Error Models & The Quantum Error Correction Conditions

Set of errors $\{E_i\}$ obtained from e.g.

$$S(\rho) = \sum_i E_i \rho E_i^\dagger$$

Amplitude-damping $(T=0)$

$$E_0 = \frac{1}{2}(1+\sqrt{1-\gamma})I + \frac{1}{2}(1-\sqrt{1-\gamma})Z$$

$$E_1 = \sqrt{\gamma}\,\sigma_-$$

Phase-damping

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p} \end{pmatrix} ; \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}$$

($\rightarrow$ equivalent to phase flip channel)

Together form typical single-qubit error model:

T_1, T_2

$$\rho \mapsto \begin{pmatrix} 1 - \rho_{11} e^{-t/T_1} & \rho_{01} e^{-t/T_2} \\ \rho_{01}^* e^{-t/T_2} & \rho_{11} e^{-t/T_1} \end{pmatrix}$$

Let's look at some bosonic error models ...

Bosons $[a, a^\dagger] = I$ ③

Quadrature operators $\hat{q} = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $\hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a)$

$$[\hat{q}, \hat{p}] = iI$$

$$\hat{n} = a^\dagger a$$

1. Photon loss (& Photon Gain)
2. Photon loss + Kerr nonlinearity
3. Shift error basis

1. $$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \mathcal{L}(\rho) \qquad H = \omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

$$\mathcal{L}(\rho) = \kappa\, a \rho a^\dagger - \tfrac{1}{2}\kappa \{a^\dagger a, \rho\}$$

$$\Longrightarrow \quad S_t(\rho) = \sum_{\ell=0}^{\infty} E_\ell\, \rho\, E_\ell^\dagger \qquad E_\ell = \left(\frac{\gamma}{1-\gamma}\right)^{\ell/2} \frac{a^\ell}{\sqrt{\ell!}} (1-\gamma)^{\hat{n}/2}$$

loss rate $\gamma = 1 - e^{-\kappa t}$

II

Exercise III (not easy): can you derive this closed form expression with these $\{E_\ell\}$.

Small δt, from Lindblad equation $\rho(t) = e^{\mathcal{L}t}(\rho(0))$

$$\approx \mathcal{I} + \mathcal{L}t\,(\rho(0))$$

$$= S_t(\rho(0))$$

gives

$$\left.\begin{aligned} E_0 &\approx I - \tfrac{1}{2}\,\delta t\,\hat{n} \\ E_1 &\approx \sqrt{\delta t}\; a \end{aligned}\right\} \quad E_0^\dagger E_0 + E_1^\dagger E_1 = I + O((\delta t)^2)$$

2. $H = \omega a^\dagger a + \underbrace{\frac{K}{2}(a^\dagger a)^2}_{= H_K}$ ← Kerr nonlinearity, by itself a systematic error for which one can compensate....

But combine with photon loss...

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leads to dephasing

$$e^{-iH_k t}\, a\, e^{iH_k t} = e^{i\kappa t/2}\, a\, e^{i\kappa t a^\dagger a}$$

"Don't know when loss happens, don't know how much extra rotation $\exp(i\kappa t\, a^\dagger a) \to$ dephasing"

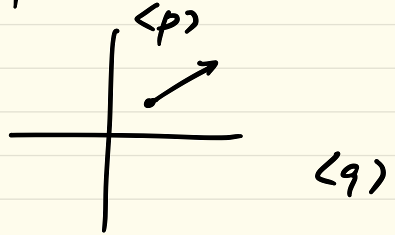
At lowest order in $\varkappa t$ and κt one has

$$E_0 \simeq I - iH_k t - \tfrac{1}{2}\varkappa t\, a^\dagger a \qquad \text{"evolve \& lose no photon"}$$

$$E_1 \simeq \sqrt{\varkappa t}\, a \qquad \text{"lose photon"}$$

Error basis for bosonic mode: displacements.

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \qquad \alpha \in \mathbb{C}$$



$$D(\alpha)\,|vac\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

coherent state.

