Perception:

\[ \mathbf{2} = \left( \sum_{i=1}^{N} \frac{1}{\mu} \sum_{\mu=2}^{N} e^{-\beta \mathcal{U}(\mathbf{x}_i)} \right) \]

Spheres:

\[ \mathbf{x} = \{ \mathbf{x}_1, \ldots, \mathbf{x}_N \} \]

\[ \mathbf{x}_i \in V \subset \mathbb{R}^d \]

\[ \mathbf{Z} = \left( \sum_{i=1}^{N} \frac{1}{\mu} \sum_{\mu=2}^{N} e^{-\beta \mathcal{U}(\mathbf{x}_i)} \right) \]

Thermodynamic limit:

\[ \frac{N_1 R \rightarrow \infty}{N, V \rightarrow \infty} \quad \text{or} \quad \frac{N_1 R \rightarrow \infty}{N \rightarrow \infty} \]

\[ \varphi = \frac{N V d}{\mathcal{Z}_1(\beta/t)} \]

\[ \varphi = \frac{N V d (\beta/t)}{\mathcal{Z}_1(\beta/t)} \]

One control parameter \( \varphi \).

\[ \mathbf{S} \rightarrow 0 \]

\[ e \sim (\varphi - \varphi_{\text{cr}})^2 \]

\[ S \sim \log(1 - \varphi_{\text{cr}}) \]

Spheres as constraint satisfaction problems.

Example: \( d = 2, 3 (4, 5, \ldots) \) the solid-liquid transition is the densest packing \( \rightarrow \) crystal.

Consider soft harmonic spheres, \( \mathcal{W}(h) = \frac{\epsilon}{2} h^2 \Theta(-h) \)

\( \varphi < \varphi_{\text{cr}} \) crystal has entropic rigidity, \( P \propto T \)

\( \varphi > \varphi_{\text{cr}} \) crystal is mechanically rigid.
Solution 1
Reintroduce disorder: \( E_p(x) = |\vec{x}_i - \vec{x}_j + A_i - A_j| - \sigma \)
\( A_j \): random vector

Krishnam 1962 (liquid)
Masi-Krishnan-Krishnan 2006 (glass-jamming)
Mor-Krishnan 2010

Solution 2
Get rid of the crystal as usual (polydispersity) \( l_p = |\vec{x}_i - \vec{x}_j| - \sigma \)

Increase dimension: above \( d = 4 \) no crystallization is observed
\( \Rightarrow \) Henry Cohn, very hard to find crystals in large \( d \)

In both cases, there is a bunch of crystalline solutions, but impossible to find \( \Rightarrow \) restrict the study to amorphous ones.

Solution 3

What about replicas? Why replica if there is no quenched disorder?

Solution: Frank-Paris potential:
Reference equilibrium configuration \( \tilde{R} \)
Completed equilibrium configuration \( \tilde{X} \)

\( \rho_g(\phi, T | \phi_g, T_g, \phi_g) = -\frac{1}{N} \int d\tilde{R} e^{\frac{\phi H(\tilde{R})}{2\sigma}} \log \int d\tilde{X} e^{-\frac{\phi_H(\tilde{X})}{2\sigma}} s(\phi \cdot \phi_g, \phi_{g}) \)

Note: \( \phi, \phi_g \) are controlled by \( \sigma, \phi_g \)

Alternative solution: the Monasson method (Monasson, PRL 1995)

We are going to discuss the problem in \( d \to 0 \) using the Frank-Paris potential as our exam tool.
\( \Rightarrow \) does not need any quenched disorder.
Summary of Lecture 2

Perceptron

$\exists x \in \mathbb{R}^n, \exists z \in \mathbb{R}$, $\mathbf{p}(x) = \mathbf{1}_x \cdot \Phi_m - \sigma > 0$

$\gamma = 1, \ldots, M$

$\mathbf{a} = \frac{N}{M}$

Spheres

$X = \{x_1, \ldots, x_N\} \mid x_i \in \mathbb{R}^d$

$z_i \in \mathbb{S}^d(\mathbb{R}) \cup \mathbb{R}^{d+1}$

Satisfy

$\mathbf{b}(x) = |\mathbf{x}_i - \mathbf{x}_j| - \sigma > 0$

$m = 1, \ldots, N(N-1)/2$

Thermodynamic limit

$h = \frac{N}{V} \sum_i U(\mathbf{1}_i - \mathbf{x}_j - \sigma)$

Disorders. Differences: crystal, no disorder => close packing is ordered, hyperstable

Focus on amorphous states:

1. $b_p(x) = |x_i - x_j + Ay| - \sigma$

2. $b_l(x) = |x_i - x_j| - \sigma$

3. Park at $d > y$, amorphous

P vs $\phi$

Critical point

S vs $\phi$

Phase transitions

 saturated transition when crystal is not allowed
Spheres in infinite dimension. (book can be sent privately)

[Physicist perspective]

Liquid

[Frisch, Biers, Klein, Wyler '80s]

[Frisch-Rosser Phys'e93]

Virial expansion.

\[- \beta F[p(x)] = \int dx \rho(x) \left[ 1 - \log \rho(x) \right] + \cdots + \Delta + \Box + \cdots + \\]

Specialize to \( \rho(x) = \rho \) [eliminate crystal]

Define \( \overline{\rho} = 2^{d} \rho \) and specialize to hard spheres in SAT phase.

