Lecture 1 : Strong periodic driving

Boulder Summer School

Week 1 , July 2023

Lecture 1 : Strong periodic driving in isolated quantum systems ①

- ↳ General properties
- ↳ Frequency lattice
- ↳ Phenomena : → Optical lattices (eg of Hamiltonian eng.)
  - → Rabi oscillations
  - → Thouless pump

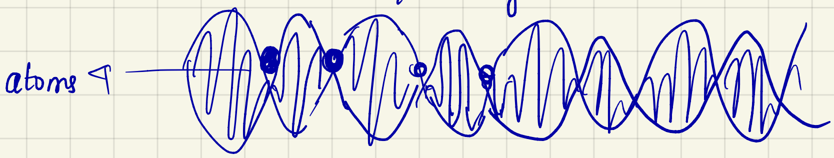
$$H(t) = \underbrace{H_0}_{\text{static Hamiltonian}} + \underbrace{H_1(\omega, t)}_{\text{periodically varying}}$$

$$H_1(\omega_1 t) = H_1(\omega_1 t + 2\pi)$$

$$\int_0^{2\pi} d(\omega_1 t)\, H_1(\omega_1 t) = 0 .$$

Concrete examples : 1) Atoms driven by an electric field

Standing wave laser field



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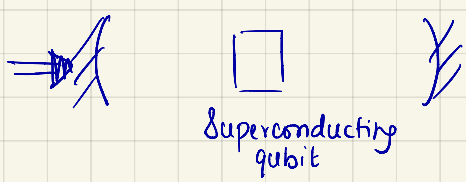
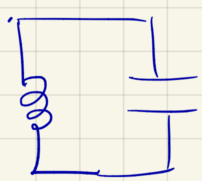


Cavity / circuit-QED system.

3)

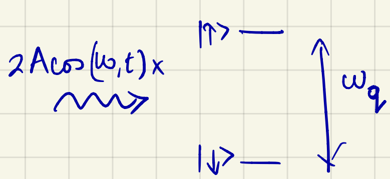


$C(t) = C_0 + C_1 \cos\omega_1 t$

Parametrically driven oscillator.

⋮

Ex 1 & 2 : Physics captured by a driven 2-level system



$$H = \frac{\omega_q}{2}\sigma_z + 2A\cos\omega_1 t\, \sigma_x$$

Even this simple model is not easily solvable.* Nevertheless, we can make headway using general properties + perturbative calculations.

(* there are formal solutions to the quantized model)

General properties

I Energy is not conserved.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -2A\,\omega_1 \sin(\omega_1 t)\, \sigma_x(t)$$

$$\underbrace{\langle \chi(0)|}_{\text{initial state}} \frac{dH}{dt} |\chi(0)\rangle = -2A\,\omega_1 \sin(\omega_1 t) \underbrace{\langle \chi(0)| \sigma_x(t) |\chi(0)\rangle}_{\text{generically non-zero.}}$$

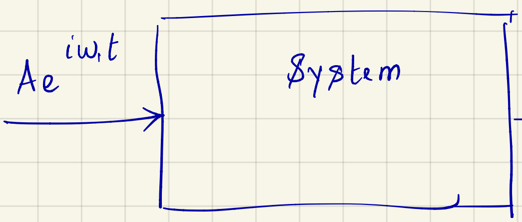
→ for the same reason $\left\langle \frac{dH_0}{dt} \right\rangle \neq 0$

→ In a few-level system, energy is bounded. One can construct time-dependent conserved quantities.

→ In a many-body system/ for systems of oscillators, the energy can grow unboundedly.

