Karlsruhe Institute of Technology

Quantum transport in disordered systems: Localization, interaction, symmetries and topologies

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Plan (tentative)

- **I. Disorder and localization**
  - disorder: diagrammatics, quantum interference, localization
  - field theory: non-linear σ-model
  - quasi-1D geometry: exact solution
- **II. Criticality and multifractality**
  - RG, metal-insulator transition, criticality
  - wave function and LDOS multifractality
  - classification of observables, properties of spectra of scaling dimensions
- **III. Symmetries and topologies**
  - symmetry classification of disordered electronic systems
  - topologies; quantum Hall effect; topological insulators
  - disordered Dirac fermions; graphene
- **IV. Interaction**
  - electron-electron-interaction: dephasing and renormalization
  - Anderson transitions in the presence of interaction;
    interplay of multifractality and interaction

Basics of disorder diagrammatics

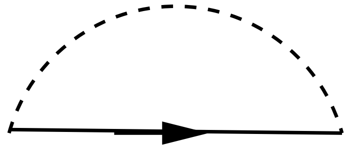
Hamiltonian $$H = H_0 + V(\mathrm{r}) \equiv \frac{(-i\nabla)^2}{2m} + V(\mathrm{r})$$

Free Green function $$G_0^{R,A}(\epsilon, p) = (\epsilon - p^2/2m \pm i0)^{-1}$$

Disorder $$\langle V(\mathrm{r})V(\mathrm{r}')\rangle = W(\mathrm{r}-\mathrm{r}')$$

simplest model: white noise $$W(\mathrm{r}-\mathrm{r}') = \Gamma\delta(\mathrm{r}-\mathrm{r}')$$

self-energy $$\Sigma(\epsilon, p)$$



$$\mathrm{Im}\,\Sigma_R = \Gamma\int(dp)\,\mathrm{Im}\,\frac{1}{\epsilon - p^2/2m + i0} = \pi\nu\Gamma \equiv -\frac{1}{2\tau}\ ,$$

τ – **mean free time**

disorder-averaged Green function $$G(\epsilon, p)$$

$$G^{R,A}(\epsilon, p) = \frac{1}{\epsilon - p^2/2m - \Sigma_{R,A}} \simeq \frac{1}{\epsilon - p^2/2m \pm i/2\tau}$$

$$G^{R,A}(\epsilon, r) \simeq G_0^{R,A}(\epsilon, r)e^{-r/2l}\ ,$$

$l = v_F\tau$ – **mean free path**

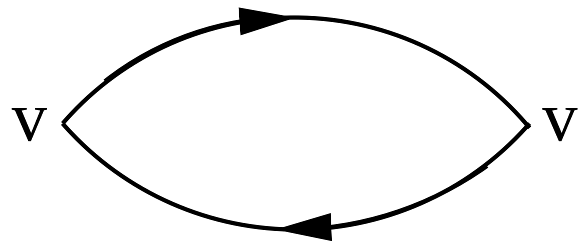
Conductivity

Kubo formula $$\sigma_{\mu\nu}(\omega) = \frac{1}{i\omega}\left\{\frac{i}{\hbar}\int_0^\infty dt \int dr e^{i\omega t}\langle[j_\mu(r,t), j_\nu(0,0)]\rangle - \frac{ne^2}{m}\delta_{\mu\nu}\right\}$$

Non-interacting electrons, $T, \omega \ll \epsilon_F$:

$$\sigma_{xx}(\omega) \simeq \frac{e^2}{2\pi V}\mathrm{Tr}\ \hat{v}_x G^R_{\epsilon+\omega}\hat{v}_x(G^A_\epsilon - G^R_\epsilon) \qquad \epsilon \equiv \epsilon_F$$

Drude conductivity:

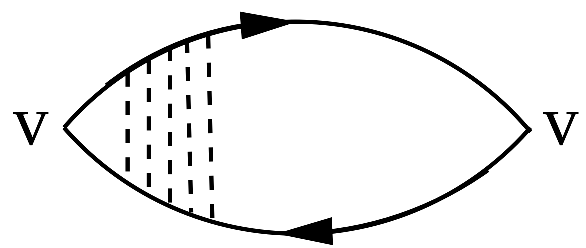


$$\sigma_{xx} = \frac{e^2}{2\pi}\int (dp)\frac{1}{m^2}p_x^2 G^R_{\epsilon+\omega}(p)[G^A_\epsilon(p) - G^R_\epsilon(p)]$$

$$\simeq \frac{e^2}{2\pi}\nu\frac{v_F^2}{d}\int d\xi_p \frac{1}{(\omega - \xi_p + \frac{i}{2\tau})(-\xi_p - \frac{i}{2\tau})} = e^2\frac{\nu v_F^2}{d}\frac{\tau}{1 - i\omega\tau}\ , \qquad \xi_p = \frac{p^2}{2m} - \epsilon$$

Finite-range disorder $\longrightarrow$ **anisotropic scattering**

$\longrightarrow$ **vertex correction** , $\tau \longrightarrow \tau_{\mathrm{tr}}$

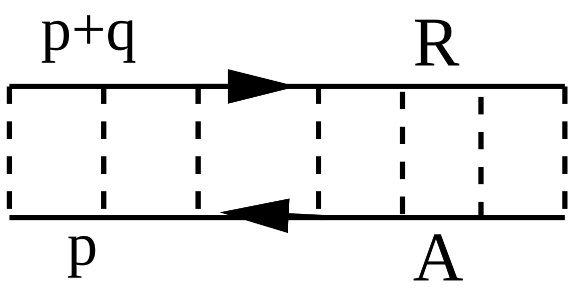


$$\frac{1}{\tau} = \nu\int\frac{d\phi}{2\pi}w(\phi) \qquad \frac{1}{\tau_{\mathrm{tr}}} = \nu\int\frac{d\phi}{2\pi}w(\phi)(1 - \cos\phi)$$

Diffuson and Cooperon

$$\mathcal{D}(q,\omega) = (2\pi\nu\tau)^{-2}\int d(r-r')\langle G^R_\epsilon(r',r)G^A_{\epsilon+\omega}(r,r')\rangle e^{-iq(r-r')}$$

Ladder diagrams (diffuson)

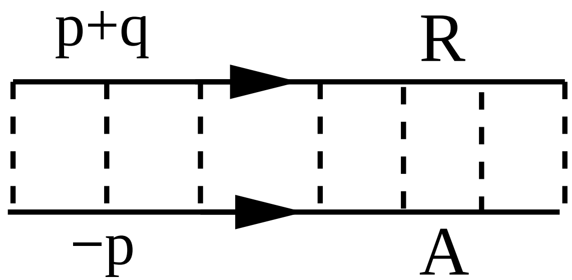


$$\frac{1}{2\pi\nu\tau}\sum_{n=0}^{\infty}\left[\frac{1}{2\pi\nu\tau}\int(dp)G^R_{\epsilon+\omega}(p+q)G^A_\epsilon(p)\right]^n$$

$$\int G^R G^A \simeq \int d\xi_p \frac{d\phi}{2\pi}\frac{1}{(\omega-\xi_p-v_F q\cos\phi+\frac{i}{2\tau})(-\xi_p-\frac{i}{2\tau})} = 2\pi\nu\tau[1-\tau(Dq^2-i\omega)]$$

$$\mathcal{D}(q,\omega) = \frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2-i\omega}$$ diffusion pole $\quad ql,\omega\tau \ll 1$

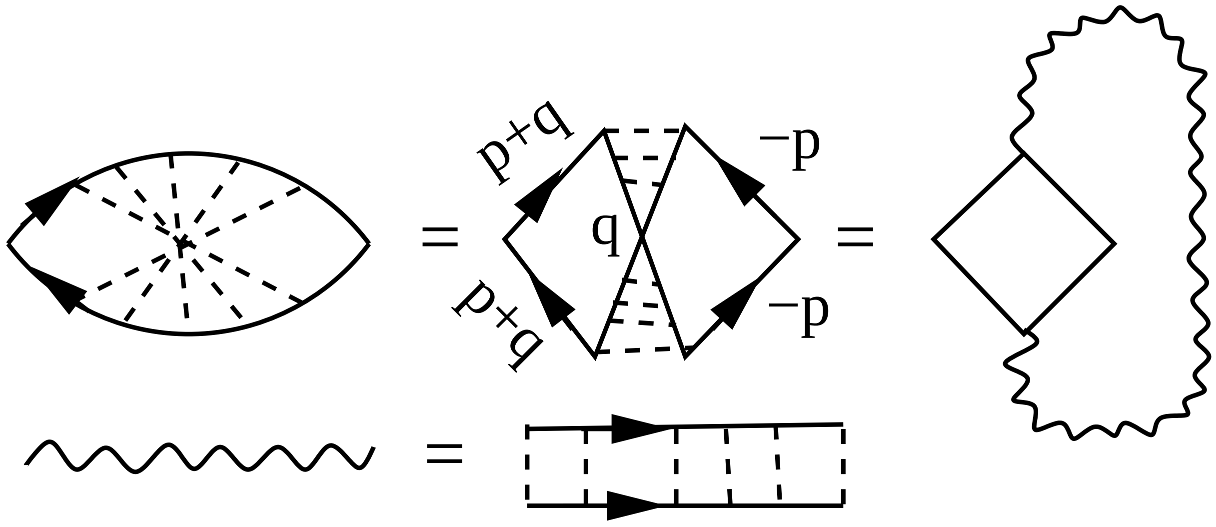
$$\left(\partial/\partial t - D\nabla_r^2\right)\mathcal{D}(r-r',t-t') = 2\pi\nu\delta(r-r')\delta(t-t')$$

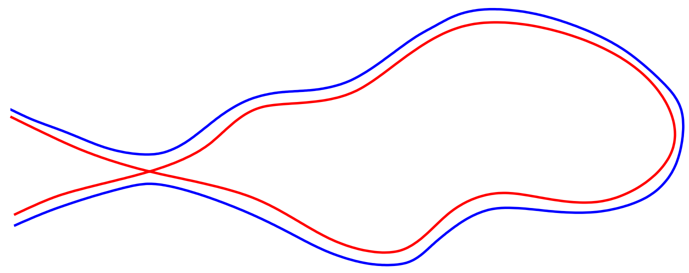


Weak. loc. correction: Cooperon $\mathcal{C}(q,\omega)$

Time-reversal symmetry preserved, no interaction $\longrightarrow$ $\mathcal{C}(q,\omega) = \mathcal{D}(q,\omega)$

Weak localization (orthogonal symmetry class)





Cooperon loop (interference of time-reversed paths)

$$\boldsymbol{\Delta\sigma_{\rm WL} \simeq -\frac{e^2}{2\pi}\frac{v_F^2}{d}\nu \int d\xi_p G_R^2 G_A^2 \int (dq) \frac{1}{2\pi\nu\tau^2}\,\frac{1}{Dq^2 - i\omega} = -\sigma_0 \frac{1}{\pi\nu}\int \frac{(dq)}{Dq^2 - i\omega}}$$

$$\boldsymbol{\Delta\sigma_{\rm WL} = -\frac{e^2}{(2\pi)^2}\left(\frac{\sim 1}{l} - \frac{1}{L_\omega}\right)}\ , \qquad \text{3D} \qquad \boldsymbol{L_\omega = \left(\frac{D}{-i\omega}\right)^{1/2}}$$

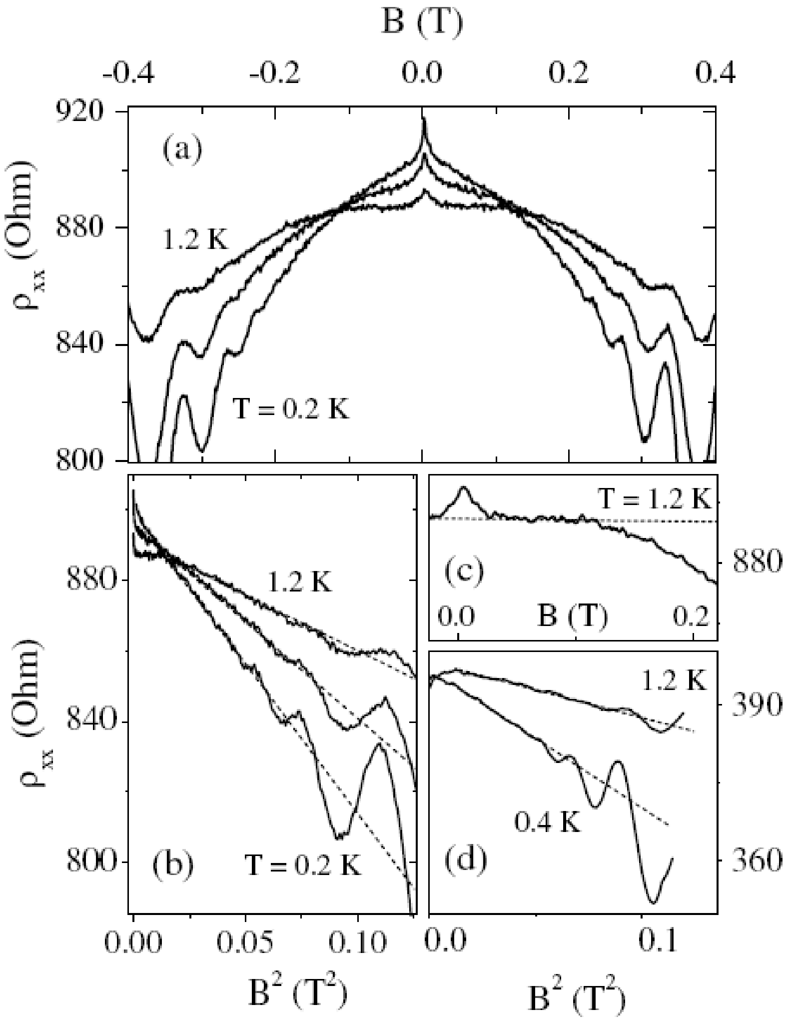
$$\boldsymbol{\Delta\sigma_{\rm WL} = -\frac{e^2}{2\pi^2}\ln\frac{L_\omega}{l}}\ , \qquad \text{2D}$$

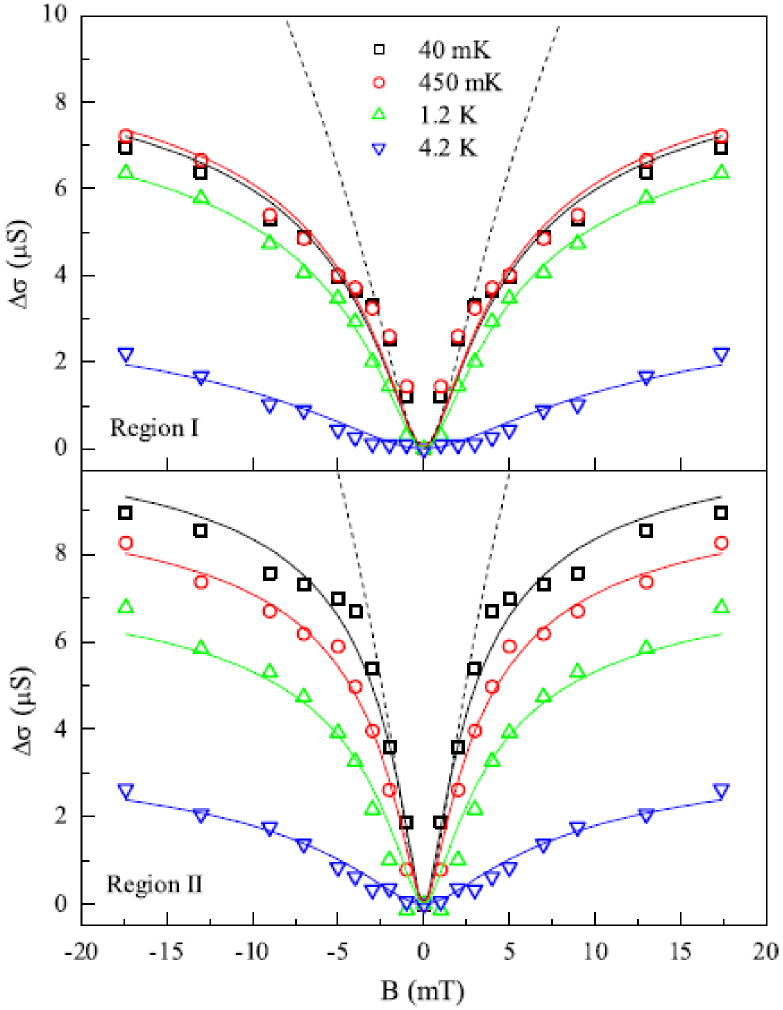
$$\boldsymbol{\Delta\sigma_{\rm WL} = -\frac{e^2}{2\pi}L_\omega}\ , \qquad \text{quasi-1D}$$

Generally: IR cutoff

$$\boldsymbol{L_\omega \longrightarrow \min\{L_\omega, L_\phi, L, L_H\}}$$

Weak localization in experiment: Magnetoresistance





Li et al. (Savchenko group), PRL'03

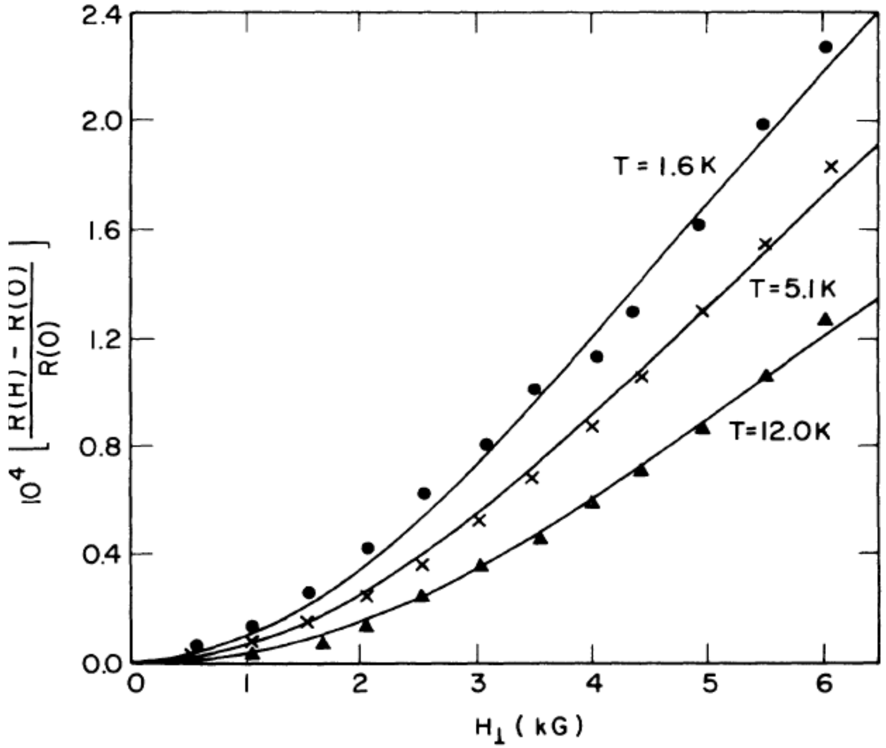
low field: weak localization

all fields: interaction correction

Gorbachev et al. (Savchenko group), PRL'07

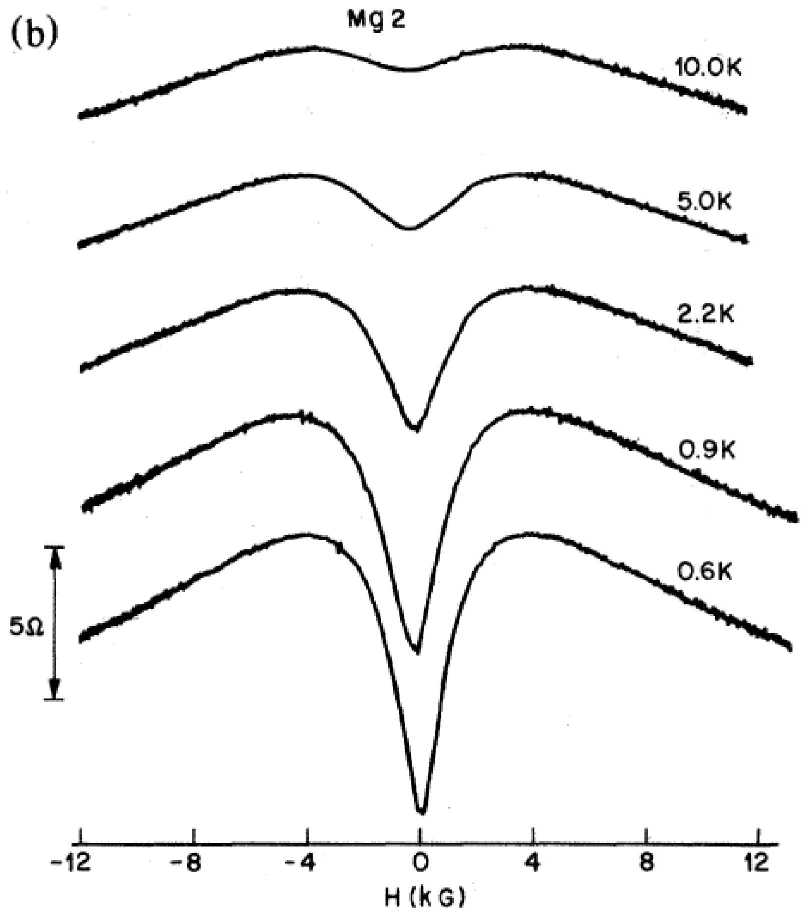
weak localization in bilayer graphene

Weak localization in experiment: Magnetoresistance (cont'd)



