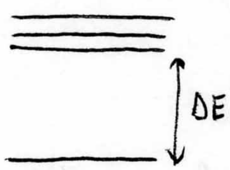
Chetan Nayak - Topological Order. (Phases)

Will not write & solve Hamiltonian - complicated, unknown, unsolvable.
Attempt to understand robust, general features, "top down"
Phases "without broken symmetry" - the classic example, QHE.

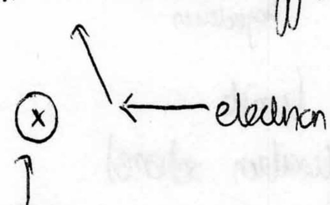
First lecture; · Introduce & define topological phases.
· Simple model - observe general features
· Motivation, generalisations.



Gapped systems (possibly degenerate ground state).

Consider states where, despite lack of low energy modes, can probe topology of space.

Expect robustness of interesting physics.
Gap cause perturbative excitations → analytic effects
Fractionalisation ⇒ discreet values
eg Aharanov Bohm effect.



Solenoid, pointing into paper. Contains flux Φ.

Measure electron scattering cross section, $\frac{d\sigma}{d\theta} = \frac{1}{2\pi p \sin^2(\theta/2)} \sin^2\left(\frac{\Phi}{2}\right)$

Observe scattering, despite no overlap with flux. Either require action at a distance (from B field) or introduce $A = \frac{\Phi}{2\pi} \cdot \frac{z \times x}{|x|^2}$

But, redundant description due to gauge freedom,
$A \to A - \nabla \chi$, $\psi \to \psi e^{i\chi}$.

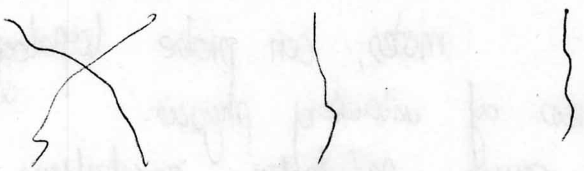
Choosing $\chi = \frac{\Phi}{2\pi} \cdot \theta$, can gauge away A, but introduce multiply valued wavefunction; Introduce non-trivial topology to the system. – missing point at origin.

Local, redundant gauge field $\longleftrightarrow$ Non trivial topology & multivalued wavefunction.

[Exactly analogous to Chern-Simons theories].

Exotic statistics.

Particles x_1^0, x_2^0, x_3^0, ... x_n^0 t_0

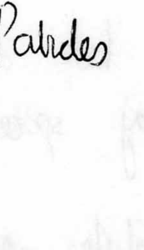


t_1

Assume distinguishable; Find amplitudes to return to original positions @ t_1. $\Rightarrow \sum_{\text{trajectories}} e^{iS_{\text{traj.}}}$

Justification. • Correct classical limit.
(matches other quantisation scheme)

In 2+1 dimensions, trajectory space is not connected

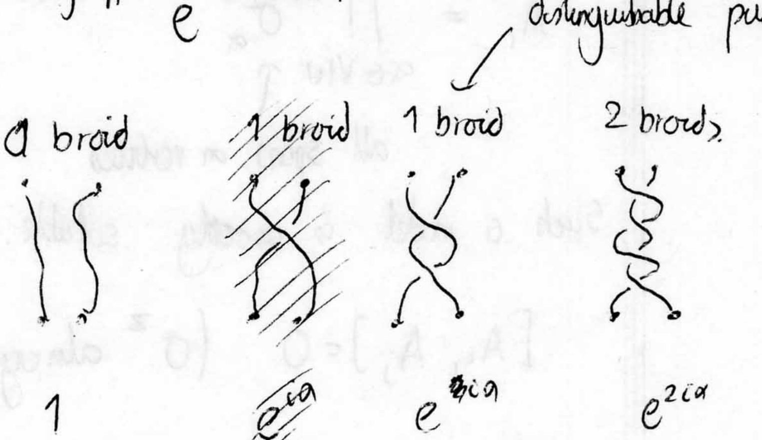


The space divides into different, disconnected sectors, each sector has an arbitrary different phase, extra freedom).