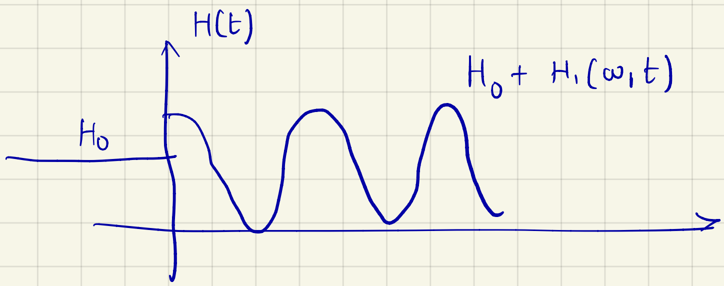
→ A way to slow down the heating is to drive at high frequency. Floquet pre-thermalization $T_h \sim \exp(\omega/J)$

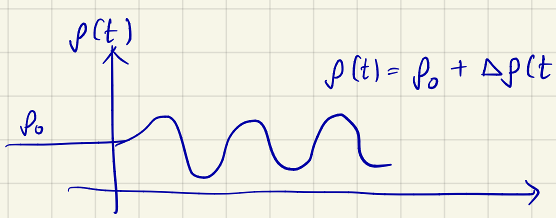
II Expect linear response for weak driving = sufficiently small drive amplitudes



$$\langle \Delta O(t) \rangle = \underbrace{A}_{\text{Linear}}\; \underbrace{\chi(\omega_1)}_{\text{complex susceptibility}}\; \underbrace{e^{i\omega_1 t}}_{\text{at the same frequency as the drive}}$$

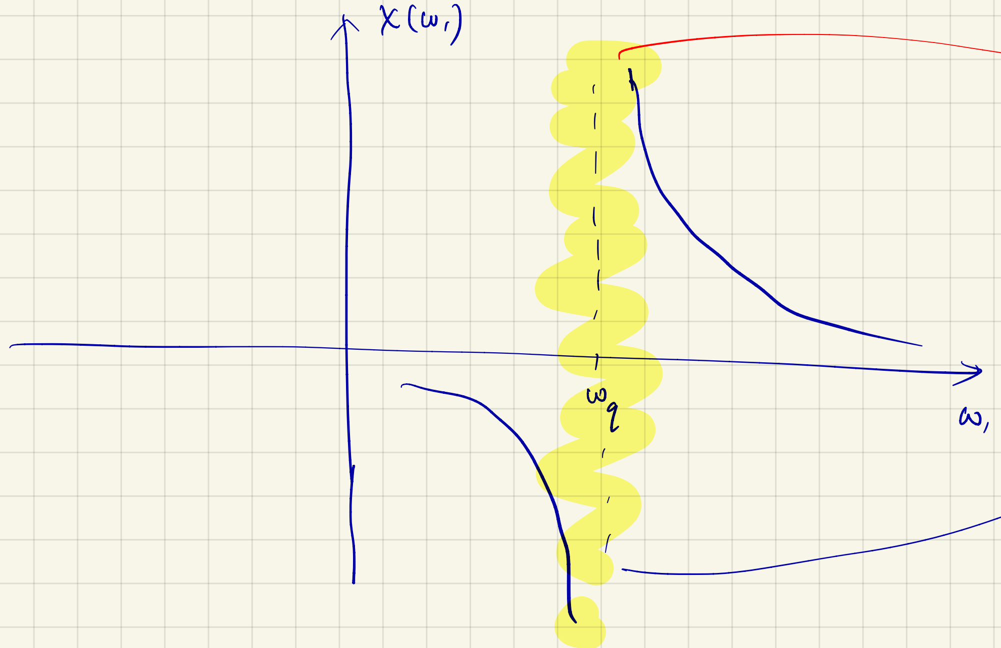
Formula for $\chi(\omega_1)$ from time-dependent perturbation theory to first order in A.





In our 2-level system example with $\rho_0 = |\uparrow\rangle\langle\uparrow|$ and $O = \sigma^x$

$$\chi(\omega_1) = \frac{4\omega_q}{\hbar}\left(\frac{1}{\omega_1^2 - \omega_q^2 + i0\,\mathrm{sgn}(\omega_1)}\right)$$



Linear response breaks down near $\omega_1 = \omega_q$.

resonance

Non-linear response important.

