabilizer codes are nice logical operators $\in C(S)$ but not in $S$

\[
C(S) = \langle S, z_1 z_2, z_2 z_3, x_1 x_2, x_2 x_3 \rangle
\]
(code $[C_4, 2, 2]$)

Centralizer of $S$ in $P_n = C(C(S)) = \frac{1}{2} P$ \( \forall S \in S \)

\[
\{ P, S \} = 0
\]

other: do not commute with errors some $S \in S$
GKP code: what is in (5)?

\[ x^{(\ell_2)} = \exp \left( i \sqrt{\pi} \hat{p} \right) = \sqrt{S_p} = \bar{x} \]
\[ \bar{z} \bar{z} = \bar{z} \bar{x} = x \]

\[ z^{(\ell_2)} = \exp \left( i \sqrt{\pi} \hat{q} \right) = \sqrt{S_q} = \bar{z} \]
\[ \exp ( -\pi (\hat{p}, \hat{q}) ) = -1 \]

By definition \( |\ell_0\rangle \) \( \bar{z} |\ell_0\rangle = |\ell_1\rangle \)

\[ x |\ell_0\rangle = |\ell_1\rangle \quad \bar{z} |\ell_1\rangle = - |\ell_1\rangle \]

1 qubit encoded

Code states are invariant under phase space translations \( S_p \) and \( S_q \) → "grid states"
What errors can stabilizer codes correct?

Quasit stabilizer codes: What Pauli errors meet the QEC conditions?

Distance of code \( d = \min_{P \in \mathbb{C}(S)/S} |P| \)

\( |P| \) weight of Pauli = \# qusites on which it acts nontrivially

Code can correct all Pauli errors of weight \( t = \left\lfloor \frac{d-1}{2} \right\rfloor \)

i.e error set \( \mathcal{E} = \{ \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k \} \)

where each \( \mathcal{E}_i \) has \( |\mathcal{E}_i| \leq t \)

Why? \( |\mathcal{E}_i + \mathcal{E}_j| < d \) for errors in \( \mathcal{E} \).

Hence \( \mathcal{E}_i + \mathcal{E}_j \) anti-commute with at least \( t \) element in \( \mathbb{S} \).
\[ E_i^+ E_j \in S \quad E_i^+ E_j P_c = P_c \]
\[ P_c E_i^+ E_j P_c = P_c \]

\[ \exists s \in S \quad E_i^+ E_j s = -s E_i^+ E_j \]

Consider \( \langle \bar{E} | E_i^+ E_j | \bar{k} \rangle \) should be \( \delta_{ij} \delta_{lk} \)

\[
\begin{align*}
\langle \bar{E} | E_i^+ E_j | s \bar{k} \rangle & = \langle \bar{E} | E_i^+ E_j | s \bar{k} \rangle \\
& = - \langle \bar{E} | E_i^+ E_j | \bar{k} \rangle = 0
\end{align*}
\]

"Correct up to half the distance" for any \( \bar{E}, \bar{k} \)

(Basically, \( \mathcal{E} \) needs to be such that no \( E_i^+ E_j \) is a logical)
How to do reversal/correction?

\[ E = \{ E_1, E_2, \ldots \} \]

Project onto error-space \( C_i \) with projectors

\[ P_i = E_i P_e E_i^+ \]

\[ \{ s : s E_i | \tilde{e} \rangle = -E_i s | \tilde{e} \rangle = -E_i | \tilde{e} \rangle \} \]

Error-space \( C_i \) characterized by eigenvalues \( \pm 1 \) of generators of stabilizer.

Thus: measure eigenvalues of generators of stabilizers.

Pauli \( P \), e.g. \( \sigma_x \), \( \sigma_y \), \( \sigma_z \), \( \sigma_x \sigma_y \), etc.
Oscillator codes...

What displacement errors meet the QEC conditions?

Any set \( \mathcal{E} = \{ P(\tilde{u}_1), P(\tilde{u}_2), \ldots \} \)

s.t. \( \forall i,j \quad P(-\tilde{u}_i) P(\tilde{u}_j) \in \text{CSS}/s. \) (up to phases)

Only difference with qudit case: \( i_{2\pi} \tilde{u}^T S \tilde{u} \)

\[
P(\tilde{u}) P(\tilde{u}) = e^{i \theta} \quad P(\tilde{u}) P(\tilde{u})
\]

Phase instead of \(+1\) or \(-1\)

\( P(\tilde{u}_1) \in S \rightarrow \) does not change syndrome

\( \exists s \in S: s P(\tilde{u}_1) \neq P(\tilde{u}_1)s \rightarrow \) will change eigenvalue of \( s. \)

Measure eigenvalues of generators of \( S. \)
GKP code: generators are $S_p$, $S_p^+$, $S_q$ and $S_q^+$

$$X(\frac{i\pi}{2}) = \exp(i \sqrt{\pi} \hat{p}) = \sqrt{S_p} = \bar{X}$$

any $X(u)$ with $u < \frac{1}{2\sqrt{2}}$

and

$$Z(\frac{i\pi}{2}) = \exp(i \sqrt{\pi} \hat{g}) = \sqrt{S_q} = \bar{Z}$$

$Z(u)$ with $u < \frac{1}{2\sqrt{2}}$

Measure eigenvalue of some $U$

= Phase estimation

If oscillator is in eigenstate $146$)

i.e. $S_p|146\rangle = e^{i6}|146\rangle$

then $P_{\pm} = \frac{1}{2}(1 \pm \cos \theta)$

does not yet determine $\theta$ ... repeat
Requires controlled-displacement between qubit and oscillator → displacement + controlled-rotation

\[ R\left(-\frac{Z\pi}{2}\right) = \exp(i\frac{\pi}{2} a + a Z) \]

