Boulder School for
Condensed Matter and Materials Physics

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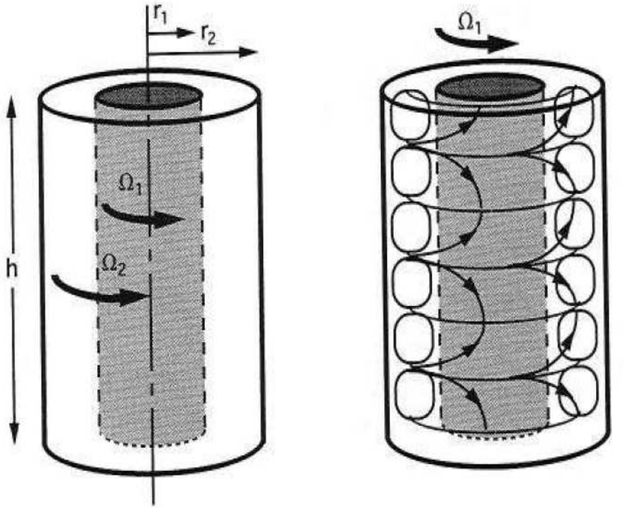
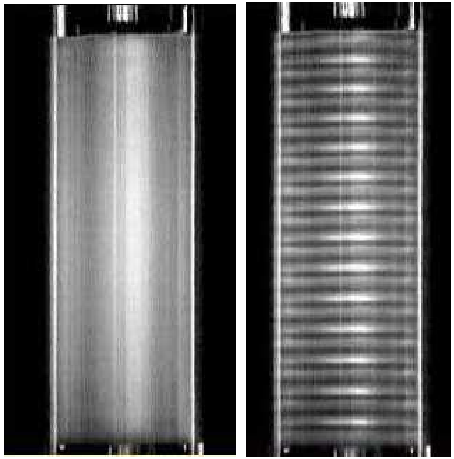
Hydrodynamic Instabilities:

A Zoo of Instabilities

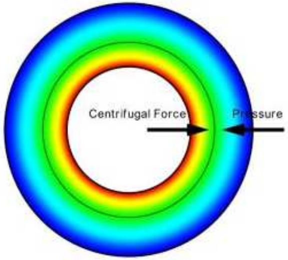
Taylor-Couette flow



R. Tagg



T.T. Lim



$$\mathrm{U} = U_\theta(r) \Longrightarrow \begin{cases} -\frac{1}{r}U_\theta^2 = -\frac{1}{\rho}\frac{dP}{dr} \\ 0 = \nu \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(rU_\theta\right)\right) \end{cases}$$

Laminar Couette profile: $U_\theta = Ar + \dfrac{B}{r}$

Rayleigh criterion: laminar Couette profile stable if

$$\frac{d}{dr}(rU_\theta)^2 > 0$$

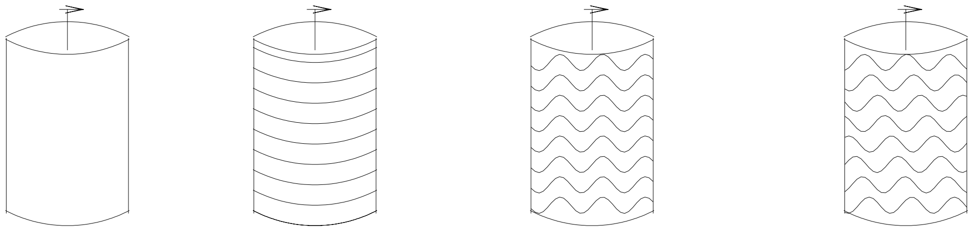
e.g. if outer cylinder rotates and inner cylinder is stationary