III Floquet theorem (Analog of Bloch's theorem in crystals)

Statement : The time-dependent Schrödinger equation for a m-level system

$$i\frac{d|\psi\rangle}{dt} = H(\omega, t)\,|\psi\rangle \qquad (\hbar = 1 \text{ henceforth})$$

with $H(\omega, t) = H(\omega, t + 2\pi)$

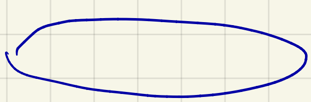
has a complete basis of solutions

$$|\psi_\alpha(t)\rangle = e^{-i\varepsilon_\alpha t}\,|\phi_\alpha(\omega, t)\rangle \quad ; \quad \alpha = 1, \dots, m.$$

ε_α : quasi-energies

$|\psi_\alpha\rangle, |\phi_\alpha\rangle$: quasi-energy states

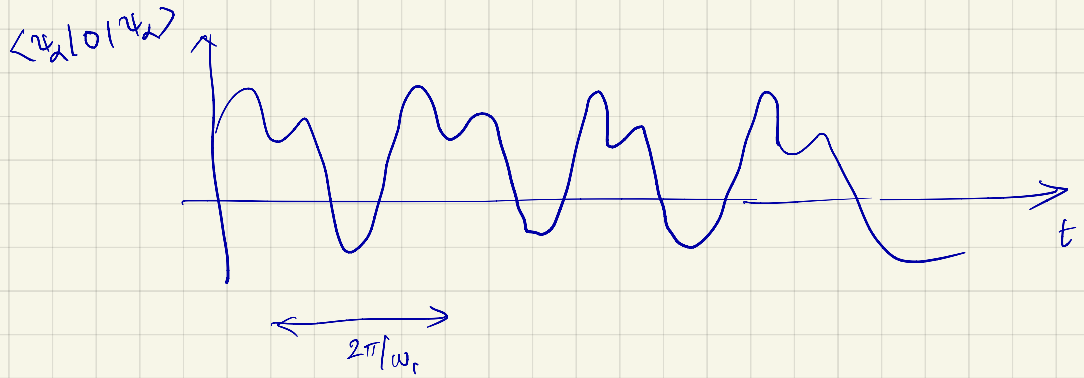
$|\phi_\alpha(\omega, t)\rangle = |\phi_\alpha(\omega, t + 2\pi)\rangle$ is a smooth periodic function.



$|\phi_\alpha(\theta)\rangle$ is a smooth map from the circle to $\mathcal{H}$.

Two takeaways :

① Periodic response in quasi-energy states



Contains higher harmonics of the drive frequency. Stronger the drive, larger the harmonic content

② Quasi-periodic response in generic states

$$|\Psi(0)\rangle = \sum_\alpha c_\alpha |\psi_\alpha(0)\rangle$$

$$|\chi(t)\rangle = \sum_\alpha c_\alpha e^{-i\varepsilon_\alpha t} |\phi_\alpha(\omega, t)\rangle$$

so that observables $\langle \chi(t) | \hat{O} | \chi(t) \rangle$ will involve fundamental frequencies $\varepsilon_\alpha - \varepsilon_\beta$ in addition to ω_r.

$\overline{\text{IV}}$ The frequency lattice allows for calculation of qe states & non-linear response.

→ The frequency lattice organizes the harmonic content of $|\phi_\alpha(\omega, t)\rangle$.

→ When the drive is dynamical (e.g. for a cavity-QED system), the frequency lattice eigenstates are eigenstates of the cavity + qubit system.

Let's start with the first way of thinking about it.

$$i\frac{d|\psi\rangle}{dt} = H(\omega,t)\,|\psi\rangle$$

Plug in $|\psi_\alpha\rangle = e^{-i\varepsilon_\alpha t}\sum_n |\tilde{\phi}_{\alpha n}\rangle e^{-in\omega_1 t}$

$$H(\omega,t) = \sum_m \tilde{H}_m e^{-im\omega_1 t}$$

$$\underbrace{\sum_m \left(\tilde{H}_{n-m} - \delta_{nm}\, n\omega_1\right)|\tilde{\phi}_{\alpha m}\rangle = \varepsilon_\alpha |\tilde{\phi}_{\alpha n}\rangle}$$

eigenvalue equation corresponding to a tight-binding model on Fourier lattice.