In 3 dimensions, if distinguishable, then connected, but crossings of identical particles cannot be connected.

In 2+1 D, $\sum_{\text{trajectories}} e^{i S_{traj}/\hbar} \, e^{i\theta_{\text{sector}}}$

Consider 2pcl sector:



Consistency condition - require 2 braids has twice the phase of one braid. This is for distinguishable particles. In 3D, two sectors. 1 exchange / 0 exchange, phase ± 1, for bosons/fermions.

Including indistinguishability, phase $e^{i\alpha}$ for one exchange, & $e^{2i\alpha}$ is one braid.

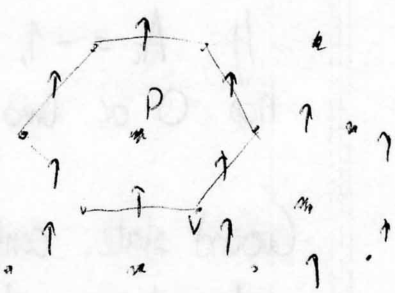
So,
$\alpha = 0$
$\alpha = \pi$
$\alpha \neq 0, \pi$ anyons

NB, if $\alpha \neq 0, \pi$, time reversal symmetry has been broken.

A simple model, Kitaev '97.

System of spins σ_z, σ_x

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



Spins on links of honeycomb lattice

$$H = J_1 \sum_{\text{vertices}, i} A_i - J_2 \sum_{\text{plaquettes}, p} B_p$$

$$A_i = \prod_{\alpha \in V(i)} \sigma^z_\alpha \qquad B_p = \prod_{\alpha \in p} \sigma^x_\alpha$$

all spins on vertices

Such a model is exactly soluble.

$[A_i, A_j] = 0$ (σ^z always commute), $[B_p, B_{p'}] = 0$ (similarly)

$[A_i, B_p] = 0$ (consider only V on plaquette, then

$$[\sigma^z_{\alpha_1}\sigma^z_{\alpha_2}, \sigma^x_{\alpha_1}\sigma^x_{\alpha_2}]$$

$$= \sigma^z_{a_1}[\sigma^z_{a_2}, \sigma^x_{a_2}]\sigma^x_{a_1} + \sigma^z_{a_2}\sigma^x_{a_2}[\sigma^z\sigma^x \quad]$$

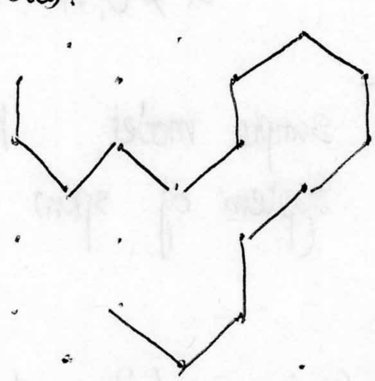
Due to anticommutation of $[\sigma^z, \sigma^x]$, this is 0.

So, simultaneously diagonalise. $A_i = \pm 1$, $B = \pm 1$.

$A_i = -1$, $B_p = +1 \Rightarrow$ all spin down?

[Consider this problem on a square lattice, $J_1 < 0$].

Consider σ_z basis, mark up spin sites.



If $A_i = -1$, each vertex has 0 or two bond,

Ground state contains only configurations that do not terminal.

A line end must be an excitation.

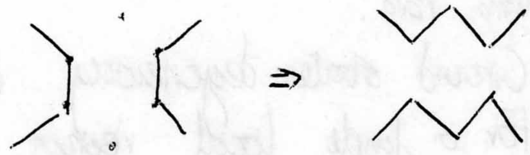
③

Effect of J_2; want to find $B_p = +1$.