Viscometer (Couette) $\not\Leftrightarrow$ **Pattern formation**

stable $\Omega_2 \uparrow$ **unstable $\Omega_1 \uparrow$**

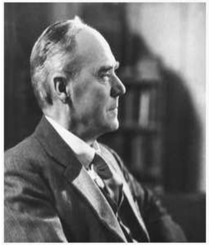
Taylor-Couette flow

for $\Omega_2 = 0$, increasing Ω_1

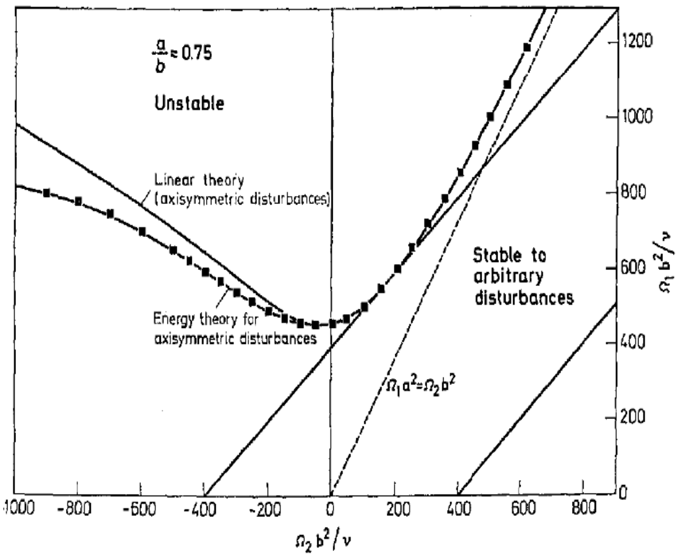


Laminar Couette $U_C(r)$ — **Taylor Vortex** $U_{TV}(r, z)$ — **Wavy Vortex** $U_{WV}(r, \theta, z, t)$ — **Modulated Wavy Vortex** $U_{MWV}(r, \theta, z, t)$

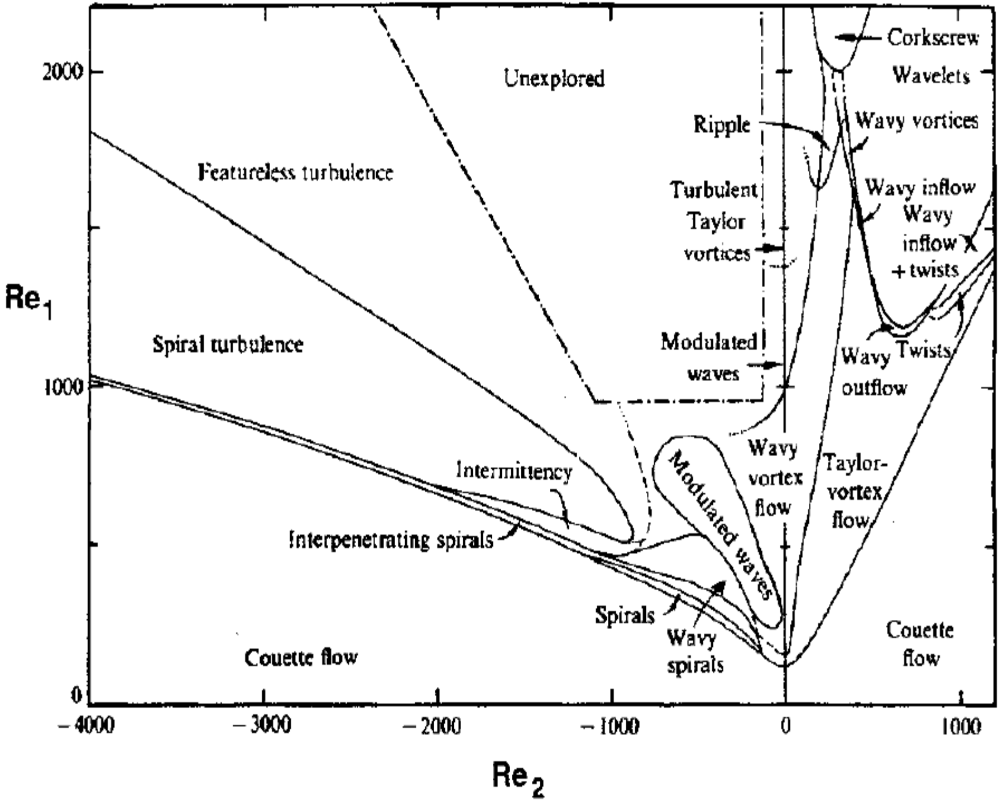
First success of linear stability analysis