$\Downarrow$ 2nd quantized notation

$$|\tilde{\phi}_\alpha\rangle = \sum_n |\tilde{\phi}_{\alpha n}\rangle |n\rangle.$$

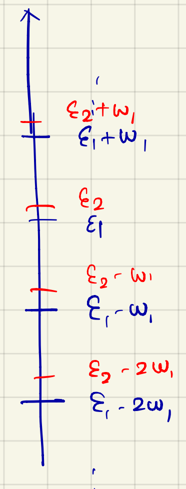
$$K_{nm} = \sum_{nm}\left(\tilde{H}_{n-m} - \delta_{nm}\, n\omega_1\right)|n\rangle\langle m|$$

$$K|\tilde{\phi}_\alpha\rangle = \varepsilon_\alpha |\tilde{\phi}_\alpha\rangle$$

K: tight-binding Hamiltonian with uniform electric field.

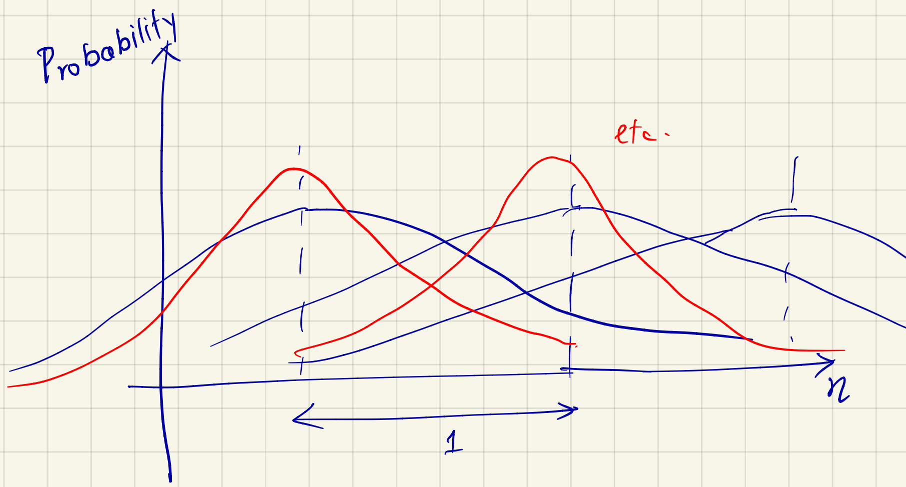
For our example, $K = \sum_n A\sigma_x \left(|n\rangle\langle n+1| + h.c\right) + \sum_n \left(\frac{\omega_q}{2}\sigma_z - n\omega_1\right)|n\rangle\langle n|$

Eigenenergies



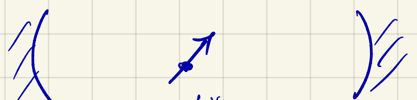
"Stark ladder".

Eigenstates



In the second way of looking at the frequency lattice

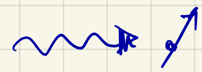
$|n\rangle =$ n photon state of drive.