Lin, Giordano, PRL'86

Au-Pd wires; weak antilocalization due to strong spin-orbit scattering



White, Dynes, Garno PRB'84

Mg films; weak antilocalization at lowest fields;
weak localization at stronger fields

Altshuler-Aronov-Spivak effect: $\Phi_0/2$ AB oscillations

The Aaronov–Bohm effect in disordered conductors

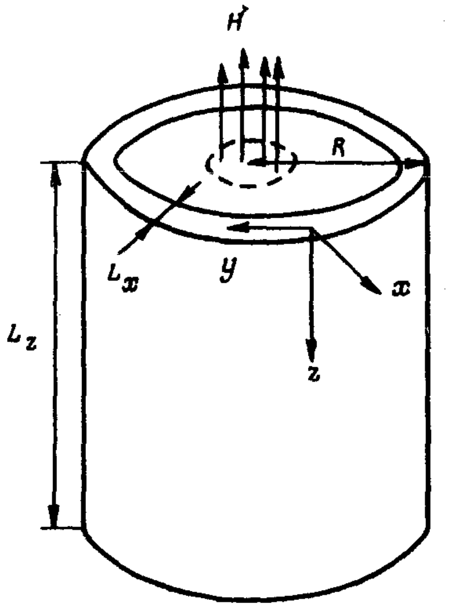
B. L. Al'tshuler, A. G. Aronov, and B. Z. Spivak
B. P. Konstantinov Institute of Nuclear Physics, USSR Academy of Sciences

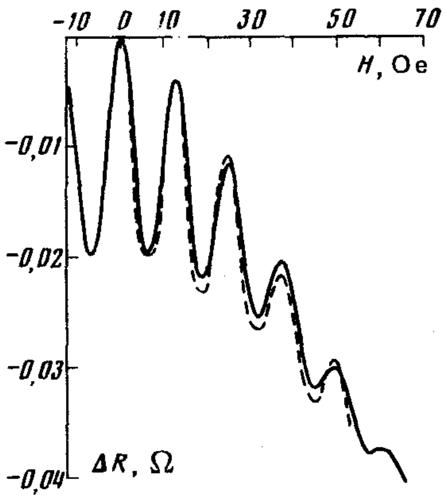
(Submitted 18 November 1980)

Pis'ma Zh. Eksp. Teor. Fiz. **33**, No. 2, 101–103 (20 January 1981)

It is shown that the Aaronov-Bohm effect, which is manifested in the oscillations of the kinetic coefficients as a function of the magnetic flux that penetrates the sample, must exist in disordered normal conductors. The period of these oscillations is $\Phi_0 = hc/2e$, i.e., it is half as large as in the ordinary Aaronov-Bohm effect.

Arkady Aronov (1939-1994)





experimental observation:
Sharvin, Sharvin '81

review: Aronov, Sharvin,
Rev. Mod. Phys.'87

Strong localization

WL correction is IR-divergent in quasi-1D and 2D; becomes $\sim \sigma_0$ at a scale

$$\xi \sim 2\pi\nu D \ , \qquad \text{quasi-1D}$$

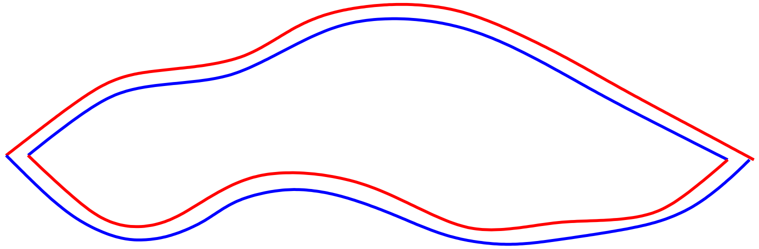
$$\xi \sim l \exp(2\pi^2 \nu D) = l \exp(\pi g) \ , \qquad \text{2D}$$

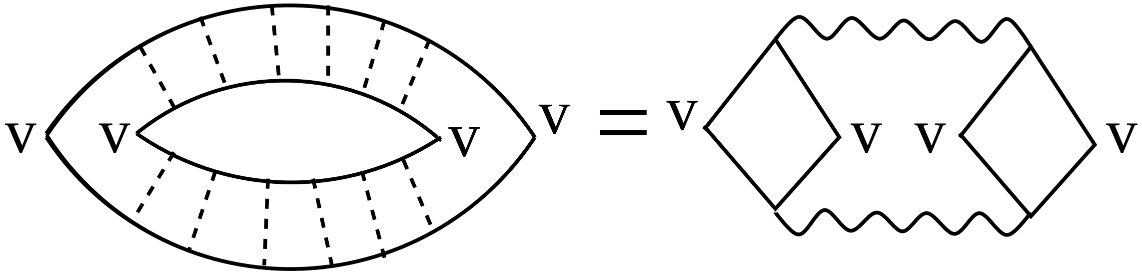
indicates strong localization, ξ – localization length

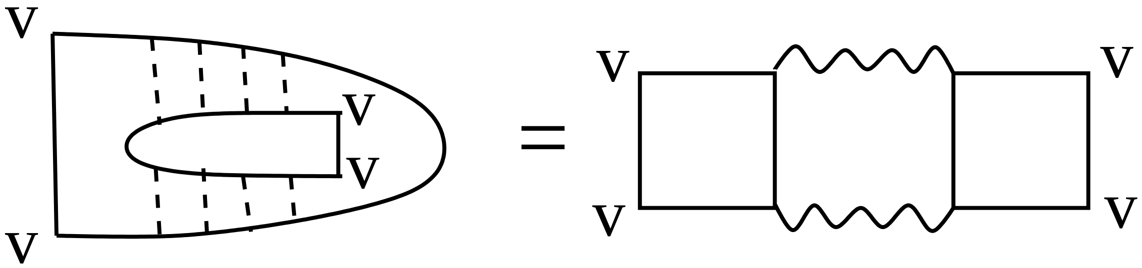
confirmed by exact solution in quasi-1D and renormalization group in 2D

Mesoscopic conductance fluctuations

$$\langle(\delta G)^2\rangle \sim \langle(\textstyle\sum_{i\neq j} A_i^* A_j)^2\rangle \sim \textstyle\sum_{i\neq j}\langle|A_i|^2\rangle\langle|A_j|^2\rangle$$







$$\langle(\delta\sigma)^2\rangle = 3\left(\frac{e^2}{2\pi V}\right)^2 (4\pi\nu\tau^2 D)^2 \sum_{\mathrm{q}} \left(\frac{1}{2\pi\nu\tau^2 D\mathrm{q}^2}\right)^2 = 12\left(\frac{e^2}{2\pi V}\right)^2 \sum_{\mathrm{q}} \left(\frac{1}{\mathrm{q}^2}\right)^2$$

$$\langle(\delta g)^2\rangle = \frac{12}{\pi^4}\sum_{\mathrm{n}}\left(\frac{1}{\mathrm{n}^2}\right)^2 \qquad n_x = 1,2,3,\ldots\ , \qquad n_{y,z} = 0,1,2,\ldots$$

$\langle(\delta g)^2\rangle \sim 1$ independent of system size; depends only on geometry!

$\longrightarrow$ universal conductance fluctuations (UCF)

quasi-1D geometry: $\langle(\delta g)^2\rangle = 8/15$

Mesoscopic conductance fluctuations (cont'd)

Additional comments:

- UCF are anomalously strong from classical point of view:

  $\langle(\delta g)^2\rangle/g^2 \sim L^{4-2d} \gg L^{-d}$

  reason: quantum coherence
- UCF are universal for $L \ll L_T, L_\phi$; otherwise fluctuations suppressed
- symmetry dependent: $8 = 2$ (Cooperons) $\times$ 4 (spin)
- autocorrelation function $\langle\delta g(B)\delta g(B+\delta B)\rangle$; magnetofingerprints
- mesoscopic fluctuations of various observables

From Metal to Insulator

Philip W. Anderson

1958 "Absence of diffusion in certain random lattices"

Disorder-induced localization

$\longrightarrow$ Anderson insulator

$D \leq 2$ all states are localized

$D \geq 3$ metal-insulator transition

Sir Nevill F. Mott

interaction-induced gap

$\longrightarrow$ Mott insulator

The Nobel Prize in Physics 1977

Metal vs Anderson insulator

Localization transition $\longrightarrow$ change in behavior of diffusion propagator,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \langle G^R_{\epsilon+\omega/2}(\mathrm{r}_1, \mathrm{r}_2) G^A_{\epsilon-\omega/2}(\mathrm{r}_2, \mathrm{r}_1)\rangle,$$

Delocalized regime: Π has the diffusion form:

$$\Pi(\mathrm{q}, \omega) = 2\pi\nu(\epsilon)/(Dq^2 - i\omega),$$

Insulating phase: propagator ceases to have Goldstone form, becomes massive,

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) \simeq \frac{2\pi\nu(\epsilon)}{-i\omega} \mathcal{F}(|\mathrm{r}_1 - \mathrm{r}_2|/\xi),$$

$\mathcal{F}(\mathrm{r})$ decays on the scale of the localization length, $\mathcal{F}(r/\xi) \sim \exp(-r/\xi)$.

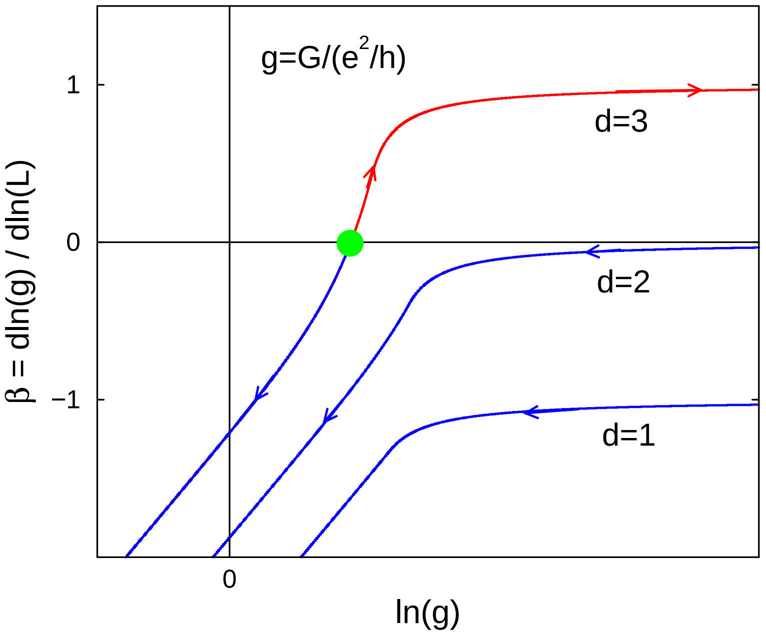
Comment:

Localization length ξ obtained from the averaged correlation function $\Pi = \langle G^R G^A \rangle$ is in general different from the one governing the exponential decay of the typical value $\Pi_{\mathrm{typ}} = \exp\langle \ln G^R G^A \rangle$.

E.g., in quasi-1D systems: $\xi_{\mathrm{av}} = 4\xi_{\mathrm{typ}}$

This is usually not important for the definition of the critical index ν.

Anderson transition

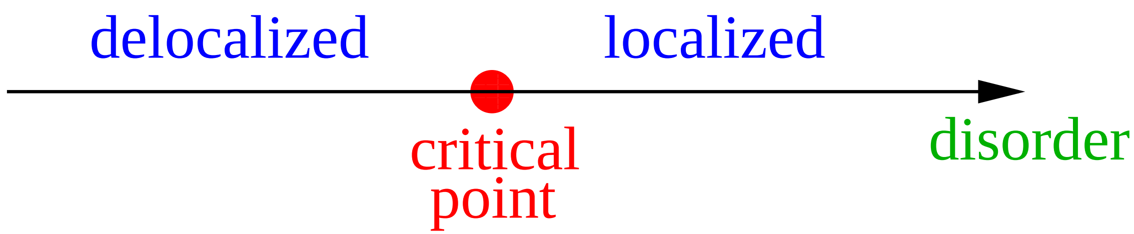


Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

Modern approach:
RG for field theory (σ-model)

quasi-1D, 2D : metallic $\rightarrow$ localized crossover with decreasing g

$d > 2$: Anderson metal-insulator transition (sometimes also in $d = 2$)



Continuous phase transition with highly unconventional properties!

review: Evers, ADM, Rev. Mod. Phys.'08

Field theory: non-linear σ-model

$$S[Q] = \frac{\pi\nu}{4} \int d^d\mathrm{r}\, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(\mathrm{r}) = 1$$

Wegner'79 (replicas); Efetov'83 (supersymmetry)

(non-equilibrium: Keldysh σ-model, will not discuss here)

σ-model manifold:

- unitary class:
  - fermionic replicas: $\mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - bosonic replicas: $\mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $n \to 0$
  - supersymmetry: $\mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$
- orthogonal class:
  - fermionic replicas: $\mathrm{Sp}(4n)/\mathrm{Sp}(2n) \times \mathrm{Sp}(2n)$, $n \to 0$
  - bosonic replicas: $\mathrm{O}(2n,2n)/\mathrm{O}(2n) \times \mathrm{O}(2n)$, $n \to 0$
  - supersymmetry: $\mathrm{OSp}(2,2|4)/\mathrm{OSp}(2|2) \times \mathrm{OSp}(2|2)$

in general, in supersymmetry:

$Q \in \{$"sphere" $\times$ "hyperboloid"$\}$ "dressed" by anticommuting variables

Non-linear σ-model: Sketch of derivation

Consider unitary class for simplicity

- introduce supervector field $\Phi = (S_1, \chi_1, S_2, \chi_2)$:

$$G^R_{E+\omega/2}(\mathrm{r}_1, \mathrm{r}_2) G^A_{E-\omega/2}(\mathrm{r}_2, \mathrm{r}_1) = \int D\Phi\, D\Phi^\dagger S_1(\mathrm{r}_1) S_1^*(\mathrm{r}_2) S_2(\mathrm{r}_2) S_2^*(\mathrm{r}_1)$$
$$\times \exp\left\{ i \int d\mathrm{r} \Phi^\dagger(\mathrm{r}) [(E - \hat{H})\Lambda + \frac{\omega}{2} + i\eta] \Phi(\mathrm{r}) \right\},$$

where $\Lambda = \mathrm{diag}\{1, 1, -1, -1\}$. No denominator! $Z = 1$

- disorder averaging $\longrightarrow$ quartic term $(\Phi^\dagger \Phi)^2$
- Hubbard-Stratonovich transformation:

quartic term decoupled by a Gaussian integral over a 4×4 supermatrix variable $\mathcal{R}_{\mu\nu}(\mathrm{r})$ conjugate to the tensor product $\Phi_\mu(\mathrm{r}) \Phi^\dagger_\nu(\mathrm{r})$

- integrate out Φ fields $\longrightarrow$ action in terms of the $\mathcal{R}$ fields:

$$S[\mathcal{R}] = \pi\nu\tau \int d^d\mathrm{r}\, \mathrm{Str}\mathcal{R}^2 + \mathrm{Str}\ln[E + (\frac{\omega}{2} + i\eta)\Lambda - \hat{H}_0 - \mathcal{R}]$$

- saddle-point approximation $\longrightarrow$ equation for $\mathcal{R}$:

$$\mathcal{R}(\mathrm{r}) = (2\pi\nu\tau)^{-1} \langle \mathrm{r} | (E - \hat{H}_0 - \mathcal{R})^{-1} | \mathrm{r} \rangle$$

Non-linear σ-model: Sketch of derivation (cont'd)

The relevant set of the solutions (the saddle-point manifold) has the form:

$$\mathcal{R} = \Sigma \cdot I - (i/2\tau)Q \ , \qquad Q = T^{-1}\Lambda T \ , \qquad Q^2 = 1$$

Q – 4×4 supermatrix on the σ-model target space

- gradient expansion with a slowly varying $Q(\mathrm{r})$ $\longrightarrow$

$$\Pi(\mathrm{r}_1, \mathrm{r}_2; \omega) = \int DQ\, Q_{12}^{bb}(\mathrm{r}_1) Q_{21}^{bb}(\mathrm{r}_2) e^{-S[Q]},$$

where $S[Q]$ is the σ-model action

$$S[Q] = \frac{\pi\nu}{4} \int d^d\mathrm{r}\, \mathrm{Str}\, [-D(\nabla Q)^2 - 2i\omega\Lambda Q],$$

- size of Q-matrix: $4 = 2$ (Adv.–Ret.) $\times$ 2 (Bose–Fermi)
- orthogonal & symplectic classes (preserved time-reversal)

$\longrightarrow$ $8 = 2$ (Adv.–Ret.) $\times$ 2 (Bose–Fermi) $\times$ 2 (Diff.-Coop.)

- product of N retarded and N advanced Green functions

$\longrightarrow$ σ-model defined on a larger manifold, with the base being a product of $\mathrm{U}(N,N)/\mathrm{U}(N) \times \mathrm{U}(N)$ and $\mathrm{U}(2N)/\mathrm{U}(N) \times \mathrm{U}(N)$

σ model: Perturbative treatment

For comparison, consider a **ferromagnet** model in an external magnetic field:

$$H[\mathrm{S}] = \int d^d r \left[\frac{\kappa}{2}(\nabla \mathrm{S}(r))^2 - \mathrm{BS}(r)\right] , \qquad \mathrm{S}^2(r) = 1$$

n-component vector σ-model. **Target manifold:** sphere $\mathrm{S}^{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$

Independent degrees of freedom: transverse part $\mathrm{S}_\perp$; $\quad S_1 = (1 - \mathrm{S}_\perp^2)^{1/2}$

$$H[\mathrm{S}_\perp] = \frac{1}{2}\int d^d r \ [\kappa[\nabla \mathrm{S}_\perp(r)]^2 + B\mathrm{S}_\perp^2(r) + O(\mathrm{S}_\perp^4(r))]$$

Ferromagnetic phase: symmetry is broken; **Goldstone modes** – spin waves:

$$\langle \mathrm{S}_\perp \mathrm{S}_\perp \rangle_q \propto \frac{1}{\kappa q^2 + B}$$

$$Q = \left(1 - \frac{W}{2}\right)\Lambda\left(1 - \frac{W}{2}\right)^{-1} = \Lambda\left(1 + W + \frac{W^2}{2} + \dots\right) ; \qquad W = \begin{pmatrix} 0 & W_{12} \\ W_{21} & 0 \end{pmatrix}$$

$$S[W] = \frac{\pi\nu}{4}\int d^d r \, \mathrm{Str}[D(\nabla W)^2 - i\omega W^2 + O(W^3)]$$

theory of "interacting" diffusion modes. Goldstone mode: diffusion propagator

$$\langle W_{12} W_{21} \rangle_{q,\omega} \sim \frac{1}{\pi\nu(Dq^2 - i\omega)}$$

σ-models: What are they good for?