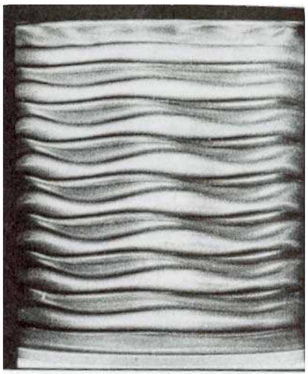

Sir Geoffrey Ingram Taylor

Taylor, 1923



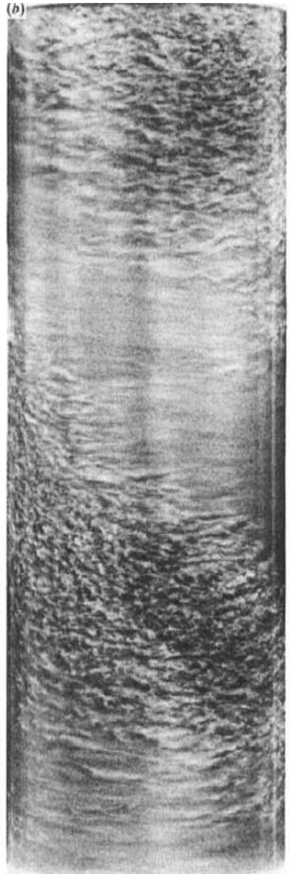
Andereck, Liu, Swinney, 1985





wavy vortices

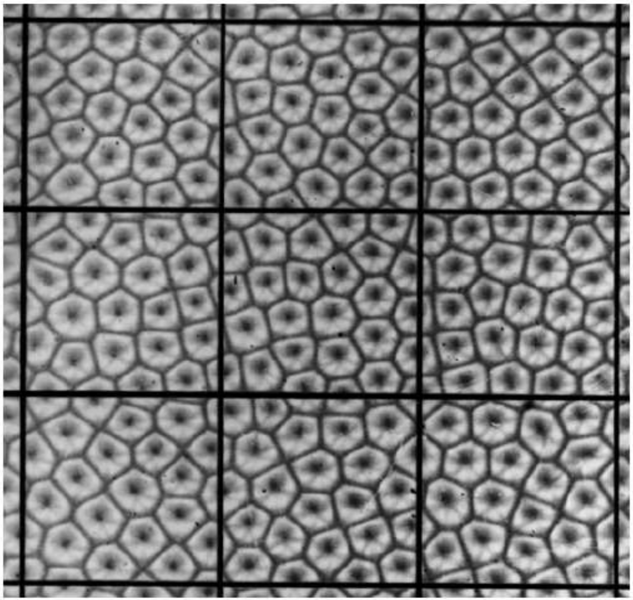
twists



spiral turbulence

Marangoni-Bénard Convection

Temperature gradient	$\Longrightarrow$	**density gradient**	$\Longrightarrow$	**Rayleigh-Bénard**
		surface tension gradient	$\Longrightarrow$	**Marangoni-Bénard**

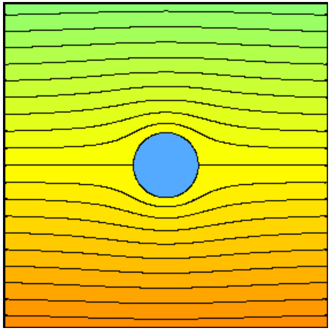


Rayleigh was mistaken about cause of Bénard's observations

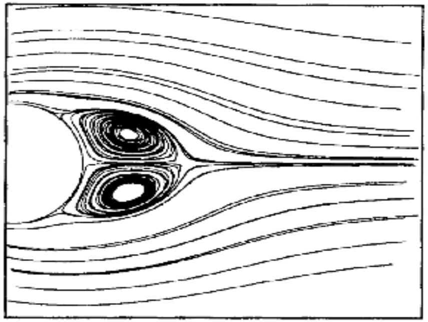
Hexagons require breaking of up-down symmetry

Cylinder wake: Bénard-von Kármán

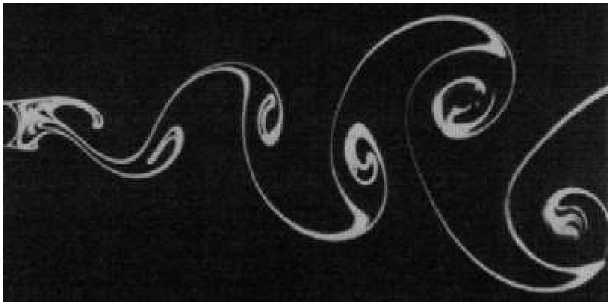
Ideal flow



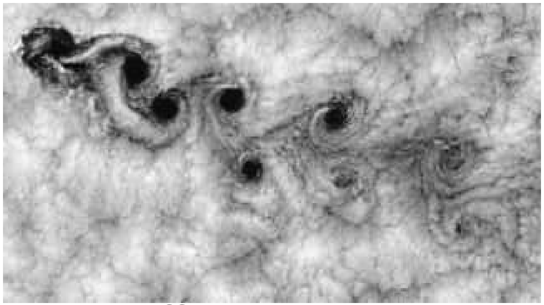
with downstream recirculation



von Kármán vortex street ($\mathrm{Re} \geq 46$)