qubit

Cavity QED

$\Longrightarrow$ Cavity in a coherent state



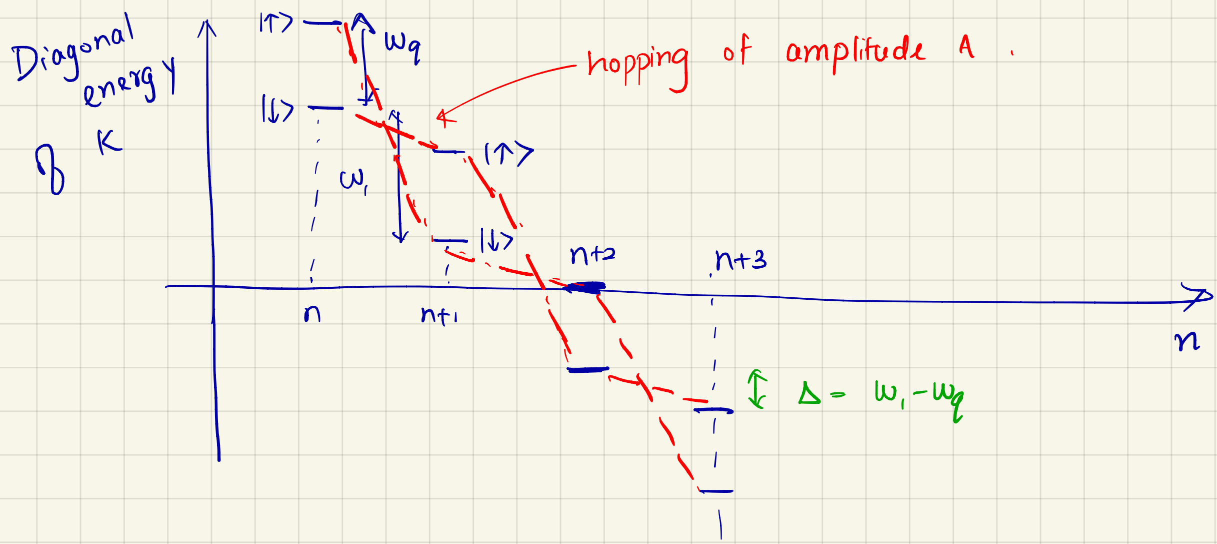
driven qubit

Application 1:

→ Physics of optical lattices from off-resonant drives

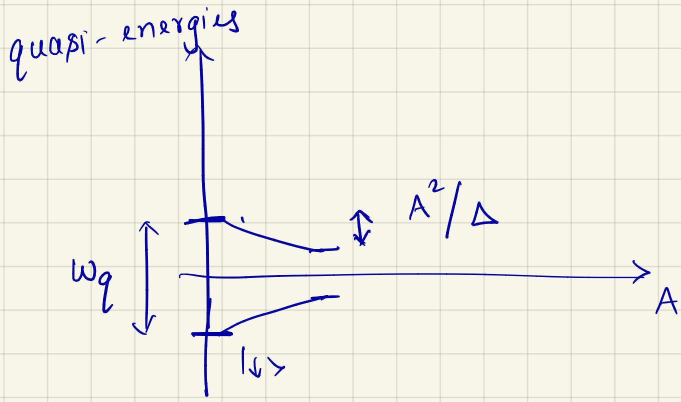
$$\omega_1, \omega_q \gg |\Delta| = |\text{detuning}| = |\omega_1 - \omega_q| \gg A$$

ω_1: Atomic freq. ω_q: Laser freq



(move this picture up)

Detuning $\gg$ hopping $\Rightarrow$ treat hopping perturbatively.



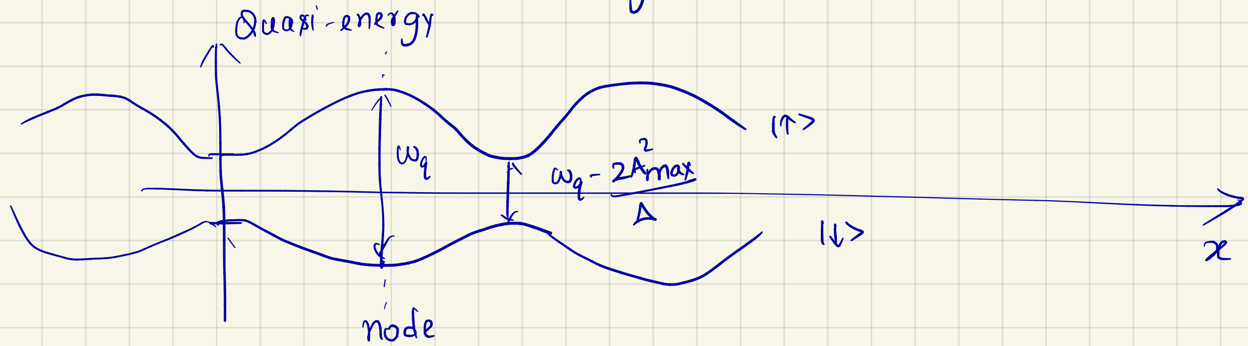
Quasi-energy gap

$$= \omega_q - \frac{2A^2}{\Delta}.$$

$\left\{ \begin{array}{ll} \uparrow & \Delta < 0 \\ \downarrow & \Delta > 0 \end{array} \right.$