- reproduce diffuson-cooperon diagrammatics . . .

  . . . and go beyond it:

- metallic samples ($g \gg 1$):

  level & wavefunction statistics: random matrix theory + deviations

- quasi-1D samples:

  exact solution, crossover from weak to strong localization

- Anderson transitions: RG treatment, phase diagrams, critical exponents

- σ-models with non-trivial topologies: Dirac fermions, topological insulators

- non-trivial saddle-points:

  nonperturbative effects, asymptotic tails of distributions

- generalizations: interaction, non-equilibrium (Keldysh)

Quasi-1D geometry: Exact solution of the σ-model

quasi-1D geometry (many-channel wire) $\longrightarrow$ 1D σ-model

$\longrightarrow$ "quantum mechanics" , longitudinal coordinate – (imaginary) "time"

$\longrightarrow$ "Schroedinger equation" of the type $\partial_t W = \Delta_Q W$, $t = x/\xi$

- Localization length, diffusion propagator Efetov, Larkin '83
- Exact solution for the statistics of eigenfunctions Fyodorov, ADM '92-94
- Exact $\langle g\rangle(L/\xi)$ and $\mathrm{var}(g)(L/\xi)$ Zirnbauer, ADM, Müller-Groeling '92-94

e.g. for orthogonal symmetry class:

$$\langle g^n\rangle(L) = \frac{\pi}{2}\int_0^\infty d\lambda \tanh^2(\pi\lambda/2)(\lambda^2+1)^{-1} p_n(1,\lambda,\lambda)\exp\left[-\frac{L}{2\xi}(1+\lambda^2)\right]$$

$$+\ 2^4 \sum_{l\in 2N+1}\int_0^\infty d\lambda_1 d\lambda_2 l(l^2-1)\lambda_1\tanh(\pi\lambda_1/2)\lambda_2\tanh(\pi\lambda_2/2)$$

$$\times\, p_n(l,\lambda_1,\lambda_2)\prod_{\sigma,\sigma_1,\sigma_2=\pm 1}(-1+\sigma l + i\sigma_1\lambda_1 + i\sigma_2\lambda_2)^{-1}\exp\left[-\frac{L}{4\xi}(l^2+\lambda_1^2+\lambda_2^2+1)\right]$$

$$p_1(l,\lambda_1,\lambda_2) = l^2+\lambda_1^2+\lambda_2^2+1,$$

$$p_2(l,\lambda_1,\lambda_2) = \frac{1}{2}(\lambda_1^4+\lambda_2^4+2l^4+3l^2(\lambda_1^2+\lambda_2^2)+2l^2-\lambda_1^2-\lambda_2^2-2)$$

Quasi-1D geometry: Exact solution of the σ-model (cont'd)

$L \ll \xi$ **asymptotics:**

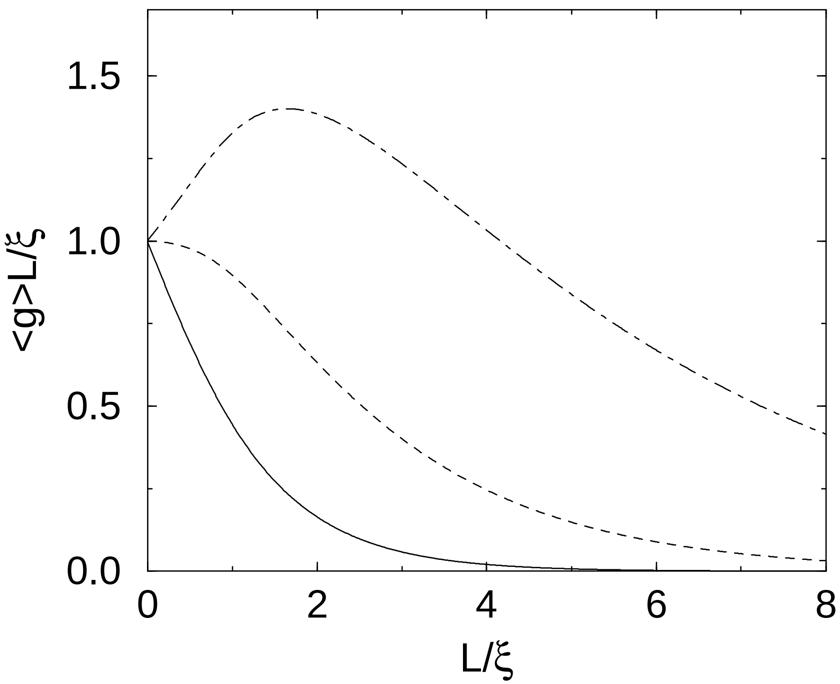
$$\langle g\rangle(L) = \frac{2\xi}{L} - \frac{2}{3} + \frac{2}{45}\frac{L}{\xi} + \frac{4}{945}\left(\frac{L}{\xi}\right)^2 + O\left(\frac{L}{\xi}\right)^3$$

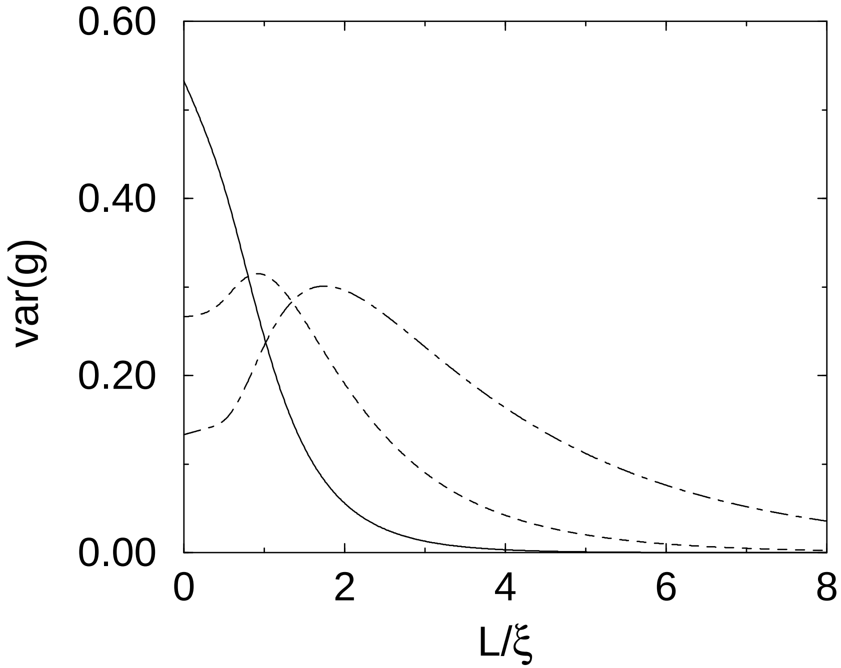
and

$$\mathrm{var}(g(L)) = \frac{8}{15} - \frac{32}{315}\frac{L}{\xi} + O\left(\frac{L}{\xi}\right)^2 .$$

$L \gg \xi$ **asymptotics:**

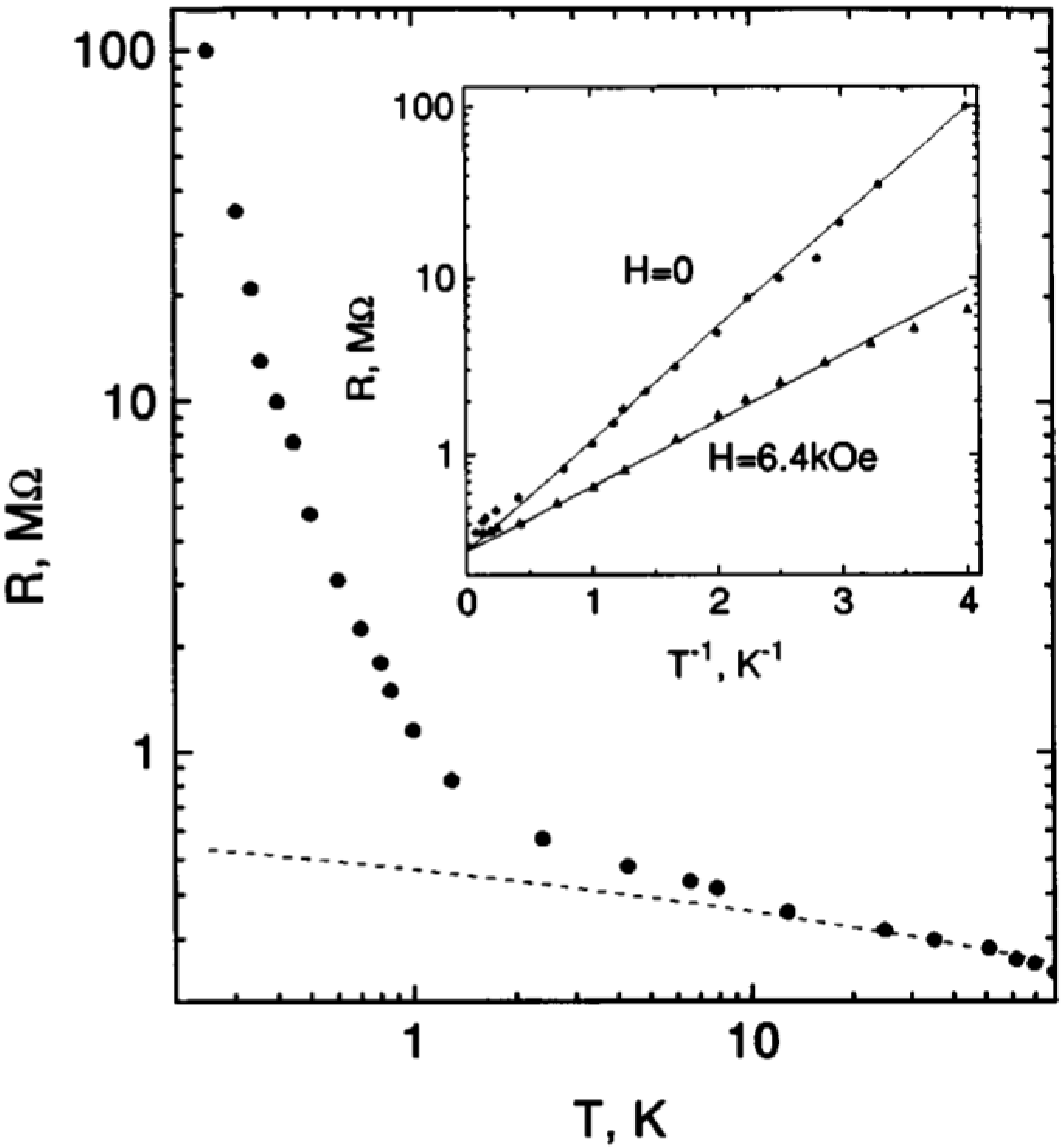
$$\langle g^n\rangle = 2^{-3/2-n}\pi^{7/2}(\xi/L)^{3/2}e^{-L/2\xi}$$





orthogonal (full), unitary (dashed), symplectic (dot-dashed)

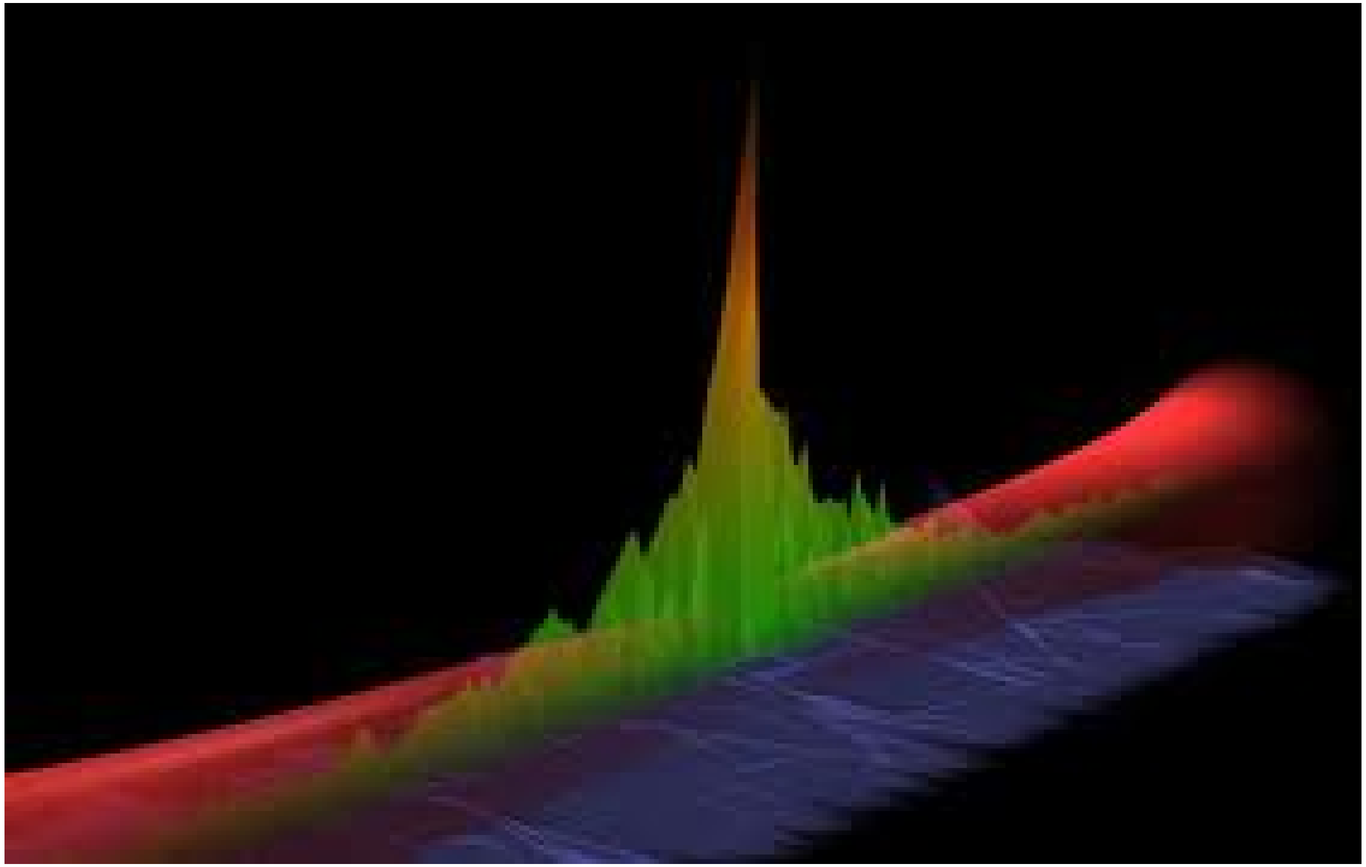
From weak to strong localization of electrons in wires

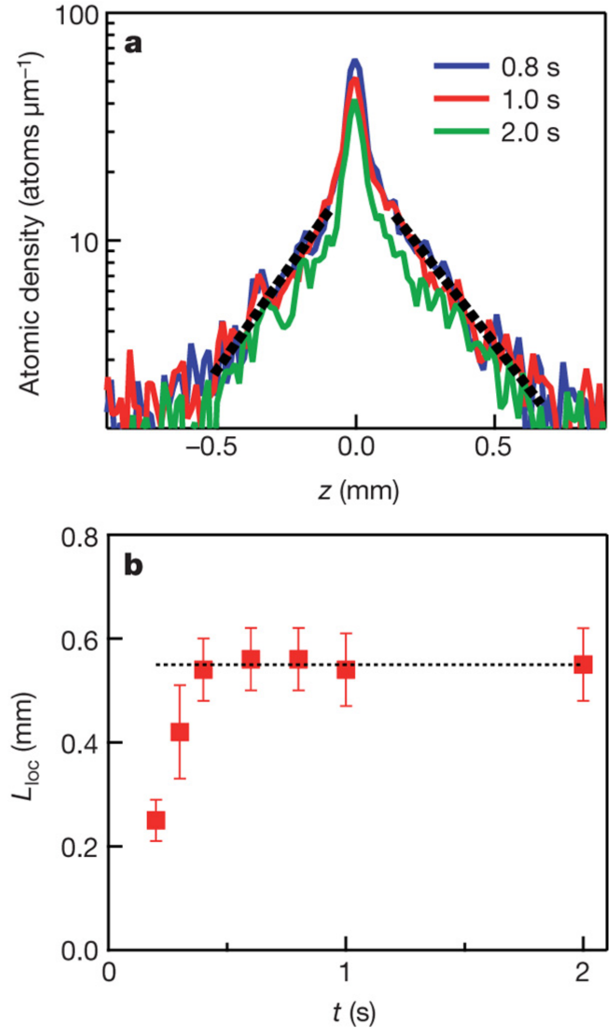


GaAs wires

Gershenson et al, PRL 97

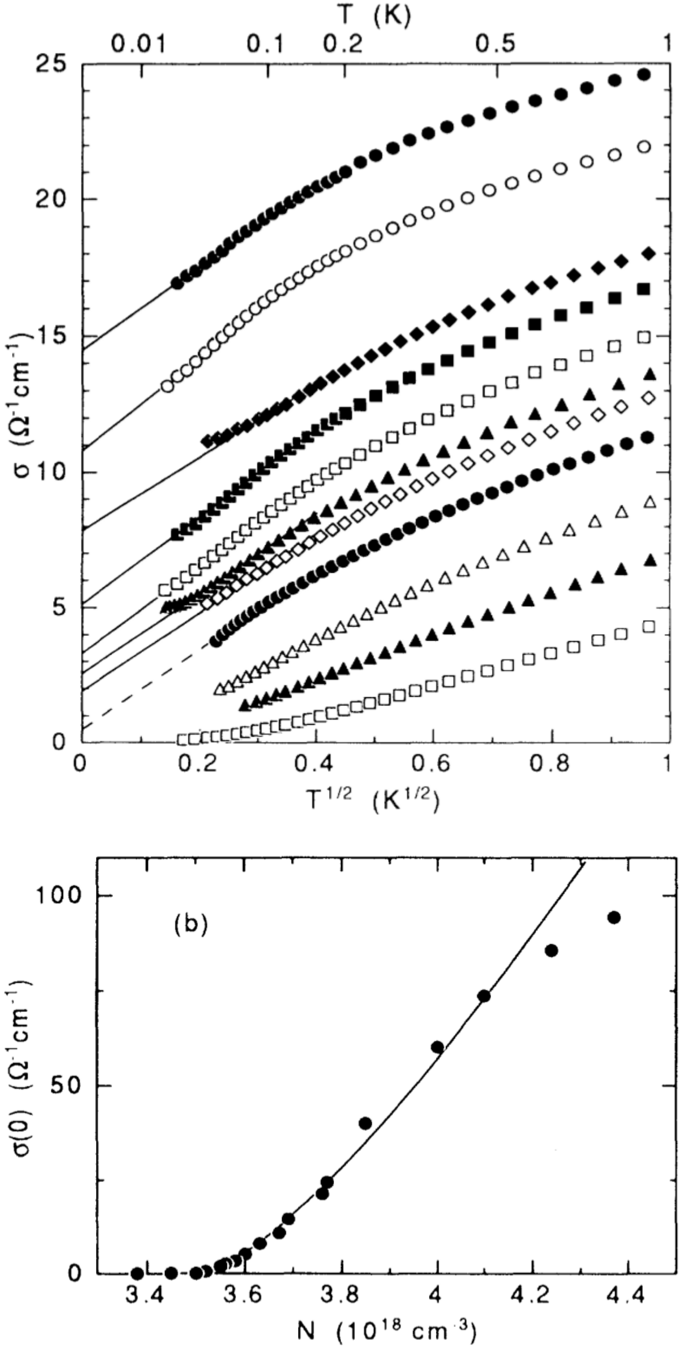
Anderson localization of atomic Bose-Einstein condensate in 1D

