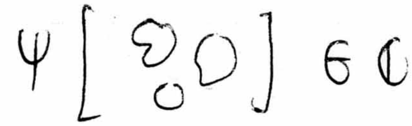
21/7

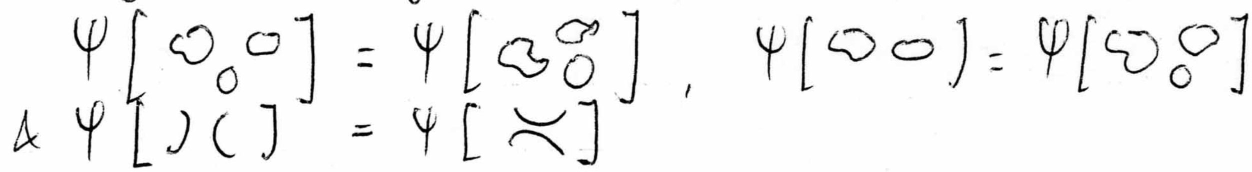
① Chetan Nayak – Lecture 3

1. Generalisation of Kitaev model.
2. Features of resulting phases – Quasiparticles, Ground state degeneracies
3. Concluding remarks

Kitaev model; wavefunction on basis of loops.



Obeying rules (for ground state).



We will discuss conditions on loops, but should remember that such conditions are imposed by the Hamiltonian, local violation of rules are excitations.

[Question about non Abelian nature:

Permutation group – trivial representation ⇒ Bosons
parity representation ⇒ Fermions

Higher dimensional representations are parastatistics, described by additional quantum numbers

BUT, here we consider representations of the braid group]

Generalising loop rules:

- Keep deformations of loops $\Psi[\bigcirc] = \Psi[\bigcirc]$
- On adding loops, add complex factor. $\Psi[\bigcirc\,\circ] = d\,\Psi[\bigcirc]$

  (This connection must exist, otherwise each number of loops is a different state).
- For the third rule;

$$\Psi[\sim] = \Psi[\Omega] = \Psi[\underline{\;\circ\;}] = d\,\Psi[\sim].$$

So, consistency demands a change of the third rule. To fix this, if $d=-1$, then use:

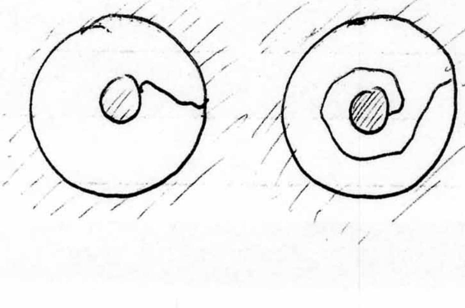
$$\left.\begin{aligned}\Psi[\circ] &= -\Psi[\quad] \\ \Psi[)(] &= -\Psi[\genfrac{}{}{0pt}{}{\smile}{\frown}]\end{aligned}\right\} d=-1$$

One might consider $\Psi\left[)\,(\right] = \frac{1}{d}\left[\genfrac{}{}{0pt}{}{\smile}{\frown}\right]$ in general,

But, no special direction, so $\Psi\left[)\,(\right] = \frac{1}{d^2}\Psi\left[)\,(\right]$

which is not consistent (unless $d^2=1$, i.e. $d=\pm 1$).

For $d=-1$, consider excitations on the annulus.



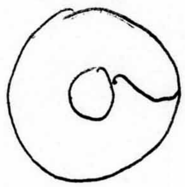
For $d=1$, we could consider excitations as point-like

(2)

Here $(d=-1)$ we need to consider the size of excitations, so, either:

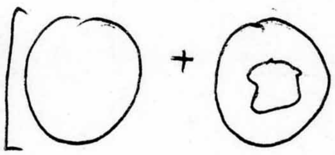
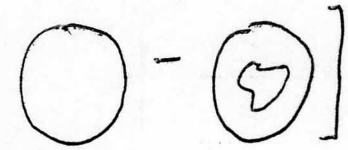
- Consider the plaquete where a line terminates

Or
- Excitations on the anulus,

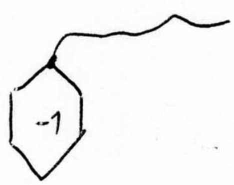


(which, at a distance looks like the previous electric particle).