Translation in phase-space

Any operator on a bosonic mode

$$E = \int d^2\alpha \;\; c(\alpha)\, D(\alpha)$$

$$c(\beta) = \frac{1}{\pi} \operatorname{Tr} E\, D(-\beta)$$

(see e.g. Cahill/Glauber 1969)

Bosonic equivalent of depolarizing toy model

Gaussian shift / displacement error model:

$$\mathcal{S}(\rho) = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \quad P_{\sigma_p}(u)\, P_{\sigma_q}(v)\, e^{-iu\hat{p}+iv\hat{q}}\, \rho\, e^{iu\hat{p}-iv\hat{q}}$$

$P_{\sigma_p}(u)$: normal distribution with variance σ_p^2

and similarly $P_{\sigma_q}(v)$

Comments

apply random $D(\alpha)$, then channel, then $D(-\alpha)$

- Displacement twirling possible (but regulate asymptotics $D(\alpha)$ for $\alpha \to \infty$)
- Photon loss, small rotation $\exp(i\theta a^\dagger a)$ has expansion in terms of small displacements?
- Photon loss followed by amplification $a \to \sqrt{G}\,a + \sqrt{G-1}\,b^\dagger$ = Gaussian displacement channel

(see Albert et al. arxiv.org 1708.05010)

π

Quantum Error Correcting Conditions

Given ρ supported on a code space C : $\bar{\rho}$

and $S(\bar{\rho}) = \sum_i E_i \bar{\rho} E_i^{\dagger}$ $\rightarrow$ error set $\mathcal{E}' = \{E_1, E_2, \ldots\}$

There exists a (trace-preserving) reversal map R

such that $R(S(\bar{\rho})) \propto \bar{\rho}$

if and only if

$$\forall i,j \qquad P_C E_i^{\dagger} E_j P_C = \alpha_{ij} P_C \qquad (*)$$

P_C = projector onto code space $\quad P_C = \sum_k |\bar{k}\rangle\langle\bar{k}|$, α_{ij} Hermitian matrix

$$\langle\bar{\ell}|\,(*)\,|\bar{k}\rangle = \langle\bar{\ell}| E_i^{\dagger} E_j |\bar{k}\rangle = \alpha_{ij}\,\delta_{\ell k}.$$

II

Exercise V : let $U^\dagger \alpha U = D$ = diagonal

and $F_m = \sum_j U_{jm} E_j$, show that error set $\{F_m\}$ satisfies QEC conditions when $\{E_i\}$ does and $P_c F_n^\dagger F_m P_c = D_{mn} P_c$

↑ diagonal.

Proof of QEC conditions in book Nielsen & Chuang:
let's do it

Some bosonic examples

error set $\mathcal{E} = \{ I, a \}$

$|\bar{0}\rangle = |n=2\rangle$

$|\bar{1}\rangle = \frac{1}{\sqrt{2}} \left(|n=0\rangle + |n=4\rangle \right)$

<u>No distortion</u>:

$$\begin{cases} \langle \bar{0} | a^\dagger a | \bar{0} \rangle = \langle \bar{1} | a^\dagger a | \bar{1} \rangle & ✓ \\ \langle \bar{0} | a^\dagger a | \bar{1} \rangle = 0 & ✓ \end{cases}$$

✓ Also: $\langle \bar{0} | a | \bar{0} \rangle = \langle \bar{1} | a | \bar{1} \rangle = 0$

✓ $\langle \bar{0} | a | \bar{1} \rangle = \langle \bar{1} | a | \bar{0} \rangle = 0$

But really $E_0 = I - \frac{\gamma}{2} a^\dagger a$, $E_1 = \sqrt{\gamma}\, a \ldots$

and E_0 leads to distortion

$$\langle \bar{0} | E_0^\dagger E_0 | \bar{0} \rangle - \langle \bar{1} | E_0^\dagger E_0 | \bar{1} \rangle$$

$$= O(\gamma^2) \left[\langle \bar{0} | (a^\dagger a)^2 | \bar{0} \rangle - \langle \bar{1} | (a^\dagger a)^2 | \bar{1} \rangle \right]$$

$\neq 0$. in this order

this is baby binomial code

arxiv.org : 1805.09072

Two-mode Version (Chuang, Leung, Yamamoto 1996!)