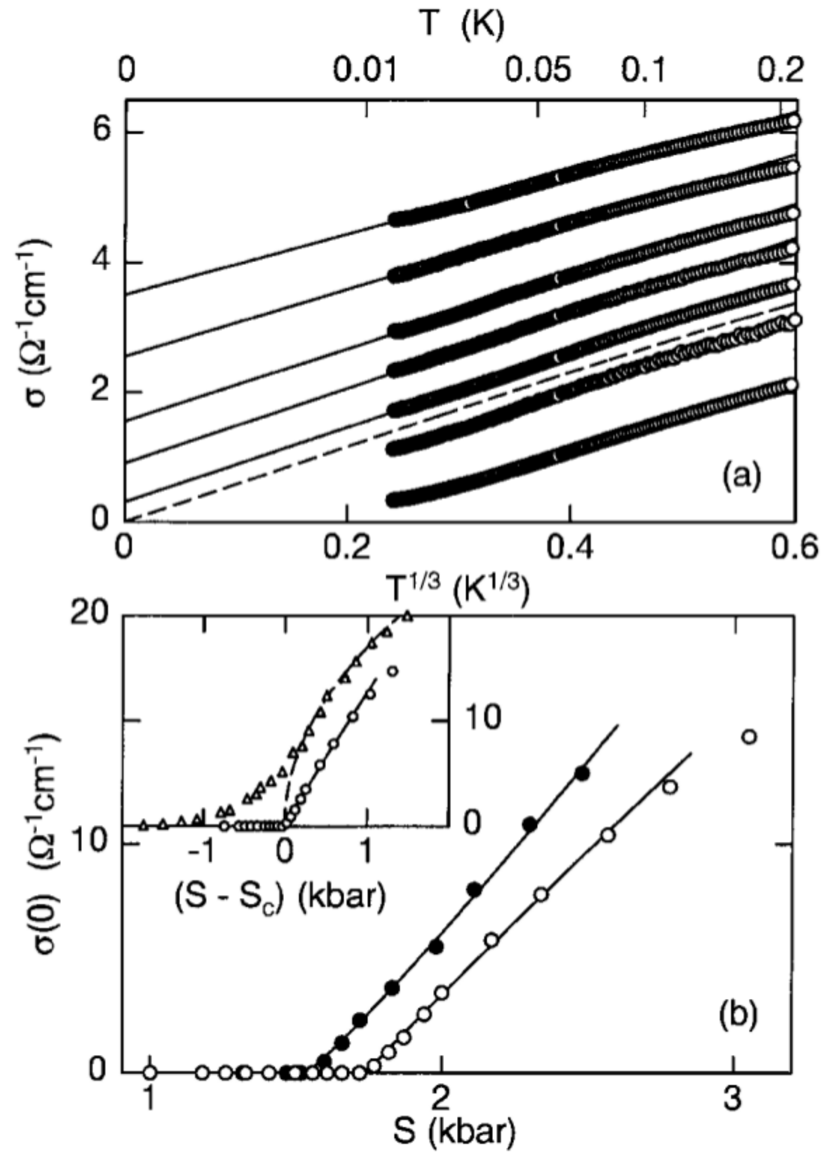




Billy et al (Aspect group), Nature 2008

3D Anderson localization transition in Si:P





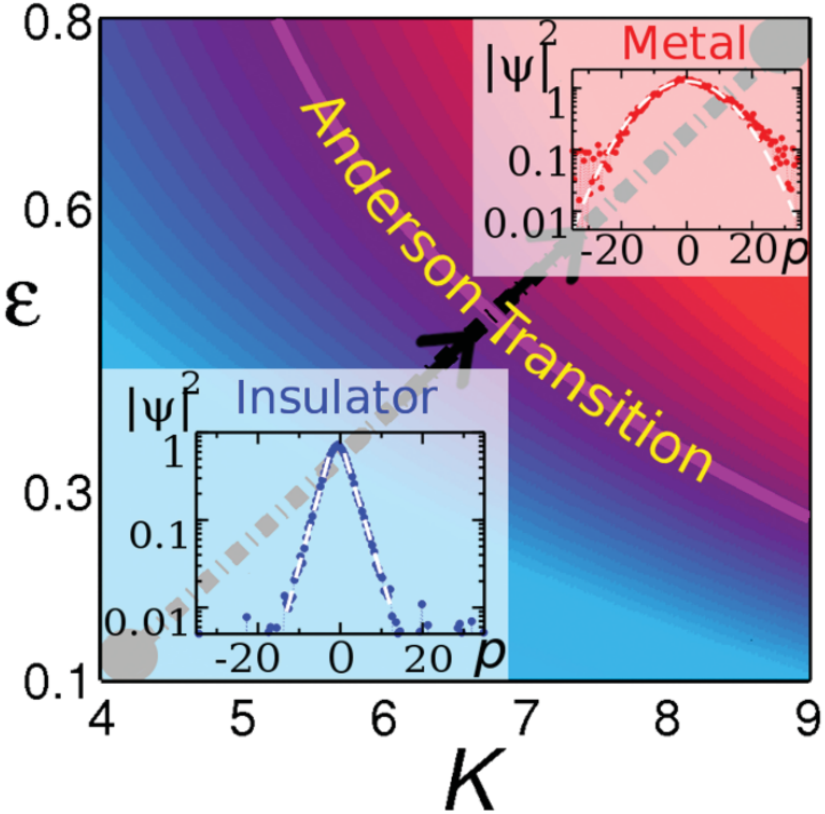
Stupp et al, PRL'93; Wafenschmidt et al, PRL'97 (von Löhneysen group)

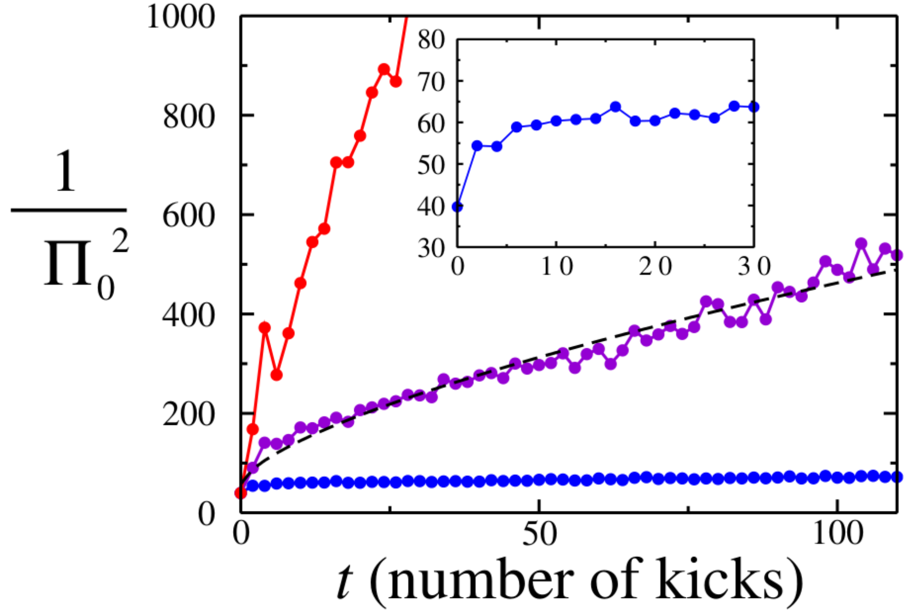
3D Anderson localization in atomic "kicked rotor"

kicked rotor $$H = \frac{p^2}{2} + K \cos x [1 + \epsilon \cos \omega_2 t \cos \omega_3 t] \sum_n \delta(t - 2\pi n/\omega_1)$$

Anderson localization in momentum space. Three frequencies mimic 3D !

Experimental realization: cesium atoms exposed to a pulsed laser beam.





Chabé et al, PRL'08

Renormalization group and ϵ–expansion

analytical treatment of Anderson transition:

RG and ϵ-expansion for $d = 2 + \epsilon$ dimensions

β-function $\quad \beta(t) = -\frac{dt}{d\ln L}; \quad t = 1/2\pi g \;, \qquad g$ – dimensionless conductance

orthogonal class (preserved spin and time reversal symmetry):

$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + O(t^6) \qquad \text{beta-function}$$

$$t_* = \frac{\epsilon}{2} - \frac{3}{8}\zeta(3)\epsilon^4 + O(\epsilon^5) \qquad \text{transition point}$$

$$\nu = -1/\beta'(t_*) = \epsilon^{-1} - \frac{9}{4}\zeta(3)\epsilon^2 + O(\epsilon^3) \qquad \text{localization length exponent}$$

$$s = \nu\epsilon = 1 - \frac{9}{4}\zeta(3)\epsilon^3 + O(\epsilon^4) \qquad \text{conductivity exponent}$$

Numerics for 3D: $\quad \nu \simeq 1.57 \pm 0.02 \qquad$ Slevin, Othsuki '99

RG for σ-models of all Wigner-Dyson classes

- orthogonal symmetry class (preserved T and S): $t = 1/2\pi g$

$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + O(t^6)\ ; \qquad t_* = \frac{\epsilon}{2} - \frac{3}{8}\zeta(3)\epsilon^4 + O(\epsilon^5)$$

$$\nu = -1/\beta'(t_*) = \epsilon^{-1} - \frac{9}{4}\zeta(3)\epsilon^2 + O(\epsilon^3)\ ; \qquad s = \nu\epsilon = 1 - \frac{9}{4}\zeta(3)\epsilon^3 + O(\epsilon^4)$$

- unitary class (broken T):

$$\beta(t) = \epsilon t - 2t^3 - 6t^5 + O(t^7)\ ; \qquad t_* = \left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3}{2}\left(\frac{\epsilon}{2}\right)^{3/2} + O(\epsilon^{5/2});$$

$$\nu = \frac{1}{2\epsilon} - \frac{3}{4} + O(\epsilon)\ ; \qquad s = \frac{1}{2} - \frac{3}{4}\epsilon + O(\epsilon^2).$$

- symplectic class (preserved T, broken S):

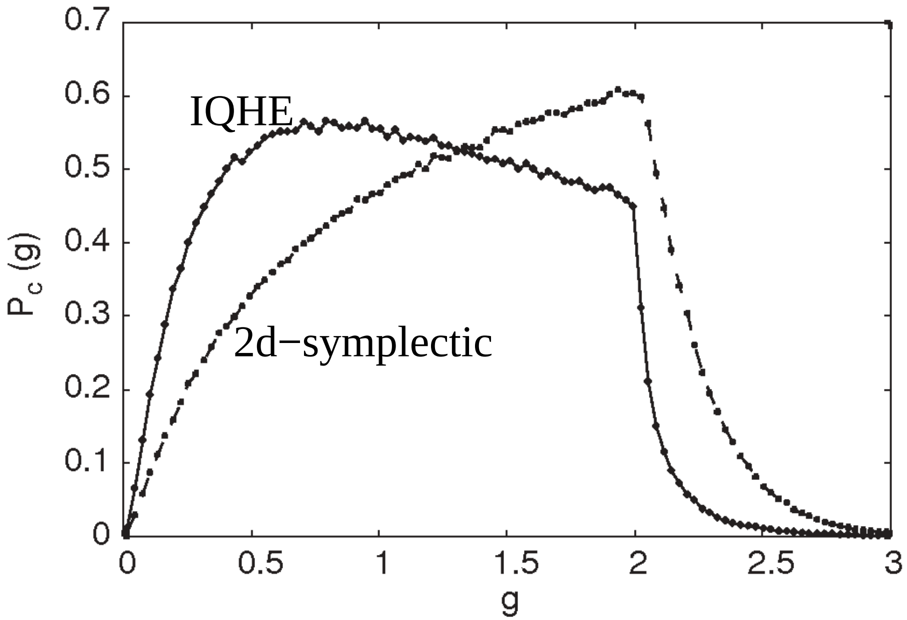
$$\beta(t) = \epsilon t + t^2 - \frac{3}{4}\zeta(3)t^5 + O(t^6)$$

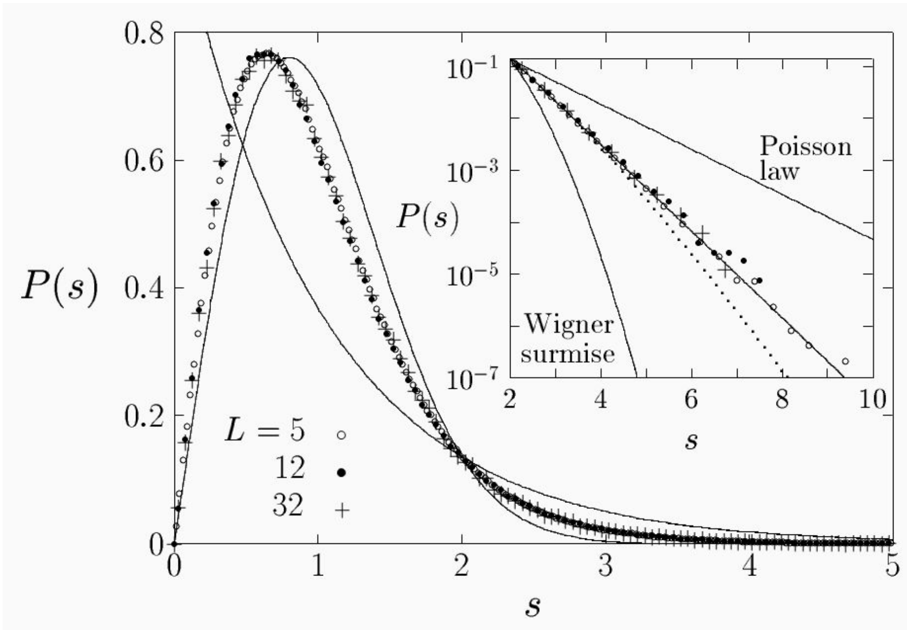
$\longrightarrow$ metal insulator transition in 2D at $t_* \sim 1$

Manifestations of criticality

$\xi \propto (E_c - E)^{-\nu}$ localization length (insulating side)

$\sigma \propto (E - E_c)^{s}$ conductivity (metallic side)

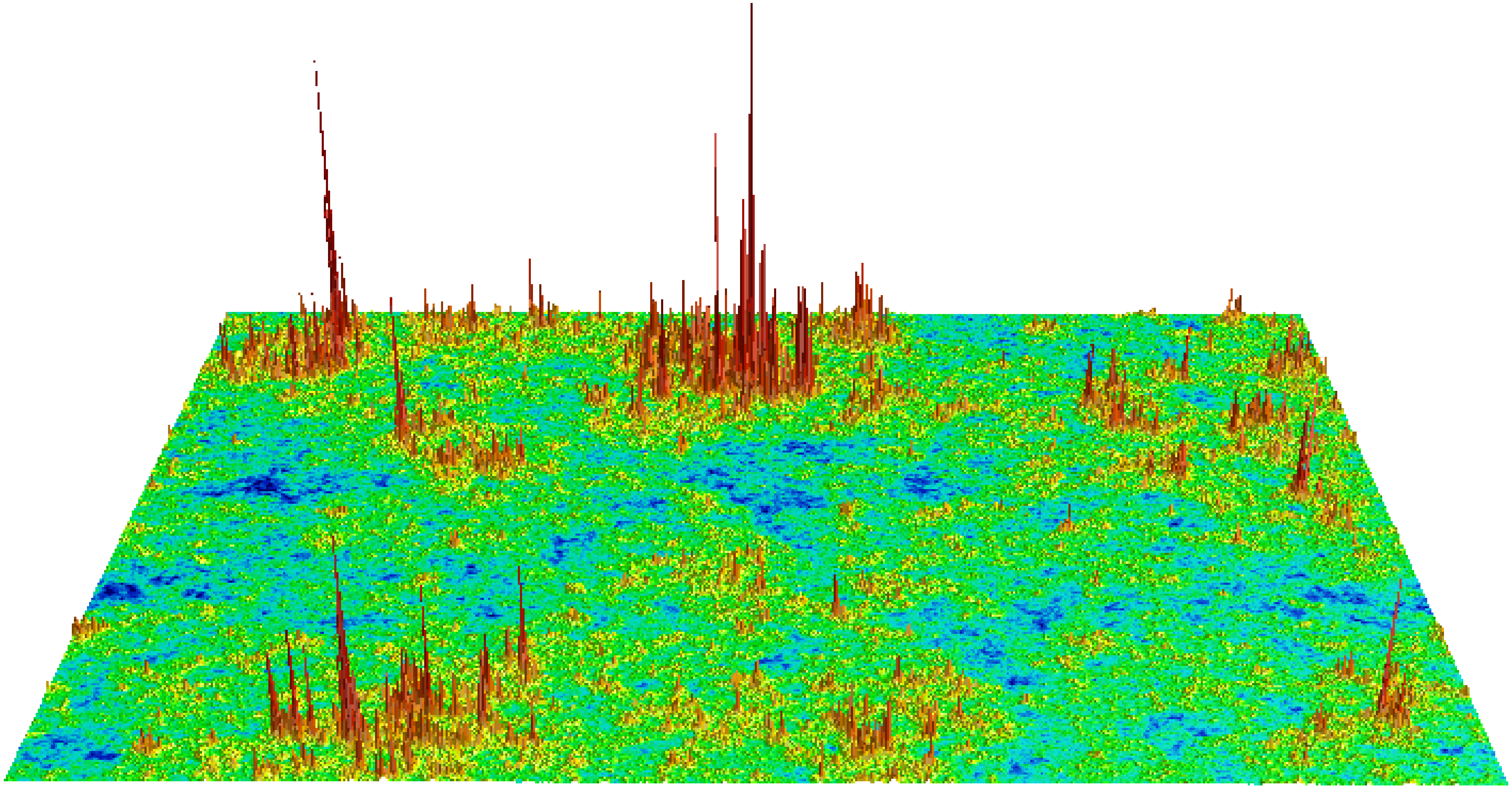




scale-invariant conductance distribution

scale-invariant level statistics

Critical wafe functions: Multifractality



Multifractality at the Anderson transition

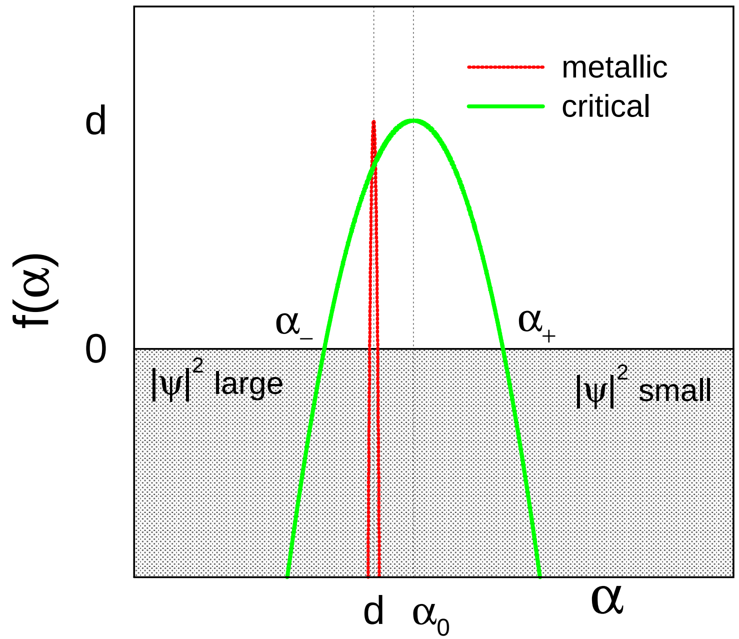
$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$$\tau_q = \underbrace{d(q-1)}_{\text{normal}} + \underbrace{\Delta_q}_{\text{anomalous}} \equiv D_q(q-1) \quad \text{multifractality}$$