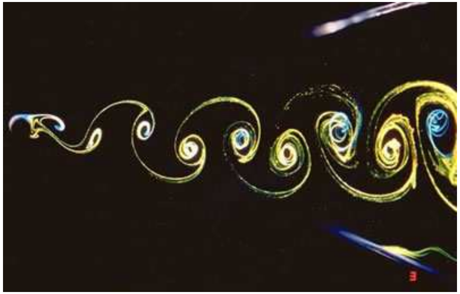


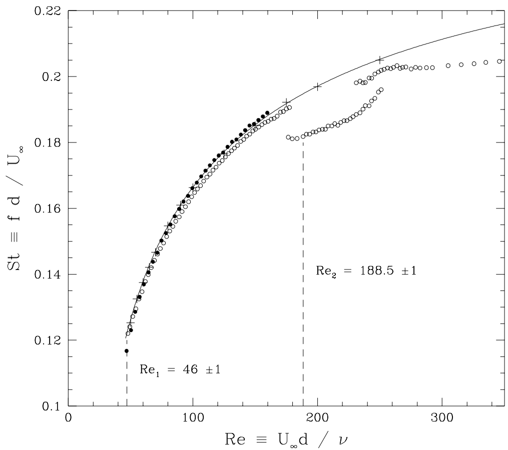
Laboratory experiment (Taneda, 1982)



Off Chilean coast past Juan Fernandez islands

von Kármán vortex street: $Re = U_\infty d/\nu \geq 46$





spatially:
two-dimensional (x, y)
(homogeneous in z)

temporally:
periodic, $St = fd/U_\infty$
appears spontaneously

$$\mathrm{U}_{2D}(x, y, t \bmod T)$$

Stability analysis of von Kármán vortex street

2D limit cycle $\mathbf{U}_{2D}(x, y, t \bmod T)$ obeys:

$$\partial_t \mathbf{U}_{2D} = -(\mathbf{U}_{2D} \cdot \nabla)\mathbf{U}_{2D} - \nabla P_{2D} + \frac{1}{Re}\Delta \mathbf{U}_{2D}$$

Perturbation obeys:

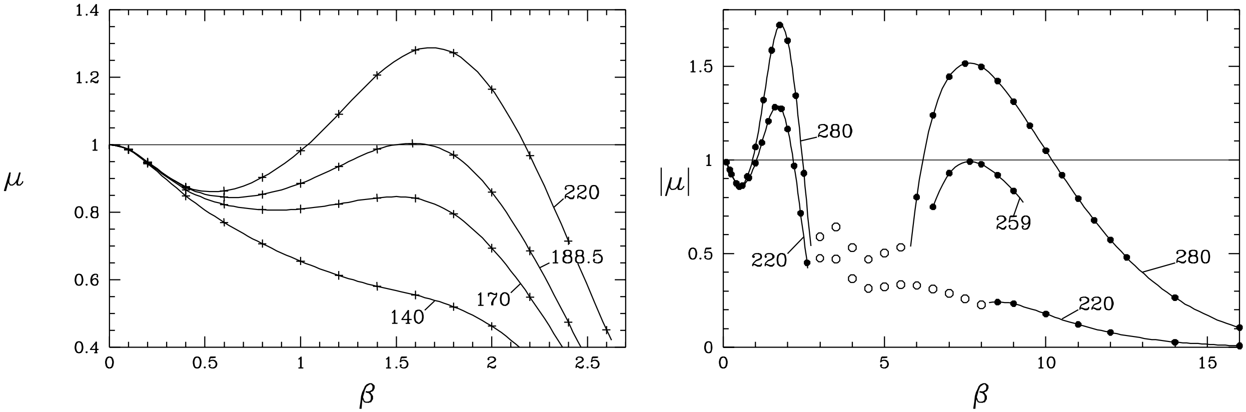
$$\partial_t \mathbf{u}_{3D} = -(\mathbf{U}_{2D}(t) \cdot \nabla)\mathbf{u}_{3D} - (\mathbf{u}_{3D} \cdot \nabla)\mathbf{U}_{2D}(t) - \nabla p_{3D} + \frac{1}{Re}\Delta \mathbf{u}_{3D}$$