But consider flipping spin all spins

On simple configuration

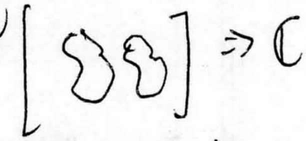
&



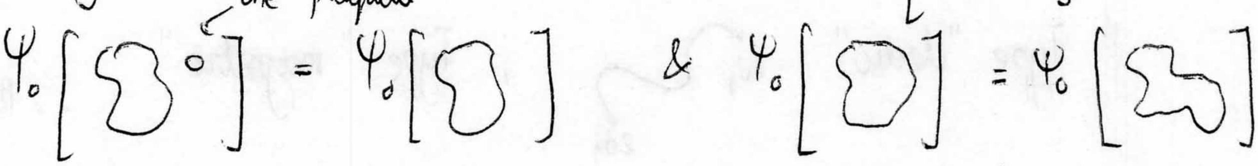
(cut & rejoin

So, since acting on only one plaquette at a time, preserves 1st property – i.e. configuration "safe" ∵ commutation.

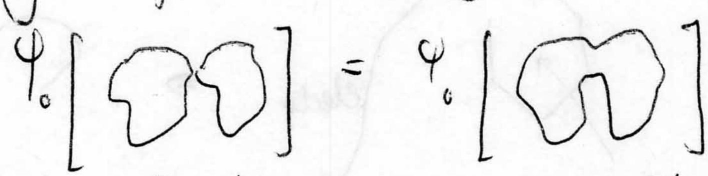
For ground state wavefunction, for map



one plaquette

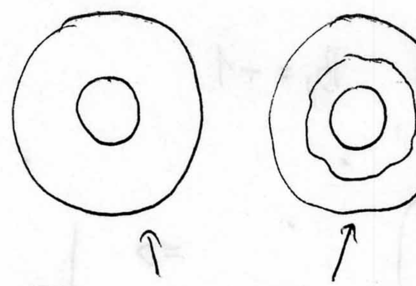


ground state is invariant under adding closed loops, deforming any loop & rejoining



Thus, all loops must exist with equal amplitudes in the ground state.

Introduce topology via boundary conditions.



[picture represents state & all these connected via B_p].

Winding numbers mod 2 only matter.

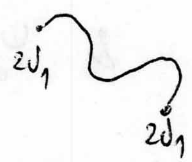
Two degenerate ground states

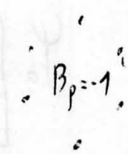
Similarly, on a torus, ~~$\mathbb{Z}_2$~~ $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ ground states.

Such systems have:

- Ground state degeneracy dependent on manifold
- For a finite local region, no way to distinguish the actual ground state.

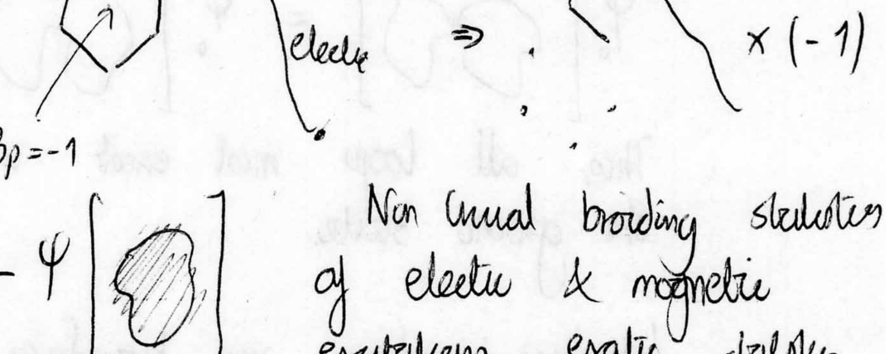
Excitations; gap $\Delta E = 4J_1$ to create a line / break a loop.
similarly, $2J_2$ for $B_p = -1$.

Type "electric" 

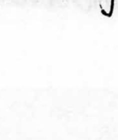
; Type "magnetic" 

(w.f changes sign for a flip)

Joining:



$\alpha_{em} = \pi/2$.

ie, Ψ [] $= -\Psi$ [ ]

Non trivial braiding statistics of electric & magnetic excitations, exotic statistics