This is also called the AC Stark shift

Optical lattice : A is position dependent because the laser field is a standing wave.



$\Delta > 0$
(blue detuned)

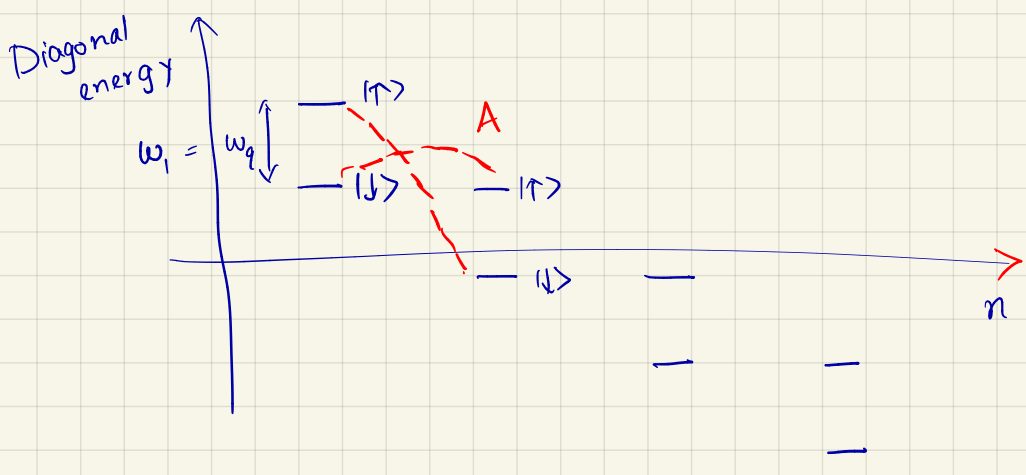
Atoms are attracted to nodes in the standing wave in the $|\downarrow\rangle$ state. They are attracted to anti-nodes in the $|\uparrow\rangle$ state.

Application 2 :

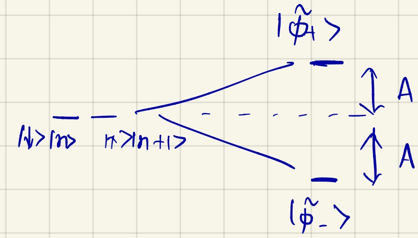
→ Physics of gates from resonant drives.

$$\omega_1, \omega_q \gg A \gg |\Delta|$$

Ex : $\Delta = 0$.