The series converges for \( \overline{\rho}_{\text{sw}} > \left( \frac{W(e/2) - 1}{W(e/2)} \right) = 0.144 \ldots \)

\( W(x) e^{W(x)} = x \) [Lebowitz][Phurrough '61]

[Remander Minkowski bound \( 2^{d+1} \rho > 1 \) and best upper bound \( \overline{\rho} \leq 2^{d+1} \left( 1 - 0.5390 \right) d = 2^{d+1} \rho_{\text{sw}} \)]

Ring diagrams dominate at each order.

\[- \beta F(p) = \int dx \rho^{d}(x) + \cdots + \Delta + \Box + \cdots + \]

\[\Rightarrow \text{exact resummation, closed expression for } \rho(p)\]

Three important result:

1. Strongly suggests that \( \overline{\rho}_{\text{sw}} = 1 \)

   i.e. the liquid realises at least the Minkowski bound.

2. The pole that makes the series divergent is at \( \overline{\rho} < 0 \)

\[\text{like } \log(1+x)\]
The first singularity at $\bar{\rho} > 0$ is at

$$\bar{\rho}_0 = \left(\frac{\rho}{\lambda}\right)^d = \frac{d}{\lambda} \frac{1}{\log(\Lambda^2) - 1} = \frac{d}{\lambda} \left[ \frac{1}{\log(\Lambda^2) - 1} \right]$$

$$= 2 \bar{\rho}_0 = 12.15 \bar{\rho}_0$$

Some kind of liquid spinodal.

Aha! The liquid might exist up to $\bar{\rho}_0$

and bent exponentially the Minkowski bound.

(Similarly this is Sel'Torgato's perspective)

Conclusion. For any $\bar{\rho} < \bar{\rho}_0$, one can discard all virial diagrams

except the first one

$$P = \int dx \rho(x) \left[ 2 - \log p(x) \right] + \frac{1}{2} \int dx dy \rho(x) \rho(y) f(x, y)$$

$$\Rightarrow \quad \rho(p) = 1 + \frac{p}{2}$$

$$\frac{A}{\bar{\rho}} = \frac{1}{1 + \sqrt{\frac{A}{2}}}$$

Liquid dynamics:

Exactly solvable [Grune'l.]

Not like, though at $\bar{\rho} = 4.8 d$

Change scale: $\bar{\rho} = d \bar{\rho}_0$  \[ \rho = 1 + \frac{\bar{\rho}}{2} - \frac{d}{\bar{\rho}} \frac{\bar{\rho}}{2} \quad \frac{\rho}{\bar{\rho}} \approx \frac{\bar{\rho}}{2} \]

Note:

$$\rho = \frac{\bar{\rho} + \Delta \bar{\rho}}{\lambda} = \bar{\rho} \frac{\rho + \Delta \rho}{2\bar{\rho}} = 1 + \frac{\bar{\rho}}{2} - \frac{d}{\bar{\rho}} \frac{\bar{\rho}}{2} \int d\rho \rho^d \left( \frac{\rho + \Delta \rho}{2\bar{\rho}} \right)^{d-1} \left[ \bar{\rho} J_0(\rho) \right]$$

$$+ \frac{d}{2 \bar{\rho}} \left( \frac{\rho + \Delta \rho}{2\bar{\rho}} \right)^{d-1} \left[ \bar{\rho} J_0(\rho) \right]^2 \frac{1}{1 + \bar{\rho} J_0(\rho)}$$
Fonzi-Parsi potential: results for the transition density

Replicas
\[ F[p(x)] = \int dx \, p(x) \left[ 1 - \log p(x) \right] + \int dxdy \, p(x)p(y) \Psi(x,y) \]

Rotational + translational invariance
+ limit case (central limit theorem)

→ exact expression of the Fonzi-Parsi potential

Out of equilibrium: Following state in density/temperature with the Fonzi-Parsi potential.

Landscape picture

All packings are in the four PSB phase.
Very slow compression: $q - d \log d$
beats best lower bound (but not exponentially)

Slow compression (but poly$(N)$): $q - \frac{7.24}{d}$
also beats some lower bounds

Very fast compression: $q - d^{1/2}$ unknown prefactor

Sampling of all solutions stops at $O(d \cdot d^{1/2})$

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(3) Criticality of jamming only found at lowest level, not true at any RRSB, \( \cdot \)

The jammed states are metastable \( \cdot \)

\[ P(h) \sim 2 \delta(h) + h^{-\gamma} \]

\[ P(f) \sim f^\beta \]

Away from jamming, $A - p^k$

Three critical exponents, two scaling relations to one independent exponent

Wegner, PRL 2012

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Conclusion:

- Mention MF theory in 3d \( [\text{Herard-Paniz}] \)

- Prove disc openning.

- Thanks to les & G-organizers

you!