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$



wave function statistics:

$$\mathcal{P}(\ln|\psi^2|) \sim L^{-d+f(\ln|\psi^2|/\ln L)}$$

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Multifractality (cont'd)

- Multifractality implies very broad distribution of observables characterizing wave functions. For example, parabolic $f(\alpha)$ implies log-normal distribution

$$\mathcal{P}(|\psi^2|) \propto \exp\{-\# \ \ln^2|\psi^2|/\ln L\}$$

- field theory language: Δ_q – scaling dimensions of operators $\mathcal{O}^{(q)} \sim (Q\Lambda)^q$ Wegner '80

- Infinitely many operators with negative scaling dimensions, i.e. RG relevant (increasing under renormalization)

- 2-, 3-, 4-, ...-point wave function correlations at criticality

$$\langle|\psi_i^2(r_1)||\psi_j^2(r_2)|\ldots\rangle$$

also show power-law scaling controlled by multifractality

- boundary multifractality

Subramaniam, Gruzberg, Ludwig, Evers, Mildenberger, ADM, PRL'06

Dimensionality dependence of multifractality

RG in $2+\epsilon$ dimensions, 4 loops, orthogonal and unitary symmetry classes

Wegner '87

$$\Delta_q^{(O)} = q(1-q)\epsilon + \frac{\zeta(3)}{4} q(q-1)(q^2-q+1)\epsilon^4 + O(\epsilon^5)$$

$$\Delta_q^{(U)} = q(1-q)(\epsilon/2)^{1/2} - \frac{3}{8} q^2(q-1)^2 \zeta(3)\epsilon^2 + O(\epsilon^{5/2})$$

$\epsilon \ll 1 \quad \longrightarrow \quad$ weak multifractality

$\longrightarrow$ keep leading (one-loop) term $\longrightarrow$ parabolic approximation

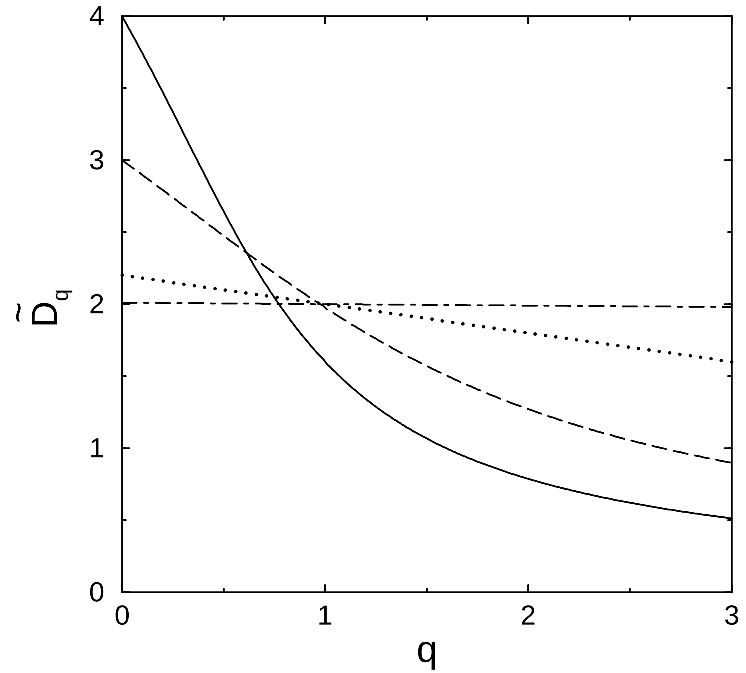
$$\tau_q \simeq d(q-1) - \gamma q(q-1), \qquad \Delta_q \simeq \gamma q(1-q)\ , \qquad \gamma \ll 1$$

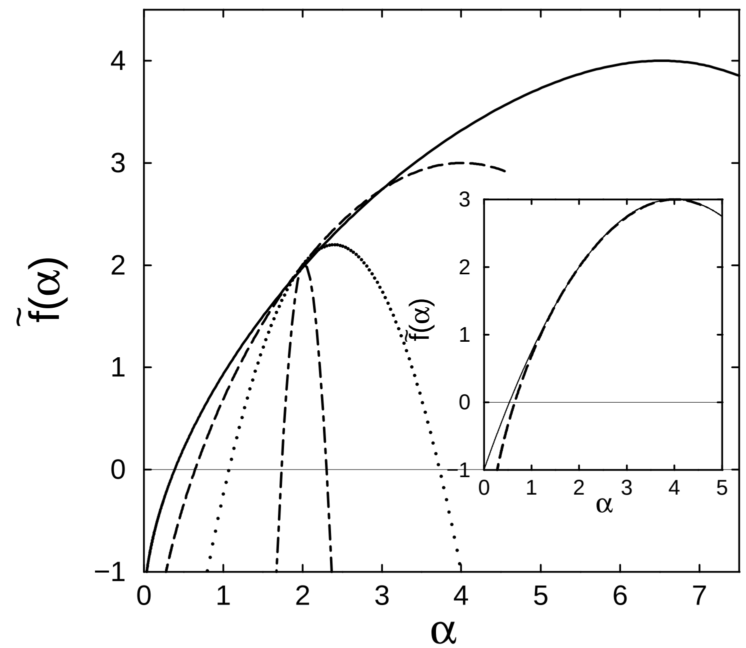
$$f(\alpha) \simeq d - \frac{(\alpha-\alpha_0)^2}{4(\alpha_0-d)}; \qquad \alpha_0 = d+\gamma$$

$\gamma = \epsilon$ (orthogonal); $\qquad \gamma = (\epsilon/2)^{1/2}$ (unitary)

$$q_\pm = \pm(d/\gamma)^{1/2}$$

Dimensionality dependence of multifractality





Analytics (2 + ϵ, one-loop) and numerics

$$\tau_q = (q-1)d - q(q-1)\epsilon + O(\epsilon^4)$$

$$f(\alpha) = d - (d+\epsilon-\alpha)^2/4\epsilon + O(\epsilon^4)$$

$d = 4$ (full)
$d = 3$ (dashed)
$d = 2 + \epsilon, \quad \epsilon = 0.2$ (dotted)
$d = 2 + \epsilon, \quad \epsilon = 0.01$ (dot-dashed)

Inset: $d = 3$ (dashed)
vs. $d = 2 + \epsilon, \ \epsilon = 1$ (full)

Mildenberger, Evers, ADM '02

Power-law random banded matrix model (PRBM)

Anderson transition: dimensionality dependence:

$d = 2 + \epsilon$: weak disorder/coupling $d \gg 1$: strong disorder/coupling

Evolution from weak to strong coupling – ?

PRBM   ADM, Fyodorov, Dittes, Quezada, Seligman '96

$N \times N$ random matrix $H = H^\dagger$ $$\langle |H_{ij}|^2 \rangle = \frac{1}{1 + |i-j|^2/b^2}$$

$\longleftrightarrow$ 1D model with $1/r$ long range hopping $0 < b < \infty$ parameter

Critical for any b $\longrightarrow$ family of critical theories!

$b \gg 1$ analogous to $d = 2 + \epsilon$ $b \ll 1$ analogous to $d \gg 1$ (?)

Analytics: $b \gg 1$: σ-model RG

$b \ll 1$: real space RG

Numerics: efficient in a broad range of b

Evers, ADM '01

Weak multifractality, $b \gg 1$

supermatrix σ-model

$$S[Q] = \frac{\pi\rho\beta}{4}\mathrm{Str}\left[\pi\rho\sum_{rr'} J_{rr'}Q(r)Q(r') - i\omega\sum_{r} Q(r)\Lambda\right].$$

In momentum (k) space and in the low-k limit:

$$S[Q] = \beta\,\mathrm{Str}\left[-\frac{1}{t}\int\frac{dk}{2\pi}|k|Q_k Q_{-k} - \frac{i\pi\rho\omega}{4}Q_0\Lambda\right]$$

DOS $\qquad \rho(E) = (1/2\pi^2 b)(4\pi b - E^2)^{1/2}\,, \qquad |E| < 2\sqrt{\pi b}$

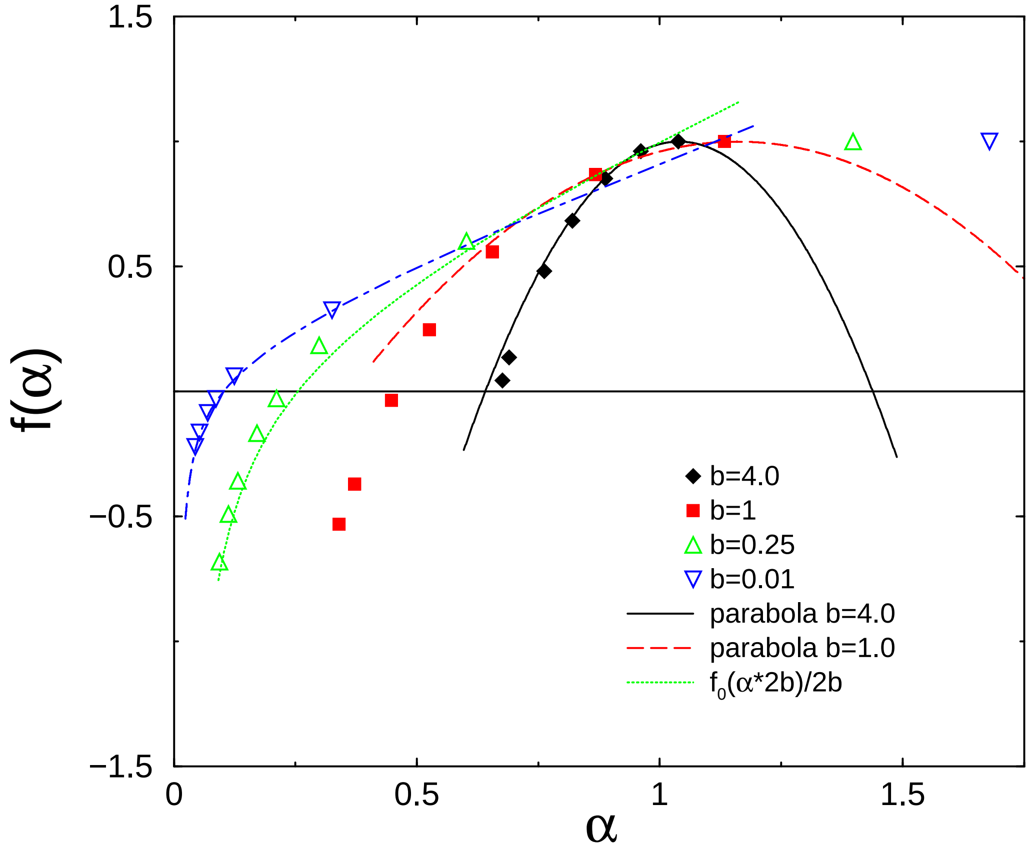
coupling constant $\qquad 1/t = (\pi/4)(\pi\rho)^2 b^2 = (b/4)(1 - E^2/4\pi b)$

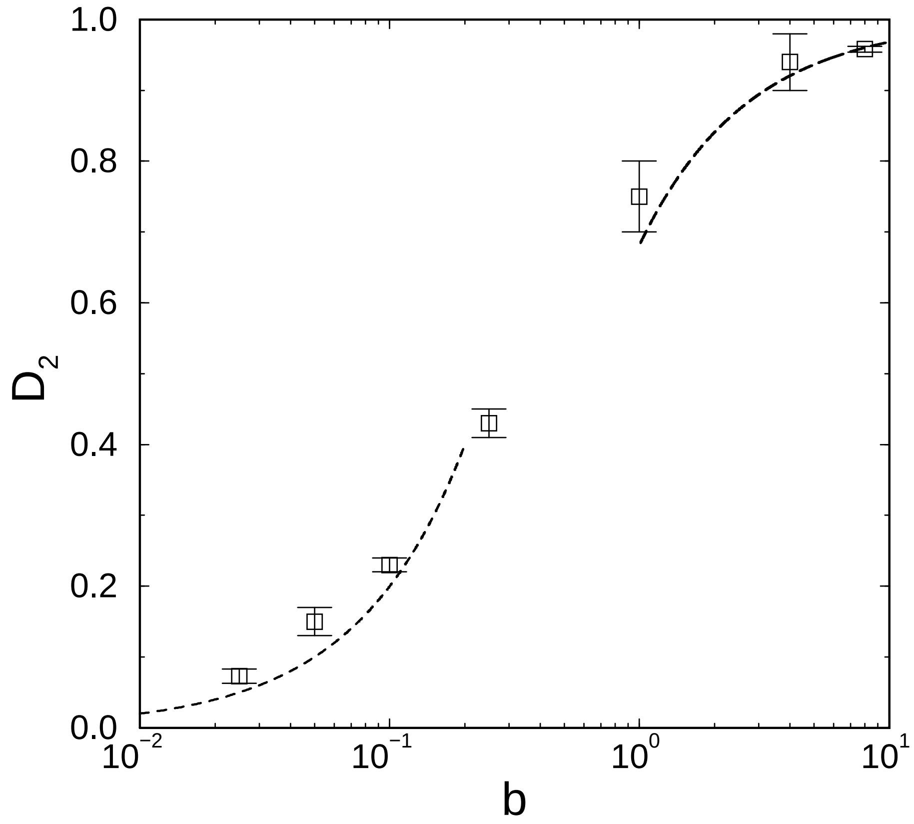
$\longrightarrow$ **weak multifractality**

$$\tau_q \simeq (q-1)(1 - qt/8\pi\beta)\,, \qquad q \ll 8\pi\beta/t$$

$$E = 0\,,\quad \beta = 1 \qquad \longrightarrow \qquad \Delta_q = \frac{1}{2\pi b}q(1-q)$$

Multifractality in PRBM model: analytics vs numerics





numerics: $b = 4,\ 1,\ 0.25,\ 0.01$

analytics: $b \gg 1$ (σ–model RG), $b \ll 1$ (real-space RG)

Strong multifractality, $b \ll 1$

Real-space RG:

- start with diagonal part of $\hat{H}$: localized states with energies $E_i = H_{ii}$
- include into consideration H_{ij} with $|i-j| = 1$

Most of them irrelevant, since $|H_{ij}| \sim b \ll 1$, while $|E_i - E_j| \sim 1$

Only with a probability $\sim b$ is $|E_i - E_j| \sim b$

$\longrightarrow$ two states strongly mixed ("resonance") $\longrightarrow$ two-level problem

$$\hat{H}_{\text{two-level}} = \begin{pmatrix} E_i & V \\ V & E_j \end{pmatrix} \ ; \qquad V = H_{ij}$$

New eigenfunctions and eigenenergies:

$$\psi^{(+)} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \ ; \qquad \psi^{(-)} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$E_{\pm} = (E_i + E_j)/2 \pm |V|\sqrt{1+\tau^2}$$

$\tan\theta = -\tau + \sqrt{1+\tau^2}$ and $\tau = (E_i - E_j)/2V$

- include into consideration H_{ij} with $|i-j| = 2$
- ...

Strong multifractality, $b \ll 1$ (cont'd)

Evolution equation for IPR distribution ("kinetic eq." in "time" $t = \ln r$):

$$\frac{\partial}{\partial \ln r} f(P_q, r) = \frac{2b}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta \cos^2\theta}$$

$$\times \left[-f(P_q, r) + \int dP_q^{(1)} dP_q^{(2)} f(P_q^{(1)}, r) f(P_q^{(2)}, r) \right.$$

$$\left. \times\ \delta(P_q - P_q^{(1)} \cos^{2q}\theta - P_q^{(2)} \sin^{2q}\theta) \right]$$

$\longrightarrow$ evolution equation for $\langle P_q \rangle$: $\quad \partial \langle P_q \rangle / \partial \ln r = -2bT(q) \langle P_q \rangle$ with

$$T(q) = \frac{1}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sin^2\theta \cos^2\theta} (1 - \cos^{2q}\theta - \sin^{2q}\theta) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(q - 1/2)}{\Gamma(q - 1)}$$

$\longrightarrow$ multifractality $\langle P_q \rangle \sim L^{-\tau_q}$, $\quad \tau_q = 2bT(q)$

This is applicable for $q > 1/2$

For $q < 1/2$ resonance approximation breaks down; use $\Delta_q = \Delta_{1-q}$

Strong multifractality, $b \ll 1$ (cont'd)

$T(q)$ asymptotics:

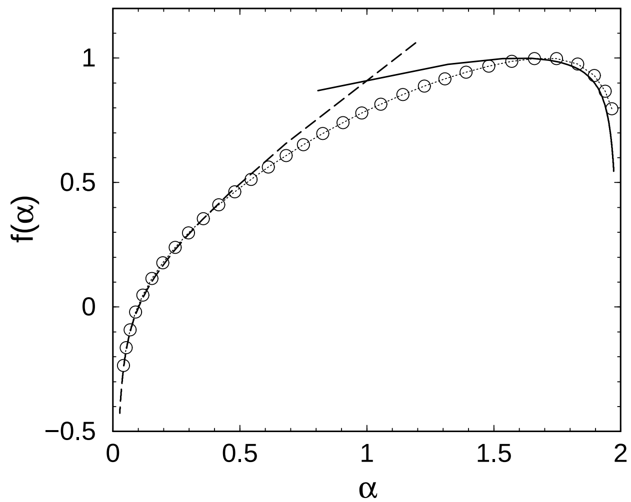
$$T(q) \simeq -1/[\pi(q-1/2)] \; , \qquad q \to 1/2 \; ;$$
$$T(q) \simeq (2/\sqrt{\pi}) q^{1/2} \; , \qquad q \gg 1$$

Singularity spectrum:

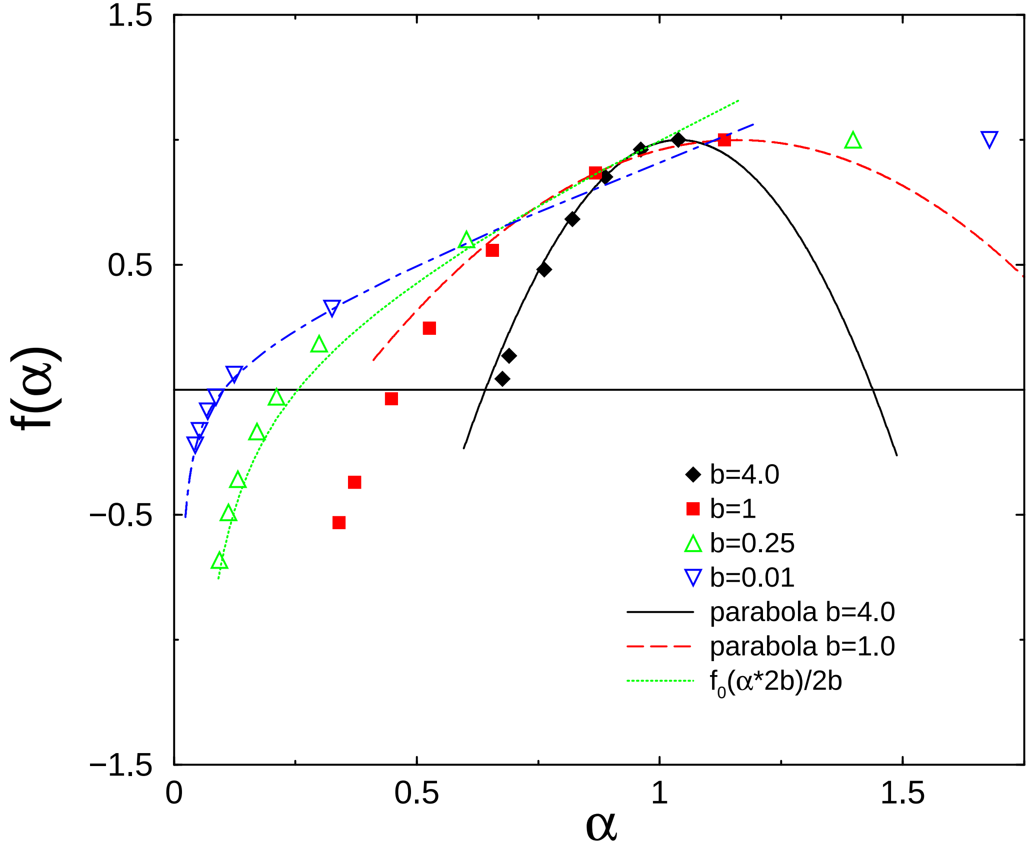
$f(\alpha) = 2bF(A)$; $\quad A = \alpha/2b$, $\quad F(A)$– Legendre transform of $T(q)$

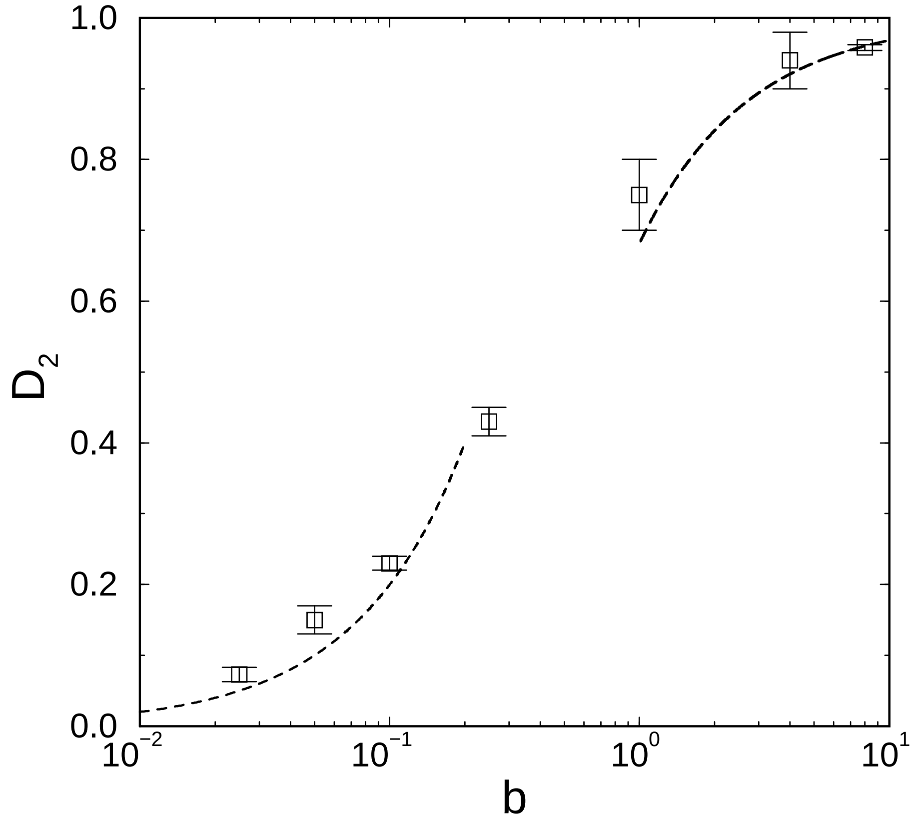
$F(A)$ asymptotics:

$$F(A) \simeq -1/\pi A \; , \qquad A \to 0 \; ;$$
$$F(A) \simeq A/2 \; , \qquad A \to \infty$$



Multifractality in PRBM model: analytics vs numerics





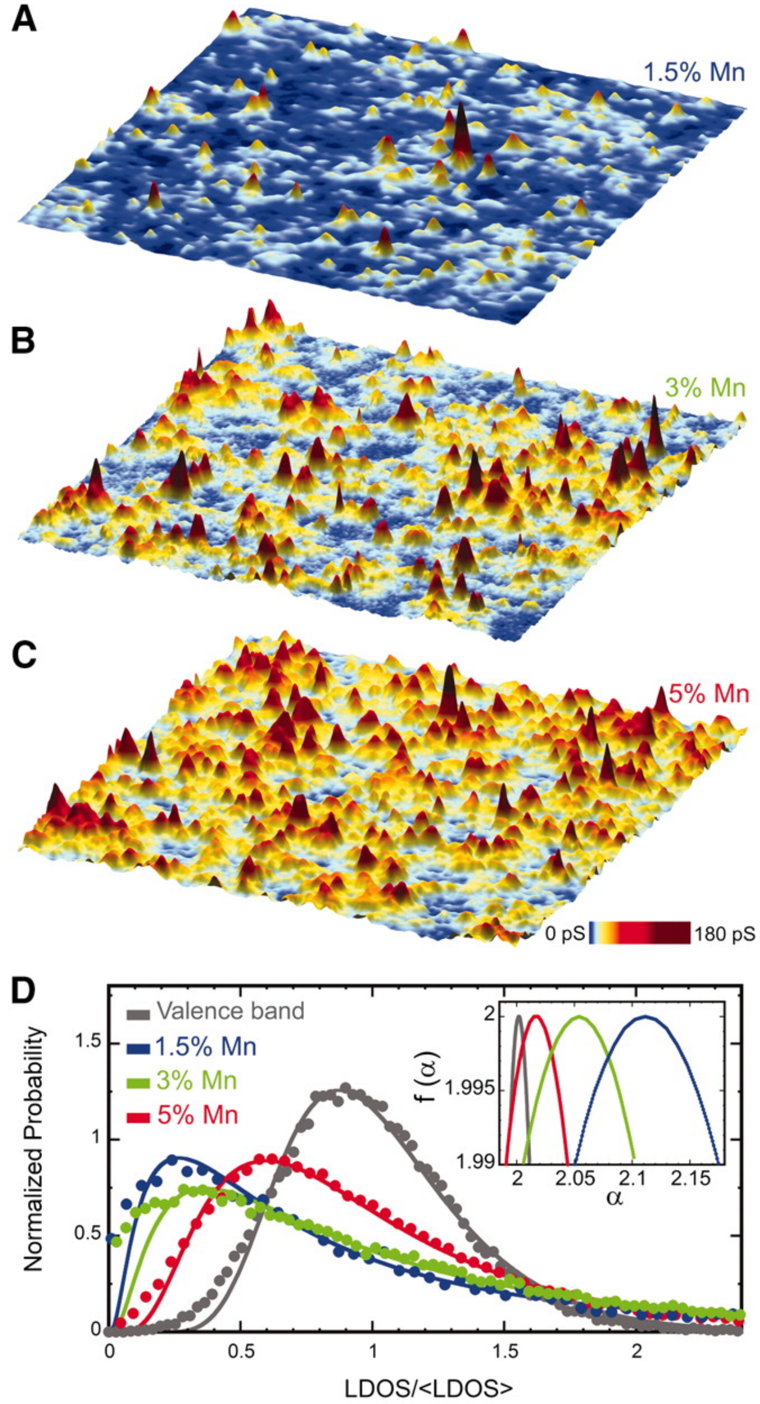
numerics: $b = 4,\ 1,\ 0.25,\ 0.01$

analytics: $b \gg 1$ (σ–model RG), $b \ll 1$ (real-space RG)

Multifractality: Experiment I

Local DOS flucutuations near metal-insulator transition in $Ga_{1-x}Mn_xAs$

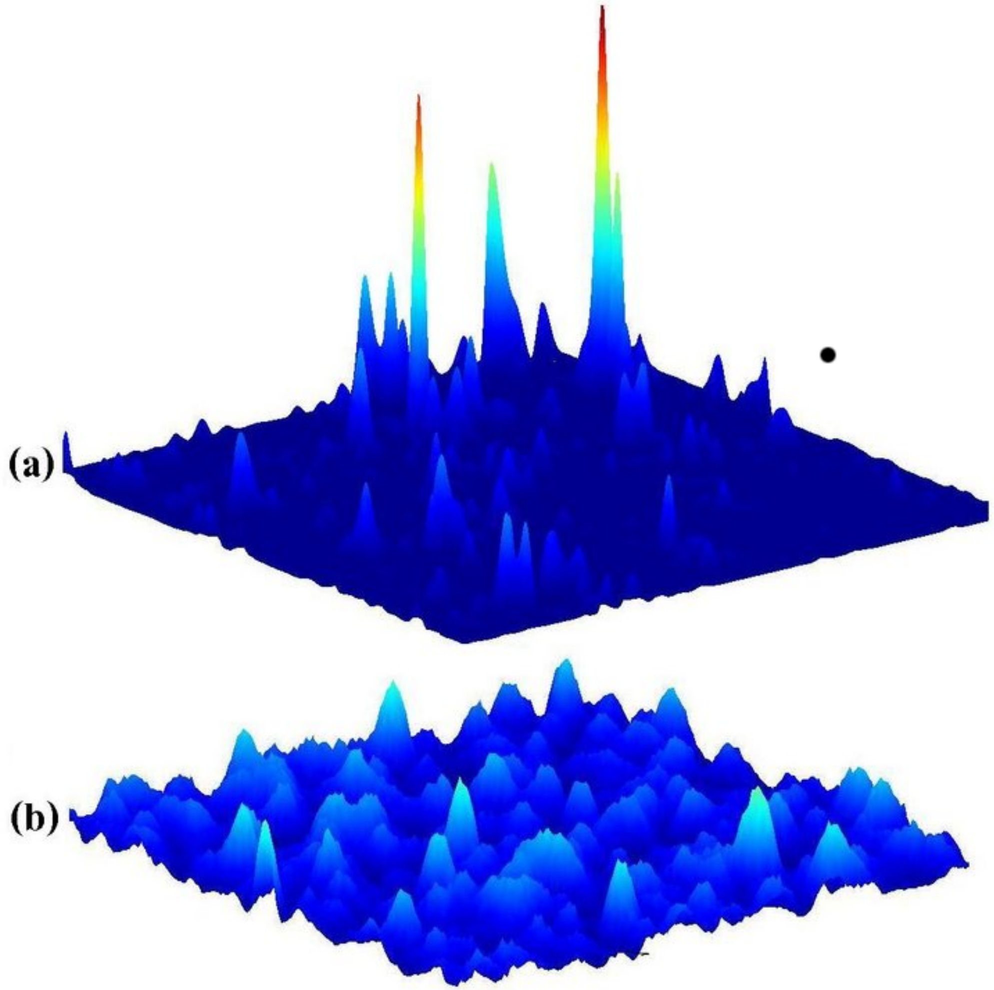
Richardella,...,Yazdani, Science '10

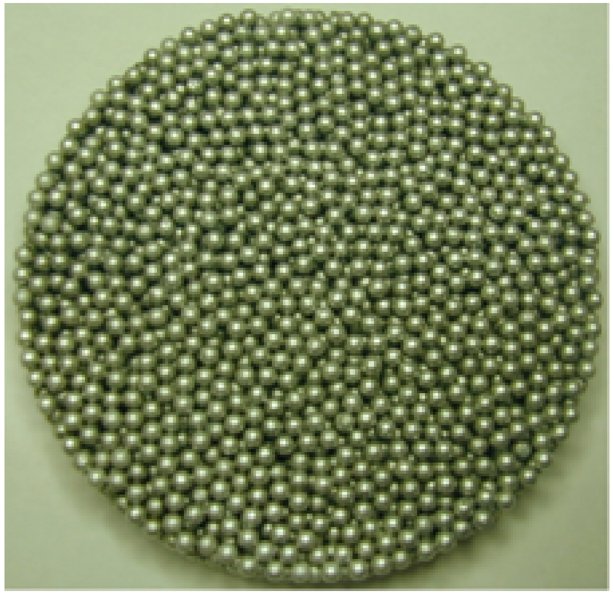


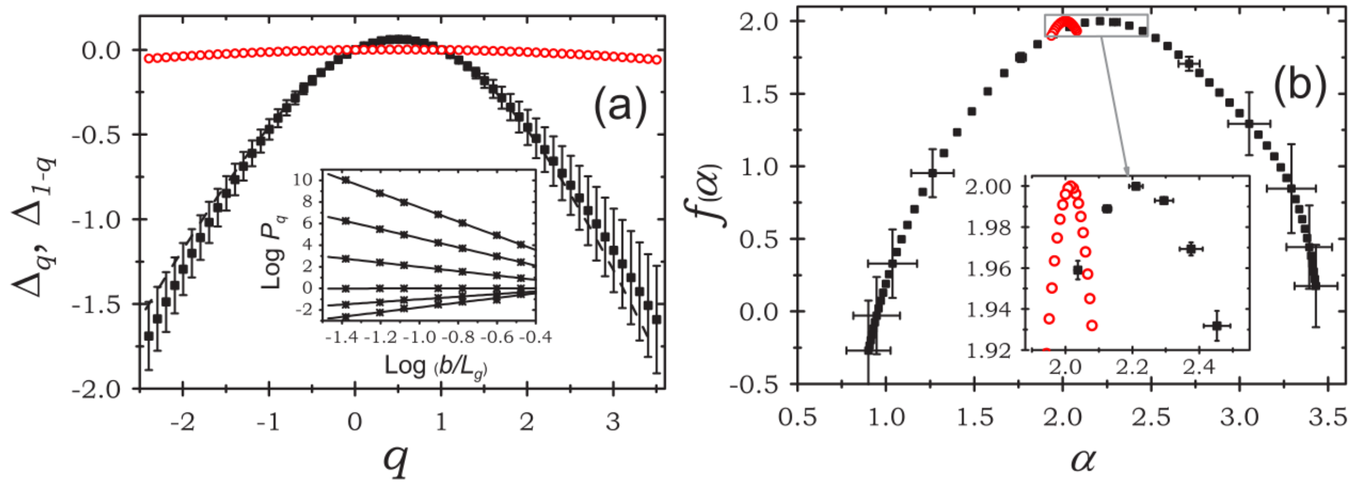
Multifractality: Experiment II

Ultrasound speckle in a system of randomly packed Al beads

Faez, Strybulevich, Page, Lagendijk, van Tiggelen, PRL'09



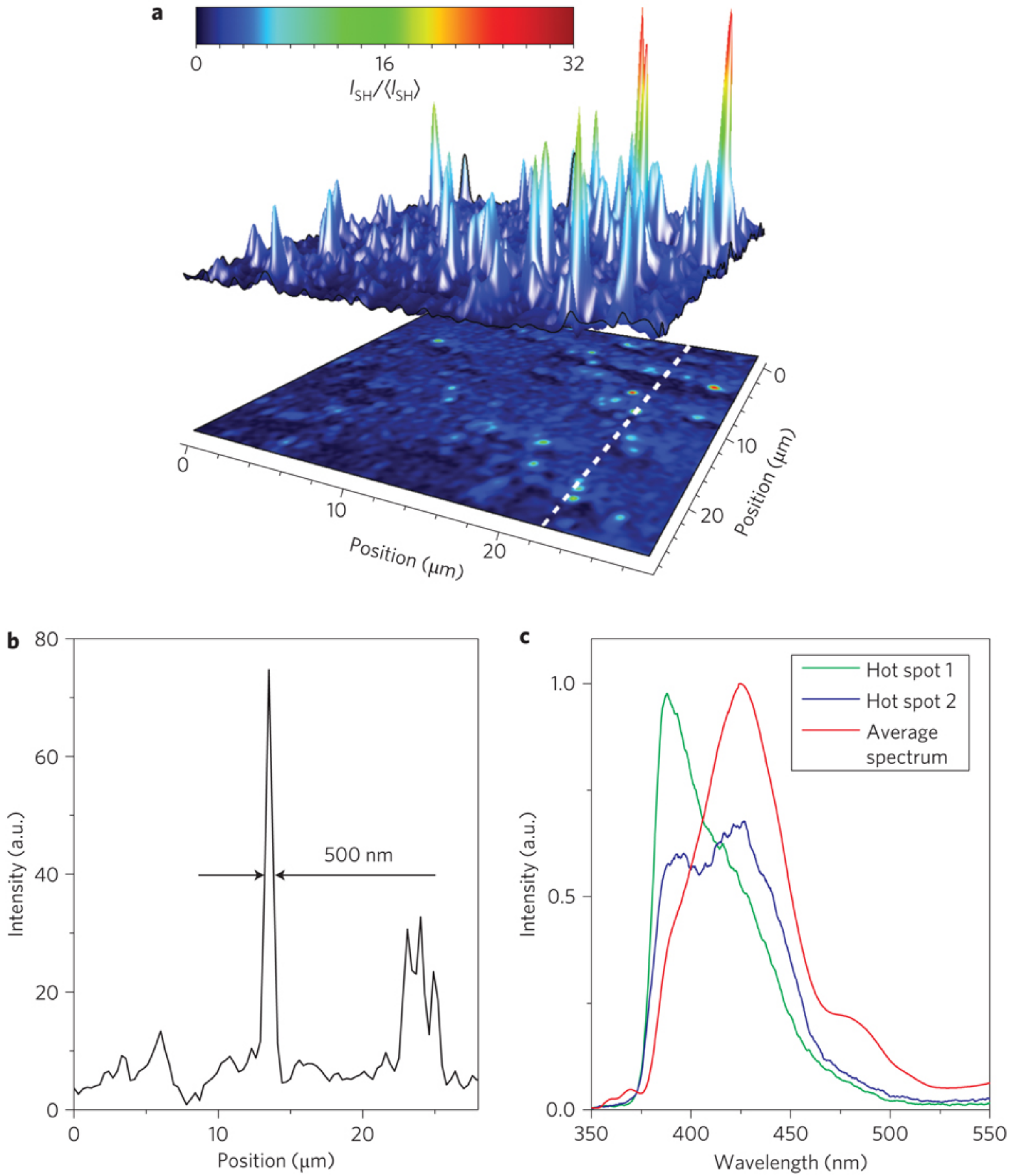




Multifractality: Experiment III

Localization of light in an array of dielectric nano-needles

Mascheck et al, Nature Photonics '12



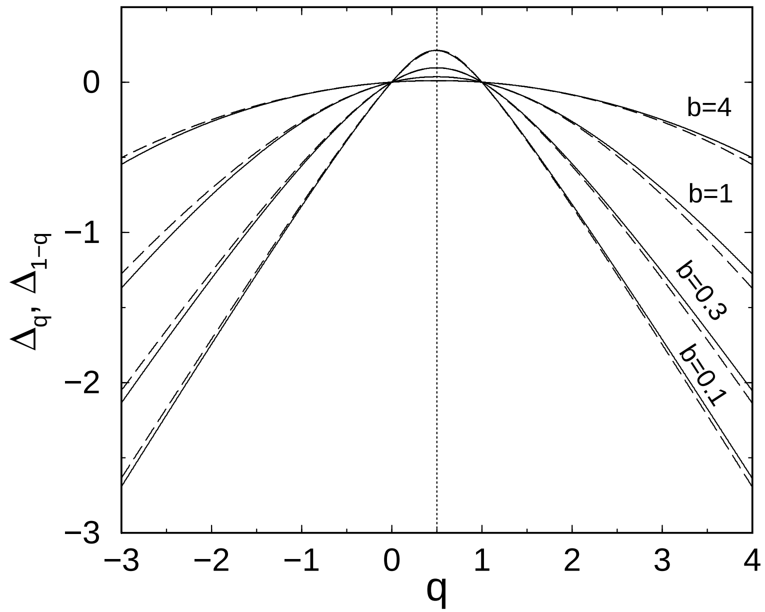
Symmetry of multifractal spectra

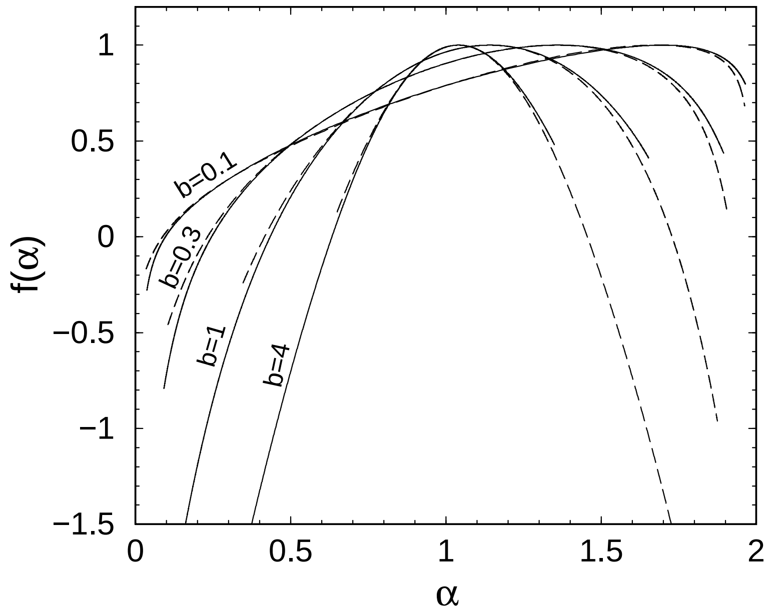
ADM, Fyodorov, Mildenberger, Evers '06

LDOS distribution in σ-model + **universality**

$\longrightarrow$ **exact symmetry of the multifractal spectrum:**

$$\Delta_q = \Delta_{1-q} \qquad f(2d-\alpha) = f(\alpha) + d - \alpha$$





$\longrightarrow$ **probabilities of unusually large and unusually small $|\psi^2(r)|$ are related !**

