Problem Set, Boulder Summer School 2023

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I. HAAR CALCULUS

The goal of this problem is to derive a few key formulas for averaging moments of random unitaries, using the Haar measure. Throughout this problem we will consider unitaries $U \in U(D)$ (with $D = d^2$ for unitary gates acting on qudits).

1. Show that

$$\mathbb{E}_{U\in U(D)} U_{\alpha\beta} U^*_{\alpha'\beta'} \equiv \int_{U\in U(D)} dU U_{\alpha\beta} U^*_{\alpha'\beta'} = \frac{1}{D}\delta_{\alpha\alpha'}\delta_{\beta\beta'}, \tag{1}$$

where as usual, the Haar measure is normalized so that $\int dU = 1$.

2. We will admit that the average of higher moments $\mathbb{E}_{U\in U(D)}\left[(U\otimes U^\dagger)^{\otimes n}\right]$ can be decomposed onto permutations of the incoming and outgoing legs of the unitaries: this is because operators commuting with the action of the unitaries on a tensor product Hilbert space are (linear combinations of) permutations of the tensor factors (a result known as 'Schur-Weyl duality'). For the second moment, we have

$$\mathbb{E}_{U\in U(D)} U_{\alpha\beta} U^*_{\alpha'\beta'} U_{\gamma\delta} U^*_{\gamma'\delta'} = C_1 \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\gamma\gamma'}\delta_{\delta\delta'} + C_2 \delta_{\alpha\alpha'}\delta_{\beta\delta'}\delta_{\gamma\gamma'}\delta_{\delta\beta'} + C_3 \delta_{\alpha\gamma'}\delta_{\beta\beta'}\delta_{\gamma\beta'}\delta_{\delta\delta'} + C_4 \delta_{\alpha\gamma'}\delta_{\beta\delta'}\delta_{\gamma\alpha'}\delta_{\delta\beta'}. \tag{2}$$

Compute the constants C_1, C_2, C_3, C_4 in terms of D.

II. PURIFICATION TRANSITION

Consider a monitored random quantum circuit with a maximally mixed initial state $\rho_0 = 1/2^L$. Use the statistical mechanics model to express the entropy of the mixed state at time t in terms of the (replica limit of) partition functions with different boundary conditions at the top and bottom boundaries.

III. LARGE d LIMIT OF MONITORED CIRCUITS AND BOND PERCOLATION

The goal of this problem is to show that in the limit of large onsite Hilbert space ($d \to \infty$), the measurement-induced phase transition (MIPT) in monitored Haar circuits discussed in class maps onto classical percolation. As we will see, spins (permutations) connected by unmeasured links are forced to be the same, whereas spins on measured links are effectively decoupled, *i.e.* measurements "break" the links connecting spins, diluting the lattice. This naturally yields a picture of the purification transition in terms of classical percolation of clusters of aligned permutation "spins".

1. Show that for large d, both $D^{C(g)}$ (with $C(g)$ the number of cycles in the permutation $g \in S_Q$) and the Weingarten function $\mathrm{Wg}_d(g)$ are proportional to a Kronecker delta function $\delta_{g,1}$ with 1 the identity permutation.

2. Explain why this implies that the replicated statistical mechanics model has an enlarged $S_{Q!}$ symmetry in this limit, with $Q = kn + 1$ the number of replicas.

3. Consider a fixed configuration $\mathbf{X}$ of measurement locations, which are in one-to-one correspondence with bond percolation configurations (with measured links being "broken"). Argue that Z_A and Z_0 (averaging over random unitaries and measurement outcomes as in the lectures, but not over measurement locations) are dominated by a single configuration for $d \to \infty$, and compute those two partition functions exactly. Show that

$$Z_A = Z_0 d^{(k+1-Q)\ell_{\mathrm{DW}}(\mathbf{X})}, \tag{3}$$

where ℓ_{DW} is the length of the minimal cut through the monitored circuit (with measured links being broken) from one end of the sub system A to the other end.

4. Compute the Renyi entropies $S_A^{(n)}$ in terms of $\ell_{\mathrm{DW}}(\mathbf{X})$. Explain why this establishes a mapping between the MIPT at large d and bond percolation, and discuss the scaling of $S_A^{(n)}$ in the percolating and non-percolating phases.