Derive this... \( Z = \pm 1 \)

\[ Z = 1 \]

\[ R\left(-\frac{\pi}{2}\right) D\left(-i\times\frac{\pi}{2}\right) R\left(+\frac{\pi}{2}\right) R\left(-\frac{\pi}{2}\right) = D\left(\times\frac{\pi}{2}\right) R\left(-\frac{\pi}{2}\right) \sim D\left(\times\frac{\pi}{2}\right) \]

\[ Z = -1 \]

\[ R\left(\frac{\pi}{2}\right) D\left(-i\times\frac{\pi}{2}\right) R\left(-\frac{\pi}{2}\right) R\left(\frac{\pi}{2}\right) = D\left(-\times\frac{\pi}{2}\right) R\left(\frac{\pi}{2}\right) \sim D\left(-\times\frac{\pi}{2}\right) \]
Approximate GKP code states

Squeezed peaks at even multiples of $\sqrt{\pi}$

Squeezed peaks at odd multiples of $\sqrt{\pi}$

How to make or describe...
Eigenstate of $\hat{S}_p = e^{i2\sqrt{\pi} \hat{p}}$ and

$$\hat{S}_q = e^{i2\sqrt{\pi} \hat{q}}?$$

Squeezed vacuum state is already approximate eigenstate of $\hat{S}_q$.

Squeezing parameter $\Delta = e^{-r}$

$$\Delta^2 = \frac{(\Delta q)^2}{(\Delta q_{\text{vac}})^2}$$

$$(\Delta q_{\text{vac}})^2 = \frac{1}{2}$$

Squeezing in dB

$$= 10 \log_{10} (\cosh^2 r)$$

$$|s_{q, \text{vac}}|^2 \propto \int dq \ e^{-\frac{q^2}{2\Delta^2}} e^{i\Delta q} \left| \langle q \rangle \right|^2 \left| \Delta \langle q \rangle \right|^2$$

one peak ... how to create multiple peaks...
Apply \(4_{\text{in}}\) repeatedly

\[ l^+ \xrightarrow{\text{post-select on } +} l^+ \]

\[ l_{\text{out}} = \frac{1}{\sqrt{2}} \left( I + S_p \right) l_{\text{in}} \]

\[ l_{\text{out}} = \frac{1}{\sqrt{2}} \left( S_p^{-1/2} + S_p^{1/2} \right) l_{\text{in}} \]

→ Binomial envelope

\[ \sim \text{ Gaussian envelope (M rounds)} \]

\[ \sim \sum_{t=-M/2}^{M/2} e^{-2t^2/M} S_p^t \]
Flühmann et al. last week (arxiv.org 1807.01033) : creation of states in motional state of Ca^{+} ion

Comments:

- no need for post-selection (Weisfuss / Terhal)
- procedure is not fault-tolerant
Fault-tolerant error correction: some options

- Stabilizer checks / generators act on few qubits, code is large: toric code

\[ S = \langle S_1(x), \ldots, S_m(x), S_{m+1}(z), \ldots, S_{2c}(z) \rangle \]
- For stabilizer codes where stabilizer use only Pauli $X$
- CSS stabilizer codes
One can use so-called Steane error correction.

4_L = \bar{4}

0_L = \bar{10}_L

(10) and (11) are logical ancillas. CNOT is logical CNOT, for qudit CSS code such gate is transversal:

\[
\begin{array}{c}
\text{Data} \\
\hline
\end{array}
\begin{array}{c}
\text{Data} \\
\hline
\end{array}
= \begin{array}{c}
\hline
\end{array}
\begin{array}{c}
\hline
\end{array}
\]
$$2 \text{ errors } |\psi\rangle_L \rightarrow \text{measure all qubits in } \mathbf{X}\text{-basis and construct eigenvalues of } \mathbf{X}\text{-checker (like } \mathbf{XXXX}) \rightarrow \text{correct } 2\text{-errors}$$
But caution...

\[ |\psi\rangle_L \xrightarrow{X} |0\rangle_L \quad \text{X errors} \]

\[ |\psi\rangle_L \xrightarrow{Z} |+\rangle_L \quad \text{Z errors} \]

Need good ancilla! "the QEC backaction problem"

\[ \overline{Z}|0\rangle = |0\rangle \quad \checkmark \]

\[ \overline{X}|+\rangle = |+\rangle \quad \checkmark \]
Same idea can be applied to GKP code.

(Forget noisy channels $N$ and $N'$) \(\Rightarrow\) homodyne measurement of $\hat{q}$

$\text{CNOT}$ (in optics language)

\(\text{CNOT} : q_{\text{target}} \rightarrow q_{\text{control}} + q_{\text{target}} ; p_{\text{control}} \rightarrow p_{\text{control}} - p_{\text{target}}\)
Conclusion:

- QEC still has to prove its worth in practice
- Bosonic codes (combined with qubit codes) an interesting avenue for exploration (superconducting qubits, ion-trap qubits)