Questions to be answered

- **Fundamental reason for symmetry of multifractal spectra – ?**
- **Generalization to other symmetry classes – ?**
- **Complete classification of gradientless composite operators**
- **Generalization of symmetries to subleading operators – ?**

Gruzberg, Ludwig, ADM, Zirnbauer PRL'11

Gruzberg, ADM, Zirnbauer, PRB'13

Classification of scaling observables

Consider n points $\mathrm{r}_1, \ldots \mathrm{r}_\mathrm{n}$ and n wave functions $\psi_1, \ldots \psi_n$.

For each $p \leq n$ define

$$A_p(\mathrm{r}_1, \ldots, \mathrm{r}_{\tilde{p}}) = |D_p(\mathrm{r}_1, \ldots, \mathrm{r}_p)|^2$$

$$D_p(\mathrm{r}_1, \ldots, \mathrm{r}_p) = \mathrm{Det} \begin{pmatrix} \psi_1(\mathrm{r}_1) & \cdots & \psi_1(\mathrm{r}_p) \\ \vdots & \ddots & \vdots \\ \psi_p(\mathrm{r}_1) & \cdots & \psi_p(\mathrm{r}_p) \end{pmatrix}$$

For any set of complex $q_1, \ldots, q_n$ define

$$K_{(q_1, \ldots, q_n)} = \langle A_1^{q_1 - q_2} A_2^{q_2 - q_3} \ldots A_{n-1}^{q_{n-1} - q_n} A_n^{q_n} \rangle \, .$$

These are pure-scaling correlators of wave functions.

The proof goes via a mapping to the sigma model.

Scaling operators in sigma-model formalism

Sigma-model composite operators corresponding to wave function correlators $K_{(q_1,\ldots,q_n)}$ are

$$\mathcal{O}_{(q_1,\ldots,q_n)}(Q) = d_1^{q_1-q_2} d_2^{q_2-q_3} \ldots d_n^{q_n} \,,$$

where d_j is the principal minor of size $j \times j$ of the matrix (block of Q in retarded-advanced and boson-fermion spaces)

$$(1/2)(Q_{11} - Q_{22} + Q_{12} - Q_{21})_{bb} \,.$$

These are pure scaling operators. Proof:

- **Iwasawa decomposition $G = NAK$.**

Functions $\mathcal{O}_{(q_1,\ldots,q_n)}(Q)$ are N-invariant spherical functions on G/K and have a form of "plane waves" on A

- **Equivalently, $\mathcal{O}_{(q_1,\ldots,q_n)}(Q)$ can be constructed as highest-weight vectors**

Iwasawa decomposition

σ-model space: G/K $\qquad$ K — maximal compact subgroup

consider for definiteness unitary class (e.g., QH transition)

$$G/K = \mathrm{U}(n,n|2n)/[\mathrm{U}(n|n) \times \mathrm{U}(n|n)]$$

Iwasawa decomposition: $G = NAK$ $\qquad$ $g = nak$

A — maximal abelian in G/K

N — nilpotent

($\longleftrightarrow$ triangular matrices with 1 on the diagonal)

Generalization of

Gram-Schmidt decomposition: matrix = triangular × unitary

Spherical functions

Eigenfunctions of all G-invariant (Casimir) operators (in particular, RG transformation) are spherical functions on G/K.

N-invariant spherical functions on G/K are "plane waves"

$$\varphi_{q,p} = \exp\Big(-2\sum_{j=1}^{n} q_j x_j - 2i\sum_{l=1}^{n} p_l y_l \Big)$$

$x_1, \ldots, x_n; y_1, \ldots, y_n$ — natural coordinates on abelian group A.

Here q_j can be arbitrary complex, p_j are non-negative integers.

For $p_j = 0$
the function ϕ_q is exactly $\mathcal{O}_{(q_1,\ldots,q_n)}(Q)$ introduced above

Symmetries of scaling exponents

Weyl group $\longrightarrow$ invariance of eigenvalues of any G invariant operator with respect to

(i) reflections

$$q_j \to -c_j - q_j \qquad\qquad c_j = 1 - 2j$$

(ii) permutations

$$q_i \to q_j + \frac{c_j - c_i}{2}; \qquad q_j \to q_i + \frac{c_i - c_j}{2}$$

This is valid in particular for eigenvalues of RG, i.e. scaling exponents

Symmetries of scaling exponents (cont'd)

- **Weyl reflection $\longrightarrow$ identical scaling dimensions of**

$$(q), \quad (1-q)$$

$\longrightarrow$ **exactly the symmetry of multifractal spectra found earlier**

- (q_1, q_2). **We can generate out of it 8 representations:**

$$\begin{aligned}&(q_1, q_2), \quad (1-q_1, q_2), \quad (q_1, 3-q_2), \quad (1-q_1, 3-q_2),\\&(2-q_2, 2-q_1), \quad (-1+q_2, 2-q_1),\\&(2-q_2, 1+q_1), \quad (-1+q_2, 1+q_1).\end{aligned}$$

All of them will be characterized by the same scaling dimension.

- **etc.**

Symmetries of multifractal spectrum of A_2

$$A_2 = V^2 |\psi_1(r_1)\psi_2(r_2) - \psi_1(r_2)\psi_2(r_1)|^2$$

$\longleftrightarrow$ **Hartree-Fock matrix element of e-e interaction**

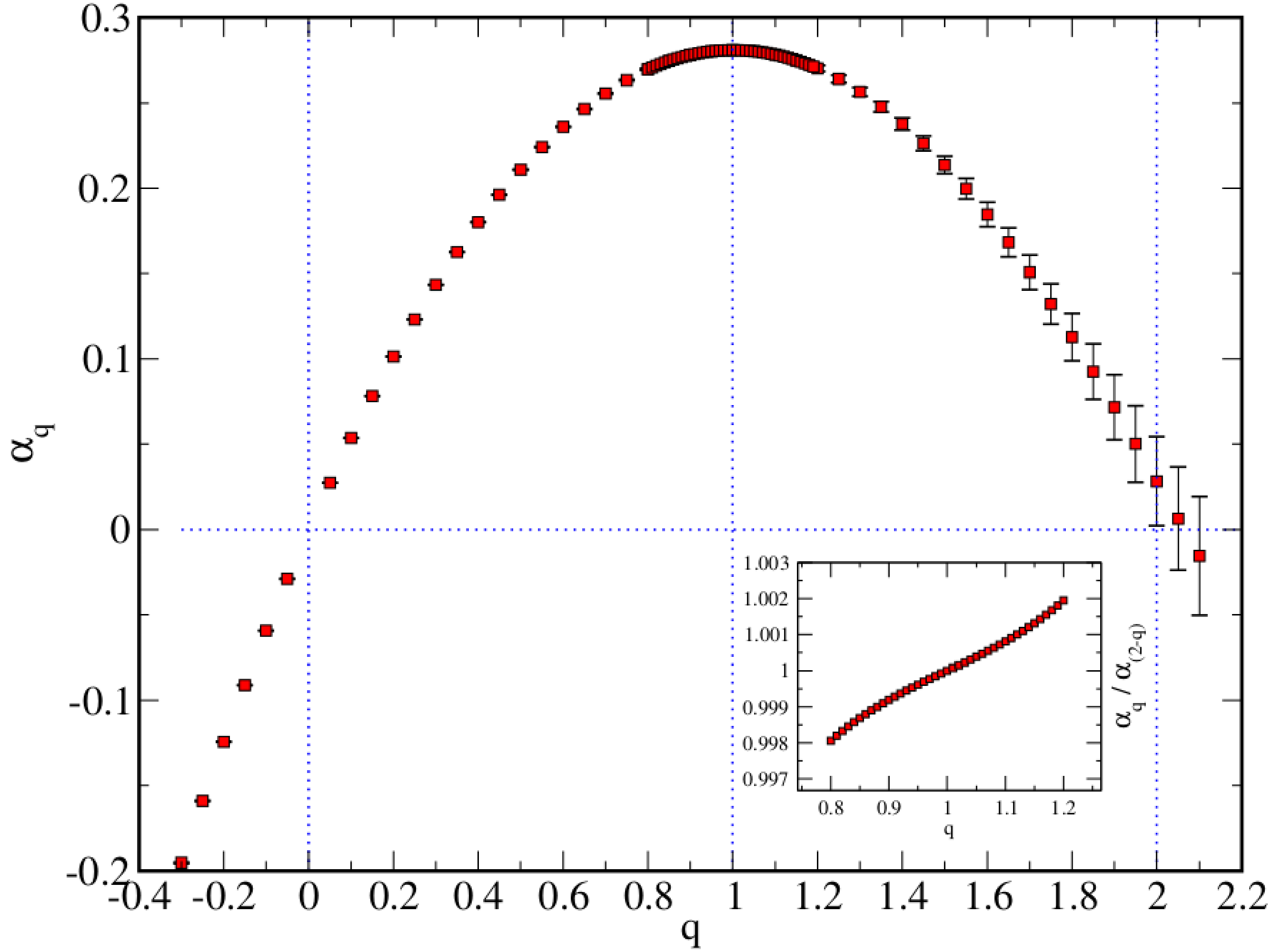
scaling: $\langle A_2^q \rangle \propto L^{-\Delta_{q,q}}$

symmetry: $\Delta_{q,q} = \Delta_{2-q,2-q}$

in particular, $\Delta_{2,2} = \Delta_0 = 0$

Multifractal spectrum of A_2 at quantum Hall transition

Numerical data: Bera, Evers, unpublished



Confirms the symmetry $q \longleftrightarrow 2 - q$

Disordered electronic systems: Symmetry classification

Altland, Zirnbauer '97

Conventional (Wigner-Dyson) classes

	T	spin rot.	symbol
GOE	+	+	AI
GUE	−	+/−	A
GSE	+	−	AII

Chiral classes

	T	spin rot.	symbol
ChOE	+	+	BDI
ChUE	−	+/−	AIII
ChSE	+	−	CII

$$\boldsymbol{H} = \begin{pmatrix} \mathbf{0} & \mathrm{t} \\ \mathrm{t}^\dagger & \mathbf{0} \end{pmatrix}$$

Bogoliubov-de Gennes classes

	T	spin rot.	symbol
	+	+	CI
	−	+	C
	+	−	DIII
	−	−	D

$$\boldsymbol{H} = \begin{pmatrix} \mathrm{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -\mathrm{h}^T \end{pmatrix}$$

Disordered electronic systems: Symmetry classification

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$\mathrm{U}(N)$	$\mathrm{GL}(N,\mathbb{C})/\mathrm{U}(N)$	AIII|AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI	GOE	$+$	$+$	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{GL}(N,\mathbb{R})/\mathrm{O}(N)$	BDI|CII	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$
AII	GSE	$+$	$-$	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{U}^*(2N)/\mathrm{Sp}(2N)$	CII|BDI	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$	A|A	$\mathrm{U}(n)$
BDI	chGOE	$+$	$+$	$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	$\mathrm{SO}(p,q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$	AI|AII	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$
CII	chGSE	$+$	$-$	$\mathrm{Sp}(2p+2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	$\mathrm{Sp}(2p,2q)/\mathrm{Sp}(2p)\times\mathrm{Sp}(2q)$	AII|AI	$\mathrm{U}(n)/\mathrm{O}(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$\mathrm{Sp}(2N)$	$\mathrm{Sp}(2N,\mathbb{C})/\mathrm{Sp}(2N)$	DIII|CI	$\mathrm{Sp}(2n)/\mathrm{U}(n)$
CI		$+$	$+$	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2N,\mathbb{R})/\mathrm{U}(N)$	D|C	$\mathrm{Sp}(2n)$
BD		$-$	$-$	$\mathrm{SO}(N)$	$\mathrm{SO}(N,\mathbb{C})/\mathrm{SO}(N)$	CI|DIII	$\mathrm{O}(2n)/\mathrm{U}(n)$
DIII		$+$	$-$	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$	C|D	$\mathrm{O}(n)$

Symmetries of scaling exponents: Other symmetry classes

- Direct generalization to 5 (out of 10) symmetry classes, with

$$c_j = 1 - 2j, \text{ class A} \qquad c_j = 1 - 4j, \text{ class C}$$
$$c_j = -j, \text{ class AI} \qquad c_j = -2j, \text{ class CI}$$
$$c_j = 3 - 4j, \text{ class AII}$$

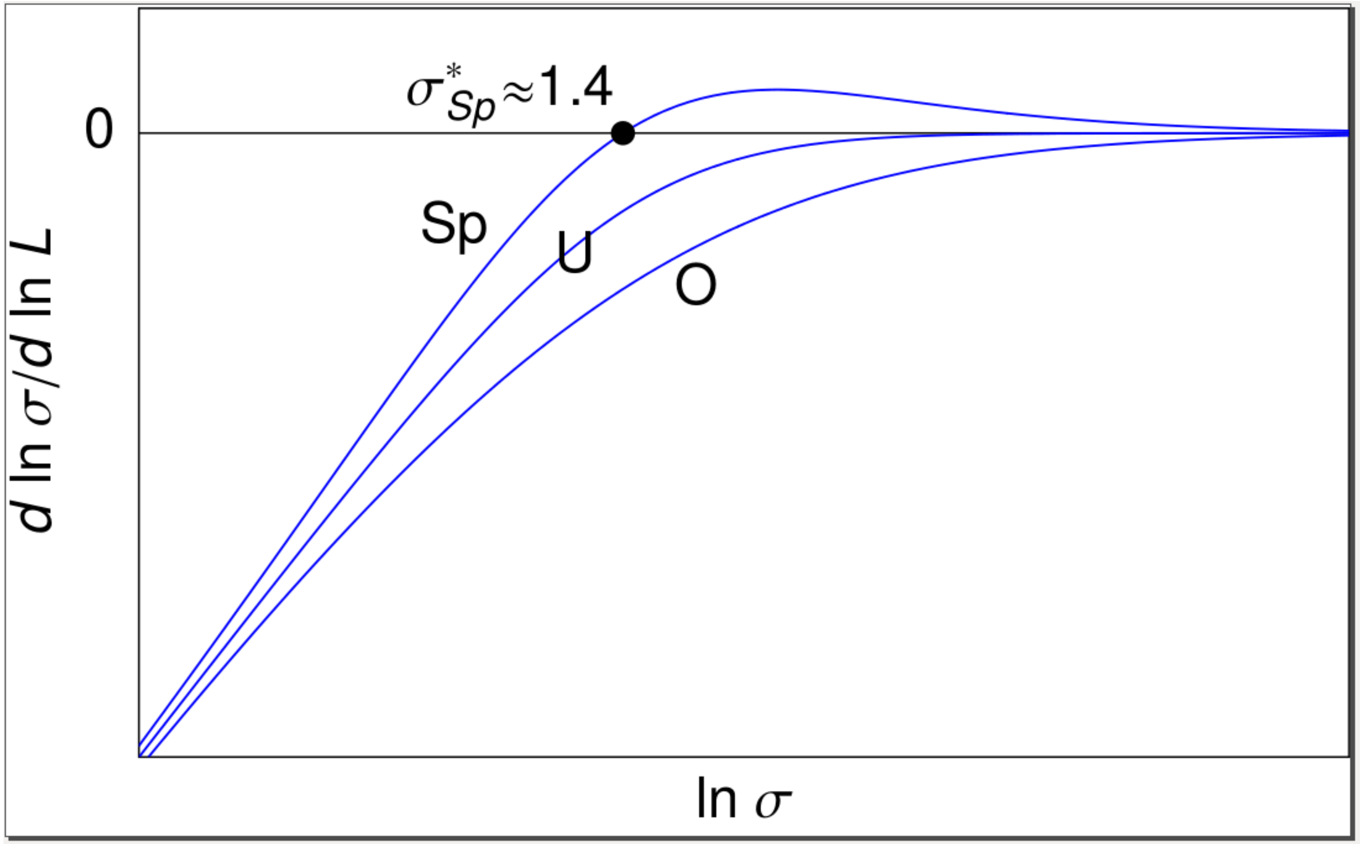
- classes D an DIII: applies if jumps between two disconnected parts of the sigma-model manifold (domain walls) are prohibited

$$c_j = 1 - j, \text{ class D}$$
$$c_j = 2 - 2j, \text{ class DIII}$$

$c_1 = 0 \quad \longrightarrow \quad$ no localization in the absence of domain walls

- chiral classes (BDI, CII, AIII) – complication: additional U(1) degree of freedom

Role of symmetry: 2D systems of Wigner-Dyson classes



Orthogonal and Unitary: localization;
parametrically different localization length: $\xi_U \gg \xi_O$

Symplectic: metal-insulator transition

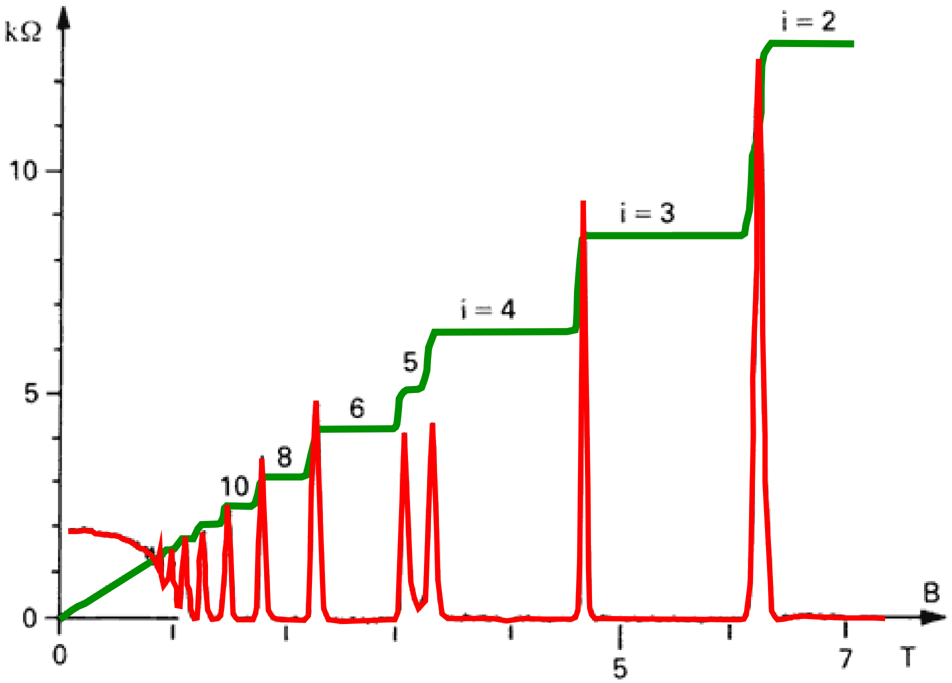
Usual realization of Sp class: spin-orbit interaction

Symmetry alone is not always sufficient to characterize the system.