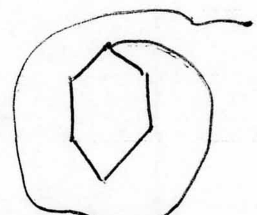
For a simply connected manifold, the ground state changes from:

$d = 1$, $\frac{1}{\sqrt{2}}$ [] to $d=-1$, $\frac{1}{\sqrt{2}}$ []

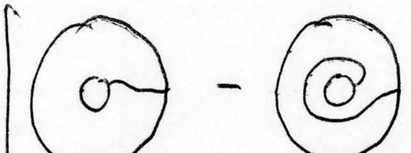
Returning to $d=1$, the electric-magnetic Pcl:

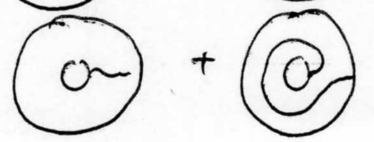


& all curves continuously connected BUT



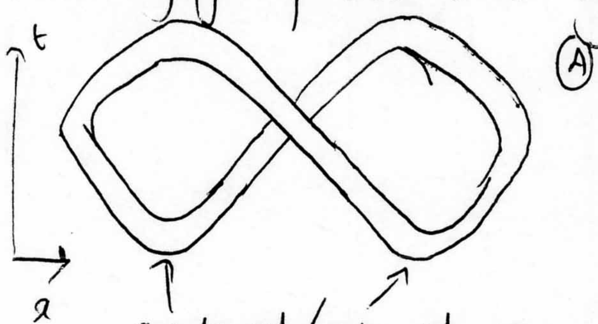
Ψ acting on these two loop pictures gives $+1$ & -1 respectively.

On the anulus, $\frac{1}{\sqrt{2}}$ [] = EM particle

$\frac{1}{\sqrt{2}}$ [] = E particle.

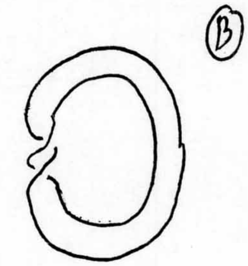
Spin Statistics theorem (for spin as rotational quantum number)

Accounting for particle size by "Wald ribbons"



create pcl/anti pcl pair.

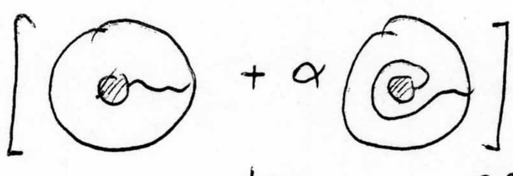
continuous deformation



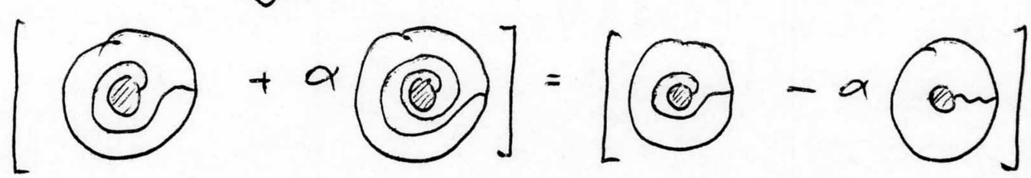
[Ribbon gives orientation, & keeps track of rotation]

Thus spin (rotation, ie B) & statistics (exchange, ie A) must be related.

Consider $\frac{1}{\sqrt{1+|\alpha|^2}}$

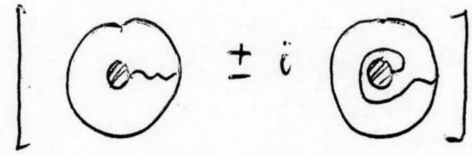


$\Downarrow$ 360° (2π) rotation.