Equation homogeneous in z, periodic in $t \Longrightarrow$

$$\mathbf{u}_{3D}(x, y, z, t) \sim e^{i\beta z} e^{\lambda_\beta t} \mathbf{f}_\beta(x, y, t \bmod T)$$

Fix β, calculate largest $\mu = e^{\lambda_\beta T}$ via linearized Navier-Stokes

From Barkley & Henderson, J. Fluid Mech. (1996)



mode A: $\mathrm{Re_c} = 188.5$ **mode B:** $\mathrm{Re_c} = 259$

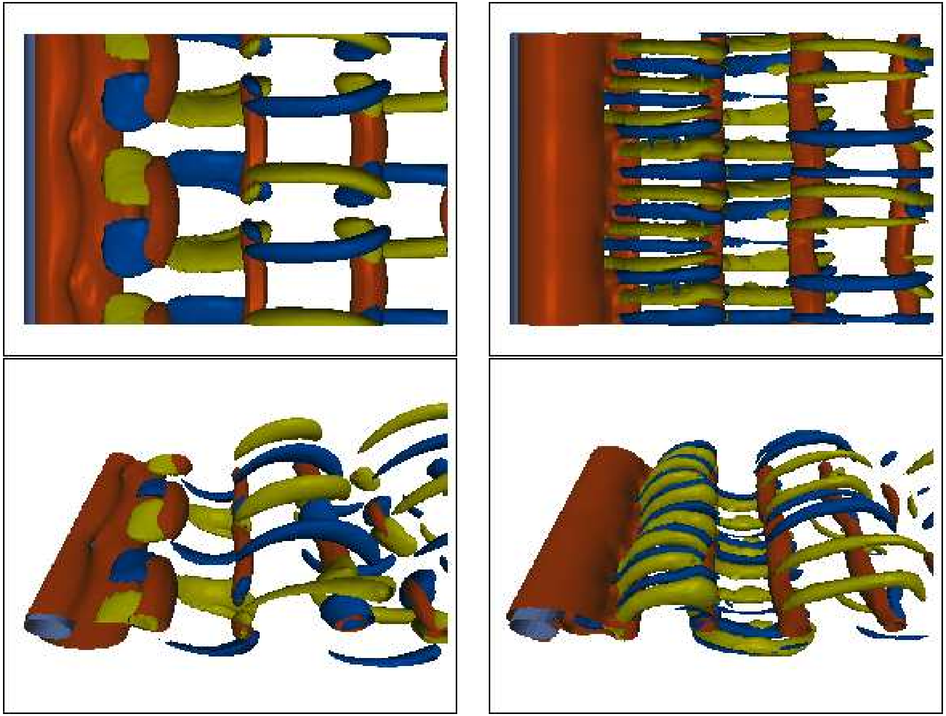
$\beta_c = 1.585 \Longrightarrow \lambda_c \approx 4$ $\beta_c = 7.64 \Longrightarrow \lambda_c \approx 1$

Temporally: $\mu = 1 \Longrightarrow$

steady bifurcation to limit cycle with same periodicity as U_{2D}

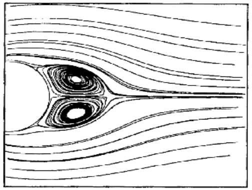
Spatially: circle pitchfork (any phase in z**)**

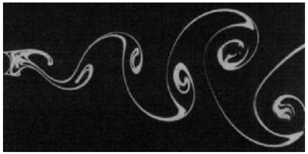
mode A at $\mathrm{Re} = 210$ **mode B at $\mathrm{Re} = 250$**

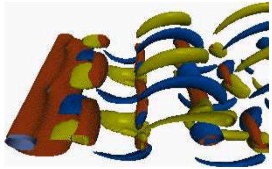


From M.C. Thompson, Monash University, Australia
(http://mec-mail.eng.monash.edu.au/∼mct/mct/docs/cylinder.html)

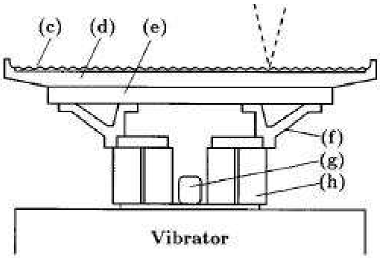
	Re=47		Re=188	
2D	$\Longrightarrow$	**2D**	$\Longrightarrow$	**3D**
steady	**Hopf**	**oscillatory**	**circle PF**	**oscillatory**