$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|40\rangle + |04\rangle)$

$|\bar{1}\rangle = |22\rangle$

$\mathcal{E} = \{ \exp(-\frac{1}{2}(\hat{n}_1 + \hat{n}_2)\gamma),$

$\sqrt{\gamma_1}\, a_1, \sqrt{\gamma_2}\, a_2 \}$

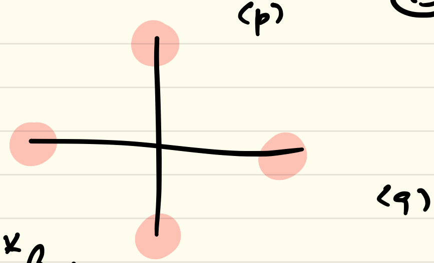
Code meets the QEC conditions!

Since $\exp(-\frac{1}{2}(\hat{n}_1 + \hat{n}_2)\gamma)\, |\bar{0}\rangle = \exp(-2\gamma)\, |\bar{0}\rangle$

" $|\bar{1}\rangle = \exp(-2\gamma)\, |\bar{1}\rangle$

no distortion!

Cat code (4-legged) $\boxed{\alpha \in \mathbb{R}}$



$|\bar{0}\rangle \propto |\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle$

$|\bar{1}\rangle \propto |\alpha\rangle + |-\alpha\rangle - |i\alpha\rangle - |-i\alpha\rangle$

orthogonal! (use $\langle\alpha|\beta\rangle = e^{-|\alpha|^2-|\beta|^2/2}\, e^{\alpha^*\beta}$)

$\mathcal{E} = \{I, a\}$

Photon parity operator $\exp(i\pi a^\dagger a)$ — $+1$ even parity; -1 odd parity

Verify that $e^{i\pi a^\dagger a}|\alpha\rangle = |-\alpha\rangle \Rightarrow$ $|\bar{0}\rangle, |\bar{1}\rangle$ even parity

$\exp(i\pi/2\, a^\dagger a)\,|\bar{0}\rangle = |\bar{0}\rangle$ photon # is 0 mod 4.

$\exp(i\pi/2\, a^\dagger a)\,|\bar{1}\rangle = -|\bar{1}\rangle$ " " " 2 mod 4.

$= \bar{Z}$ = logical Z.

//

QEC conditions

$a|\bar{0}\rangle$ has parity 3 mod 4

$a|\bar{1}\rangle$ " " 1 mod 4

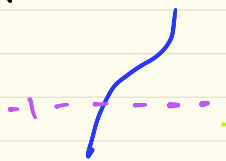
$\langle\bar{0}|a|\bar{0}\rangle = \langle\bar{1}|a|\bar{1}\rangle = 0$ ✓

$\langle\bar{0}|a|\bar{1}\rangle = \langle\bar{0}|a^\dagger|\bar{1}\rangle = 0$ ✓

$\langle\bar{0}|a^\dagger a|\bar{1}\rangle = 0$ ✓

"AMAZING."

smallest $|\alpha|^2 = 2.34$



but what about

$\langle\bar{0}|a^\dagger a|\bar{0}\rangle \overset{?}{=} \langle\bar{1}|a^\dagger a|\bar{1}\rangle$

large α? $\rightarrow$ yes, but $I \; \cancel{\approx} \; I - \frac{\gamma}{2}\hat{n}$