So, if an eigenstate $\lambda = -\alpha$, $\alpha\lambda = 1$ $\Rightarrow$ $\lambda^2 = -1$,

So, $d = -1$, states $\frac{1}{\sqrt{2}}$



are eigenstates

[This combination is an eigenstate of spin & statistics, in addition to being an eigenstate of the Hamiltonian]

③

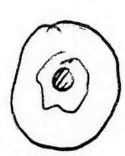
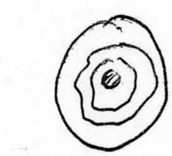
Particle zoology

$\alpha =$	0	0	π	0
$d = 1$	1	e	em	m
$d = -1$	1	s	$\bar{s}$	$s\bar{s}$
$\alpha =$	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	0

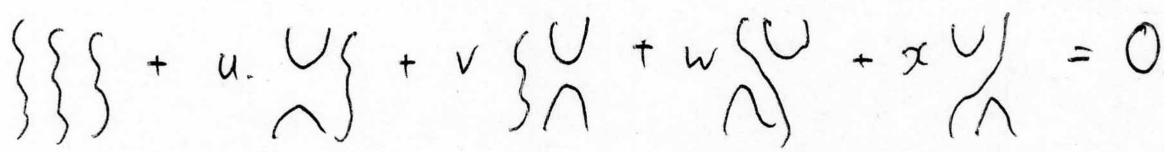
So, in $d=1$ "diagonal statistics" contain only bosons & fermions, in $d=-1$, semions/anti-semions also exist.

Further generalisation, $d \neq \pm 1$, alternate surgery relations

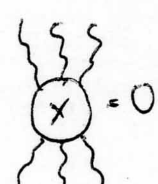
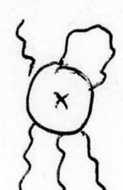
Neglecting rule 3, on the torus or annulus, the ground state is infinitely degenerate, [it has gapless excitations & so is less interesting for QC]

i.e.,  &  etc... not connected

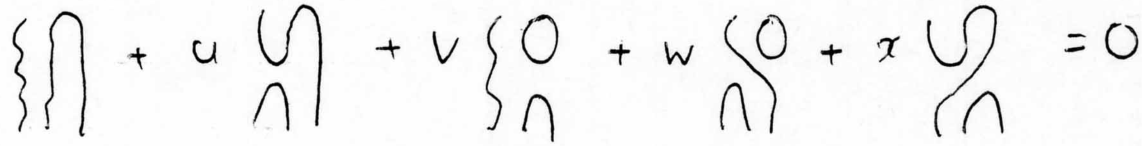
Without "two-curve surgery", try "three curve surgery".



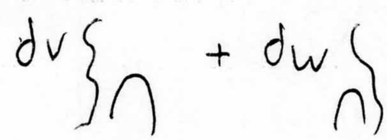
To have a condition on three strands, cannot have rules on 1, 2 strands, so, considering $d \neq \pm 1$

 $= 0 \Rightarrow$ close as  $= 0$, would relate two curves, unless closure $\Rightarrow 0$.

To rephrase, $d \neq \pm 1$ & a 3 curve relation exists, taking the general form & connecting

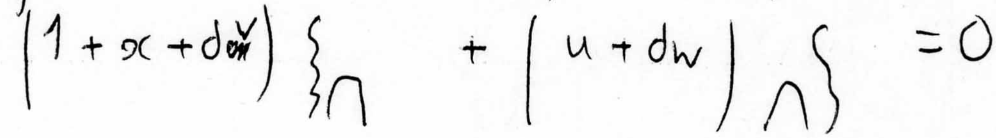


surgeon relation →



Regroup:

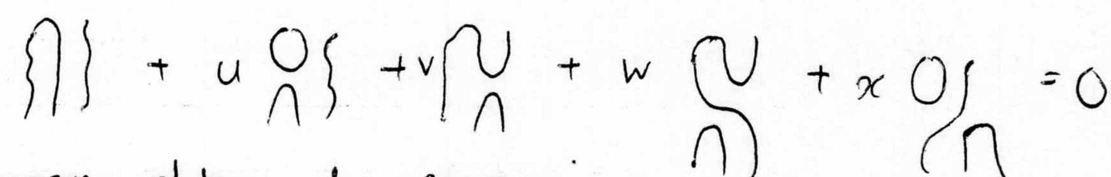
thus,



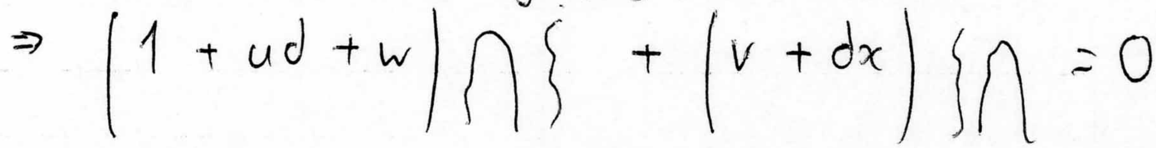
Require $\quad 1 + x + dv = 0 \quad ①$

$\qquad\qquad u + dw = 0 \quad ②$

or.



By surgeon relation, & grouping:



Hence $\quad 1 + du + w = 0 \quad ③$

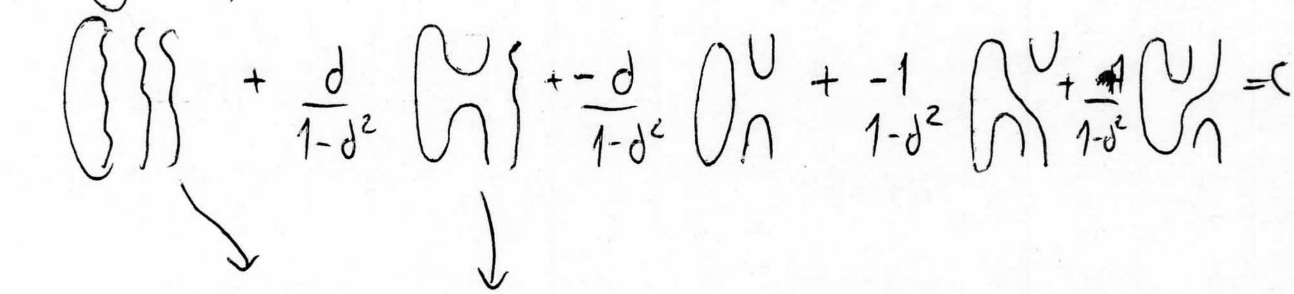
$\qquad\qquad v + dx = 0 \quad ④$

Combining ① & ④ $\quad 1 + x - d^2 x = 0.$

So, $x = \dfrac{1}{1-d^2}$, $v = \dfrac{-d}{1-d^2}$, $w = \dfrac{-1}{1-d^2}$, $u = \dfrac{d}{1-d^2}$

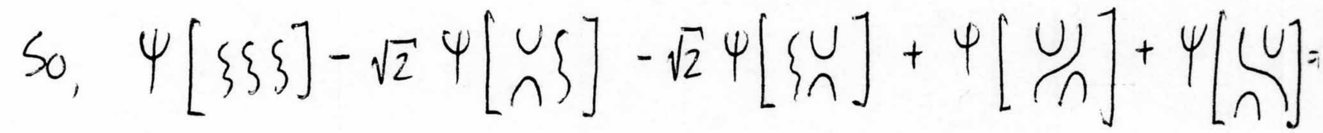
④

Connecting top & bottom.