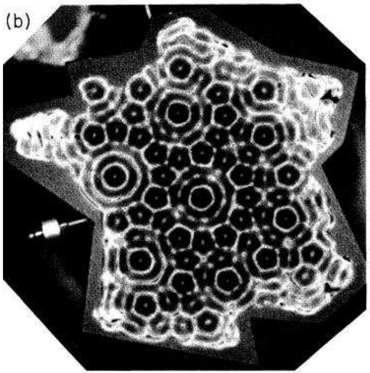


Faraday instability

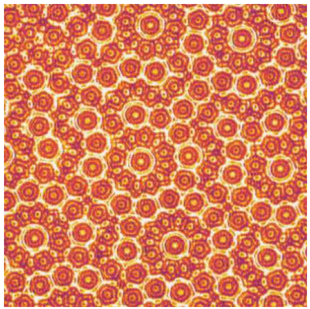


Faraday (1831): Vertical vibration of fluid layer $\Longrightarrow$ stripes, squares, hexagons

In 1990s: first fluid-dynamical quasicrystals:

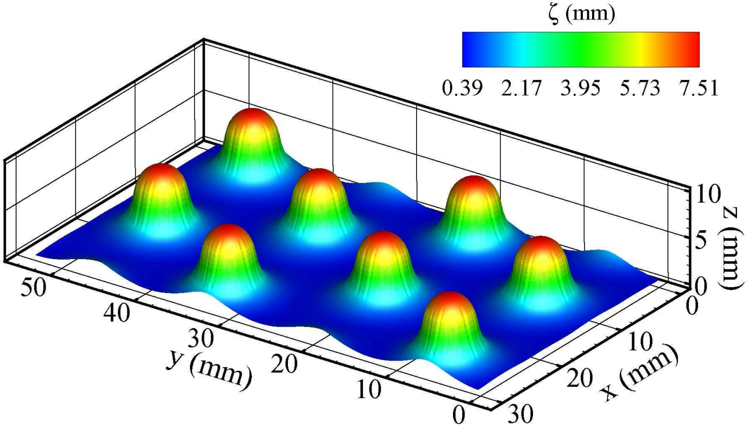


Edwards & Fauve
J. Fluid Mech. (1994)



Kudrolli, Pier & Gollub
Physica D (1998)

Hexagonal patterns in Faraday instability



From Périnet, Juric & Tuckerman, J. Fluid Mech. (2009)

Oscillating frame of reference $\Longrightarrow$ "oscillating gravity"

$$G(t) \;=\; g\,(1 - a\cos(\omega t)) \qquad (1)$$
$$G(t) \;=\; g\,(1 - a\,[\cos(m\omega t) + \delta\cos(n\omega t + \phi_0)]) \qquad (2)$$

Flat surface becomes linearly unstable for sufficiently high a

Consider domain to be horizontally infinite (homogeneous) $\Longrightarrow$ solutions exponential/trigonometric in $\mathrm{x} = (x, y)$

Seek bounded solutions $\Longrightarrow$ trigonometric: $\exp(i\mathrm{k}\cdot\mathrm{x})$

$$\textbf{Height}\;\; \zeta(x, y, t) = \sum_{\mathrm{k}} e^{i\mathrm{k}\cdot\mathrm{x}}\hat{\zeta}_k(t)$$

Oscillating gravity $\Longrightarrow$ temporal Floquet problem, $T = 2\pi/\omega$

$$\hat{\zeta}_k(t) = \sum_j e^{\lambda_k^j t} f_k^j(t)$$

Height $\zeta(x,y,t) = \sum_{\mathrm{k}} e^{i\mathrm{k}\cdot\mathrm{x}}\hat{\zeta}_k(t)$

Ideal fluids (no viscosity), sinusoidal forcing $\Longrightarrow$ Mathieu eq.

$$\partial_t^2 \hat{\zeta}_k + \omega_0^2 \left[1 - a\cos(\omega t)\right] \hat{\zeta}_k = 0$$