Degenerate PT



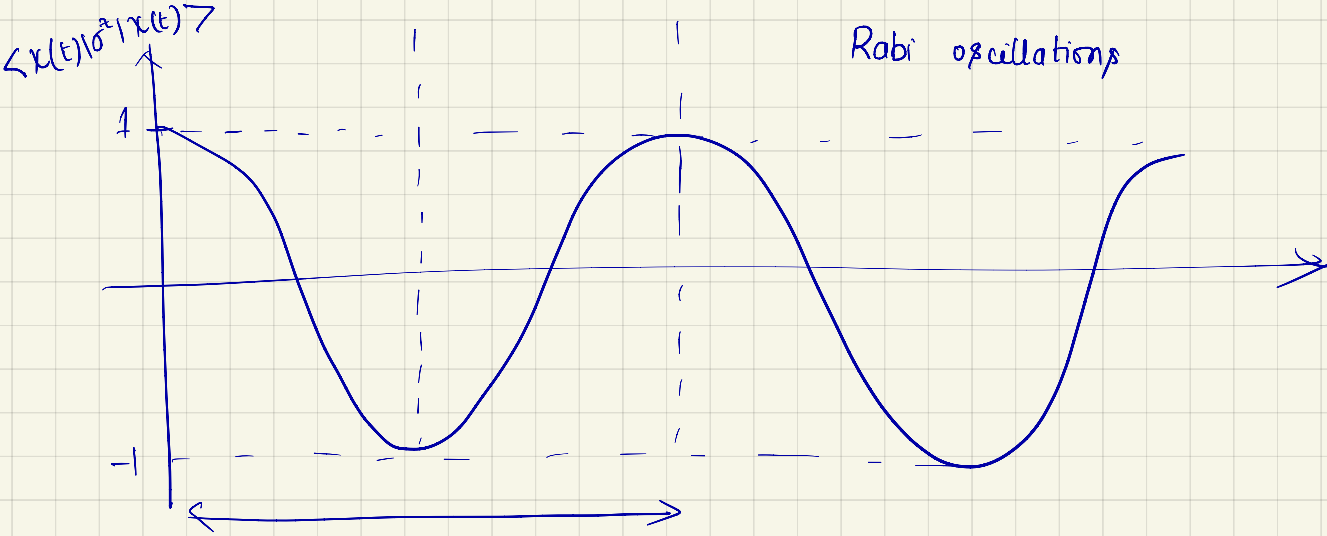
$|\downarrow\rangle|n\rangle$ and $|\uparrow\rangle|n+1\rangle$ have the same diagonal energy.

Degenerate perturbation theory $\Rightarrow$ $\varepsilon_+ - \varepsilon_- = 2A$

$$|\tilde{\phi}_\pm\rangle = \frac{|\downarrow\rangle|n\rangle \pm |\uparrow\rangle|n+1\rangle}{\sqrt{2}}.$$

When you work out the dynamics starting from $|\chi(0)\rangle = |\uparrow\rangle$

Rabi oscillations

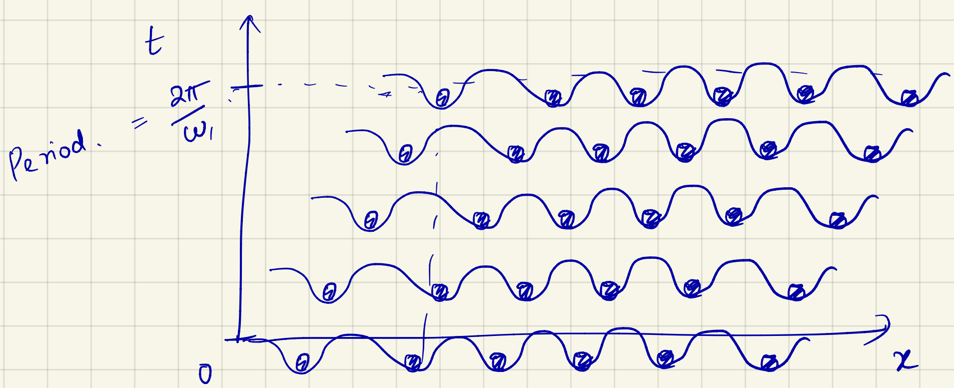


Period $= \dfrac{2\pi}{2|A|}$; $\Omega = 2|A| =$ Rabi rate.

If you stop the drive at $t = \dfrac{\pi}{2\Omega}$, Hadamard gate.
" " " " $t = \dfrac{\pi}{\Omega}$, X gate.

<u>Application 3</u> : Thouless pump (1981)

= Insulating wire pumps charge at an average quantized rate.



Configuration back to itself in the bulk.

Charge transferred across the wire in a period $= 1$ (= integer in general).

<u>Regime</u> : $\omega_1 \ll$ instantaneous band gap of the insulator.
[adiabatic regime]

The origin of this effect can be easily understood on the frequency lattice that we will introduce in lecture 2.

If we don't come back to it, remind me!