We find $d + \frac{d}{1-d^2} = 0$ for {{ structure

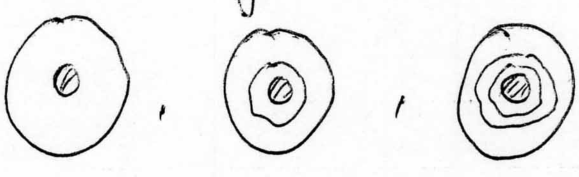
Thus, either $d = 0$, or $2-d^2=0$, $d = \pm\sqrt{2}$



So, we have a set of operators that are consistent, but no longer commute. For some cases, one may find a Hamiltonian whose ground state is described by such relations & which has a ground state manifold gapped from ~~expect~~ excitations.

Assume this does describe a gapped topological phase.

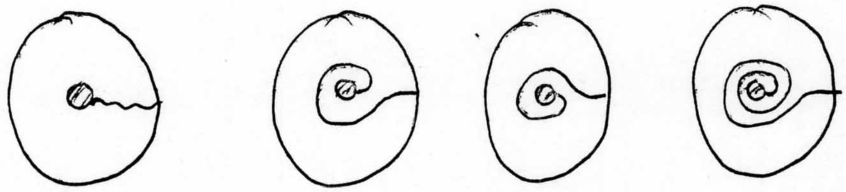
Ground states of $d=\sqrt{2}$ on the annulus.



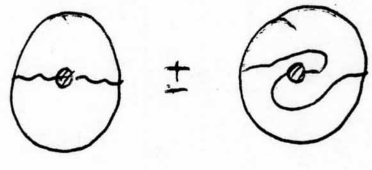
(3 loops reduces to 1 or 0 by surgery)

& similarly 9 states on the torus.

Excitation spectrum:



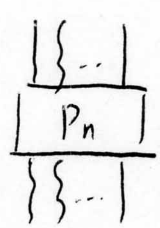
To find ground states of spin/statistics, combine with phase shifts of $\pm e^{i \pm \pi/8}$.

But we also have 

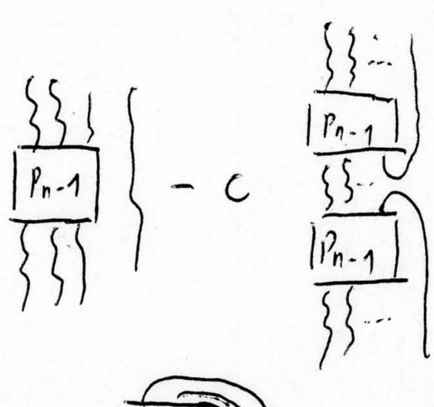
Due to such phases, & surgery relations, braiding patterns leads to states which may be equivalent.

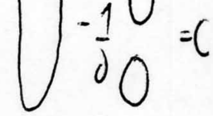
The rules above describe a system which is non-Abelian, but not computationally universal (but in some strict sense is "almost" universal). — thus, consider 4 strands.

Recursive definitions.



$= 0$

$P_n =$ 

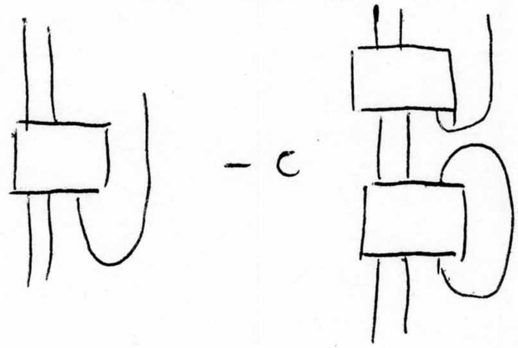
eg P_2: 

effect of closure

If $\Delta_n =$ 

then $c = \Delta_{n-2}/\Delta_{n-1}$

The 2^{nd} condition (on c) guarantees



Finally, to close top & bottom, require:

$$\Delta_n(d) = d\,\Delta_{n-1}(d) - \Delta_{n-2}(d)$$

This is the relation for Chebyshev polynomials, $d = 2\cos\left(\frac{\pi}{K+2}\right)$

(where K is the number of curves).

Concluding remark:

What physical systems can display features of such solvable models.