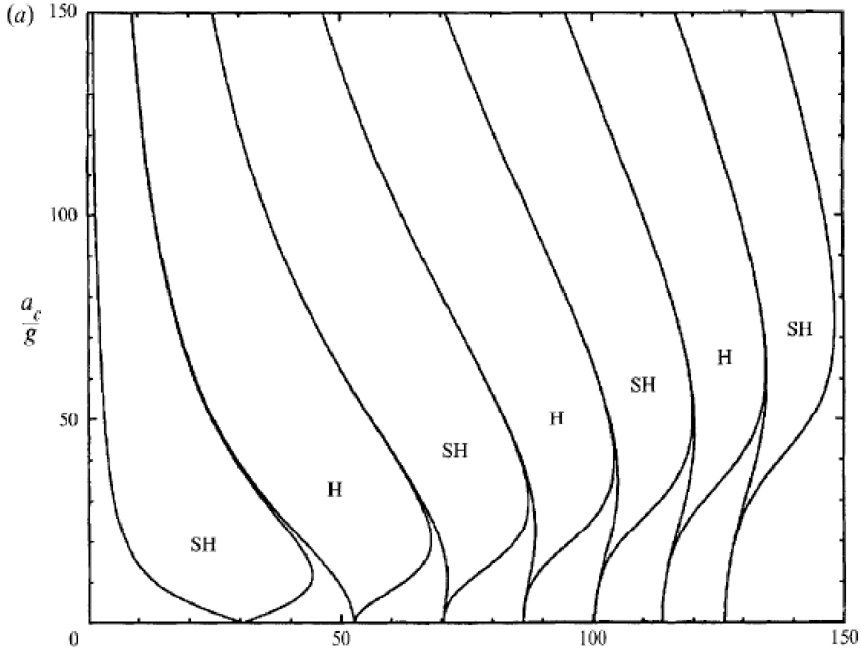
ω_0^2 **combines** g**, densities, surface tension, wavenumber** k

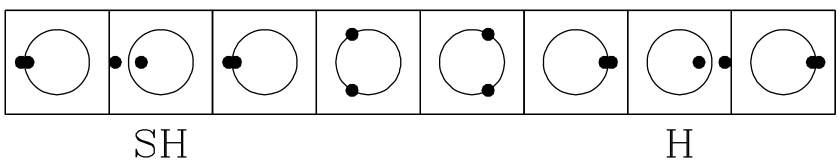
$$\hat{\zeta}_k(t) = \sum_{j=1}^{2} e^{\lambda_k^j t} f_k^j(t)$$

Re$(\lambda_k^j) > 1$ **for some** $j, k \Longrightarrow \hat{\zeta}_k \nearrow \Longrightarrow$ **flat surface unstable**

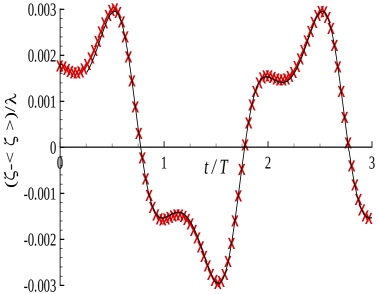
$\Longrightarrow$ **Faraday waves with wavelength** $2\pi/k$

Im(λ_k^j)	$e^{\lambda T}$	**waves**	**period**
0	**1**	**harmonic**	**same as forcing**
π/ω	-1	**subharmonic**	**twice forcing period**

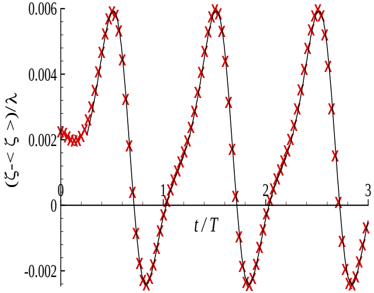

(a)
150
100
50
0
$\frac{a_c}{g}$
50
100
150
SH
H
SH
H
SH
H
SH


SH
H

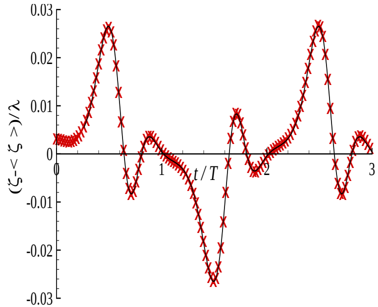
Floquet functions



within tongue 1 /2
subharmonic
$\mu = -1$



within tongue 2 /2
harmonic
$\mu = +1$



within tongue 3 /2
subharmonic
$\mu = -1$

From Périnet, Juric & Tuckerman, J. Fluid Mech. (2009)