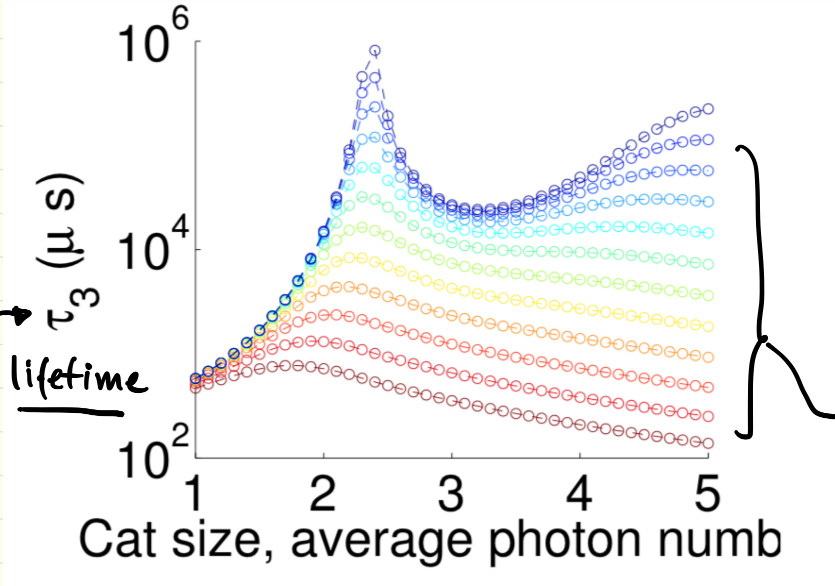
At SWEET SPOT $\boxed{\tan\alpha^2 = -\tanh\alpha^2}$ we meet all QEC conditions for $\{I, a\}$

Can you verify sweet spot, using that normalization of states is

$$|\bar{0}\rangle = \frac{|\alpha\rangle + |-\alpha\rangle + |i\alpha\rangle + |-i\alpha\rangle}{4\sqrt{2(\cosh^2\alpha + \cos\alpha^2)}}$$

$$|\bar{1}\rangle = \frac{|\alpha\rangle + |-\alpha\rangle - |i\alpha\rangle - |-i\alpha\rangle}{4\sqrt{2(\cosh^2\alpha - \cos\alpha^2)}}$$

Effect of being at sweet spot:



lifetime

$$S_\gamma(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

$$E_0 = I - \frac{\gamma}{2}\hat{n}$$

$$E_1 = \sqrt{\gamma}\, a$$

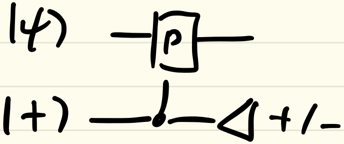
different T: different time to corrections.

$$|\psi\rangle \text{ (qubit)} - \boxed{\text{encode}} \xrightarrow{\bar{\psi}} \boxed{S_\gamma^{\,T}} - \boxed{R} - \boxed{\text{decode}} - \rho \text{ (qubit)} \;\hat{=}\; \psi - \boxed{S_{\text{eff}}} - \rho$$

instantaneous perfect

Measuring photon parity

P is unitary with ± 1 eigenvalues



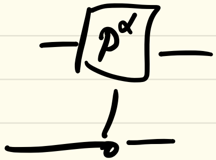
projective measurement of P

(e.g. Pauli)

Use controlled - $\exp(i\pi a^\dagger a)$ $\rightarrow$ $|1\rangle\langle 1|\, a^\dagger a$ interaction.

But not fault - tolerant!

Qubit decay during 

gives 

$\alpha < 1$.

$\Rightarrow$ measuring eigenvalue of say $P^{1/2} = e^{i\pi a^\dagger a/2} = \bar{Z}$

with some probability $\longrightarrow$ bad

II

Existence of faultless recovery R
does not imply existence of faulty recovery
which leads to a qubit which is better
than the most basic bosonic code $|\bar{0}\rangle = |n=0\rangle$

$|\bar{1}\rangle = |n=1\rangle$

which uses no quantum error correction.

Fancy Overhead-efficient Codes can be
useless ... for stabilizer codes, recovery
can be simple, fault-tolerant constructions
exist, including some easy logical gates....

II