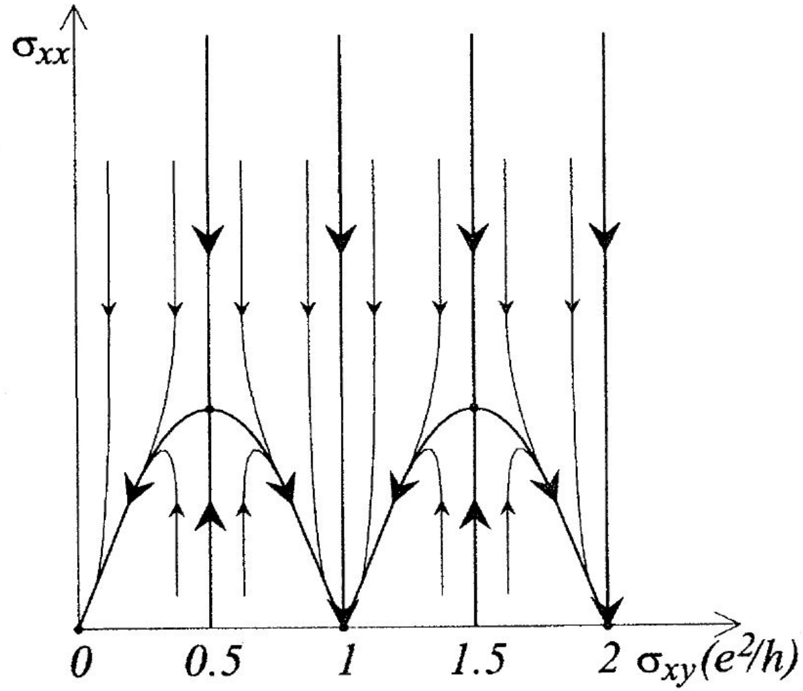
There may be also a non-trivial topology.

It may protect the system from localization.

Integer quantum Hall effect

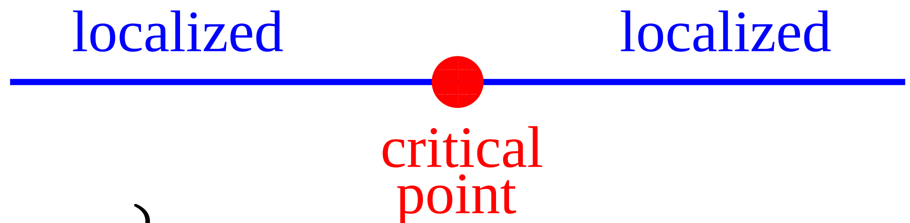


von Klitzing '80 ; Nobel Prize '85



IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84



Field theory (Pruisken):

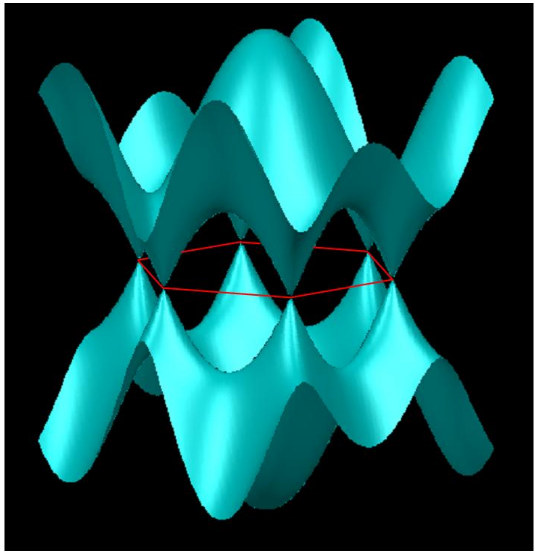
σ-model with topological term

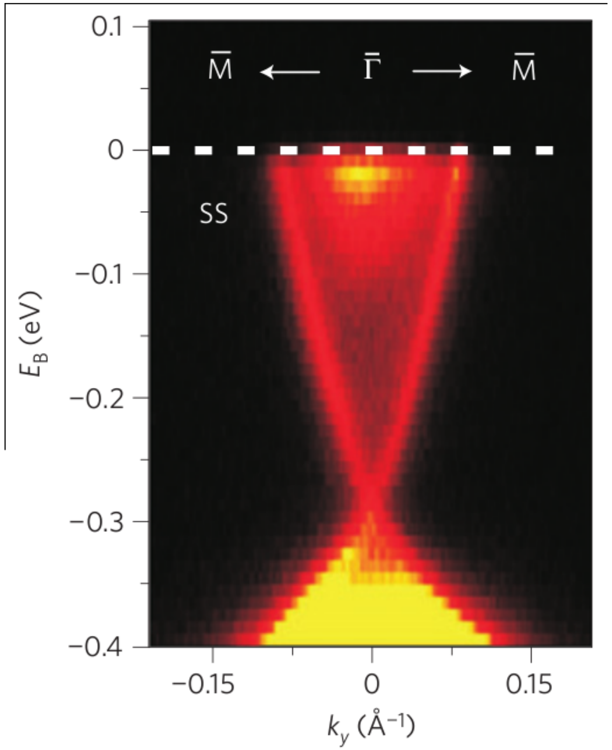
$$S = \int d^2 r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ protected edge states

$\longrightarrow$ $\mathbb{Z}$ topological insulator

2D massless Dirac fermions





Graphene

Geim, Novoselov'04

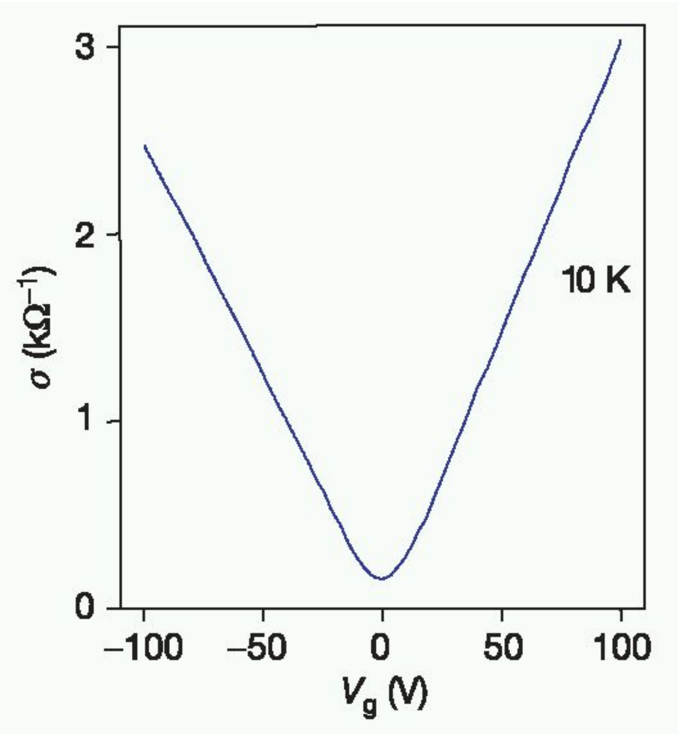
Nobel Prize'10

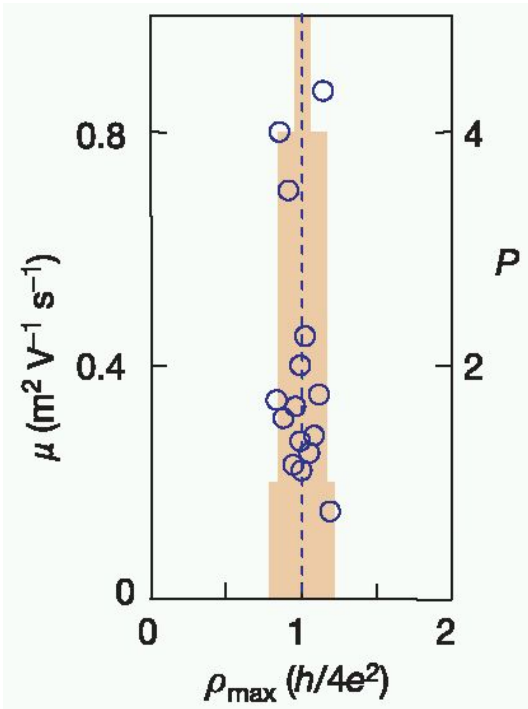
Surface of 3D topological insulators

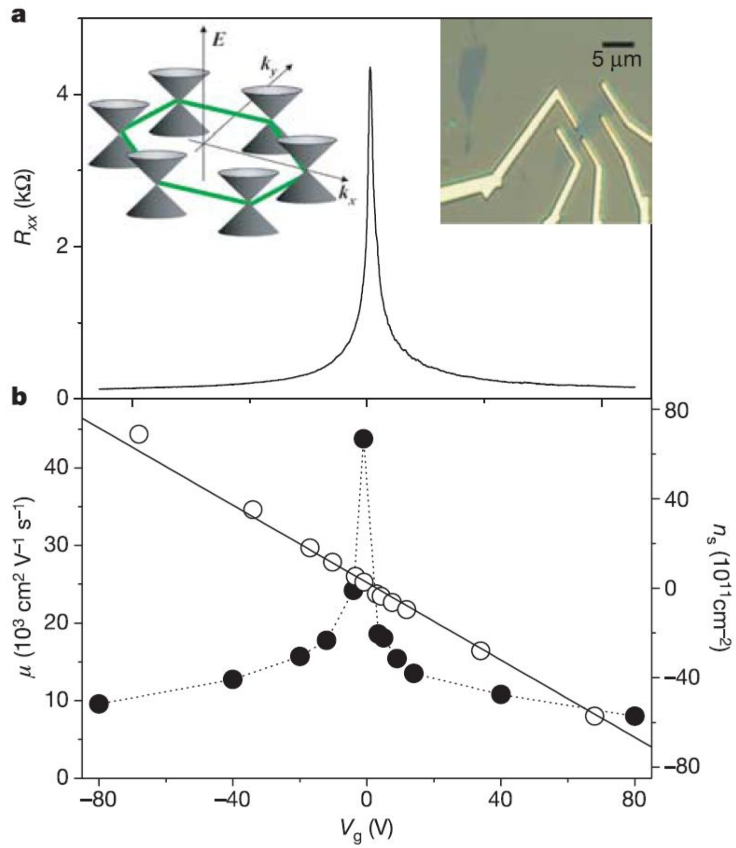
BiSb, BiSe, BiTe **Hasan group '08**

Experiments on transport in graphene

Novoselov, Geim et al; Zhang, Tan, Stormer, and Kim; Nature 2005





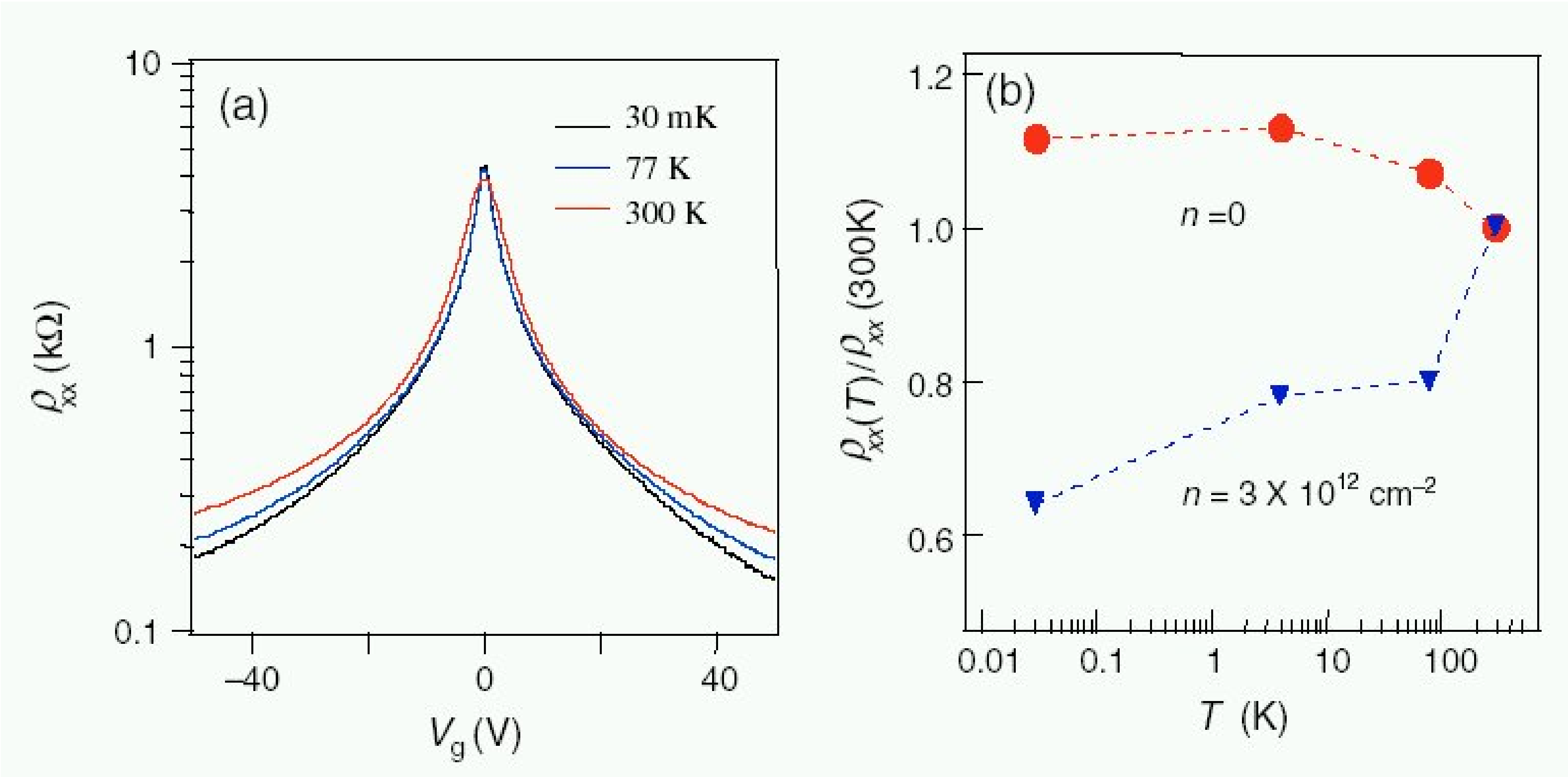


- linear dependence of conductivity on electron density ($\propto V_g$)
- minimal conductivity $\sigma \approx 4e^2/h$ ($\approx e^2/h$ per spin per valley)

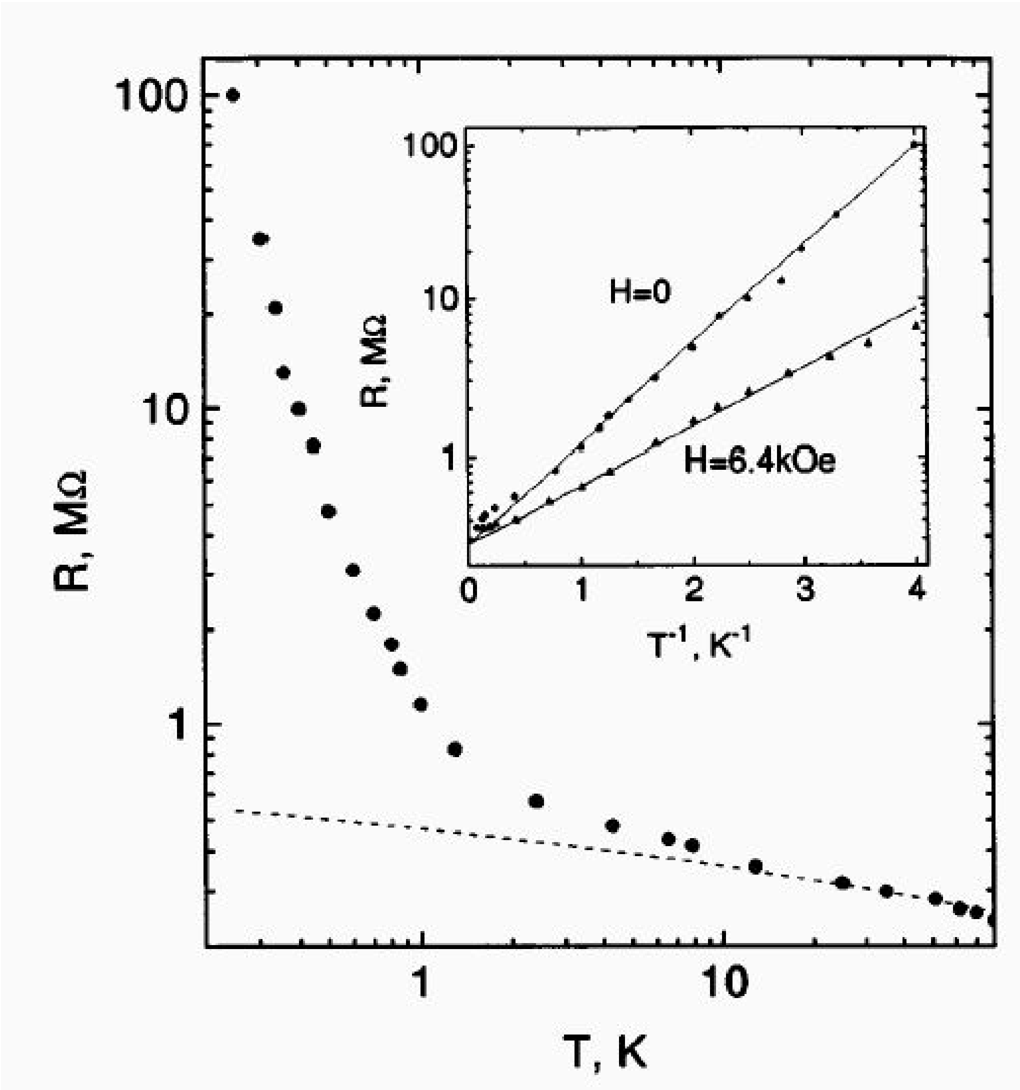
  T–independent in the range $T = 30$ mK $\div$ 300 K

T-independent minimal conductivity in graphene

Tan, Zhang, Stormer, Kim '07 $\qquad T = 30$ **mK** $\div$ **300 K**

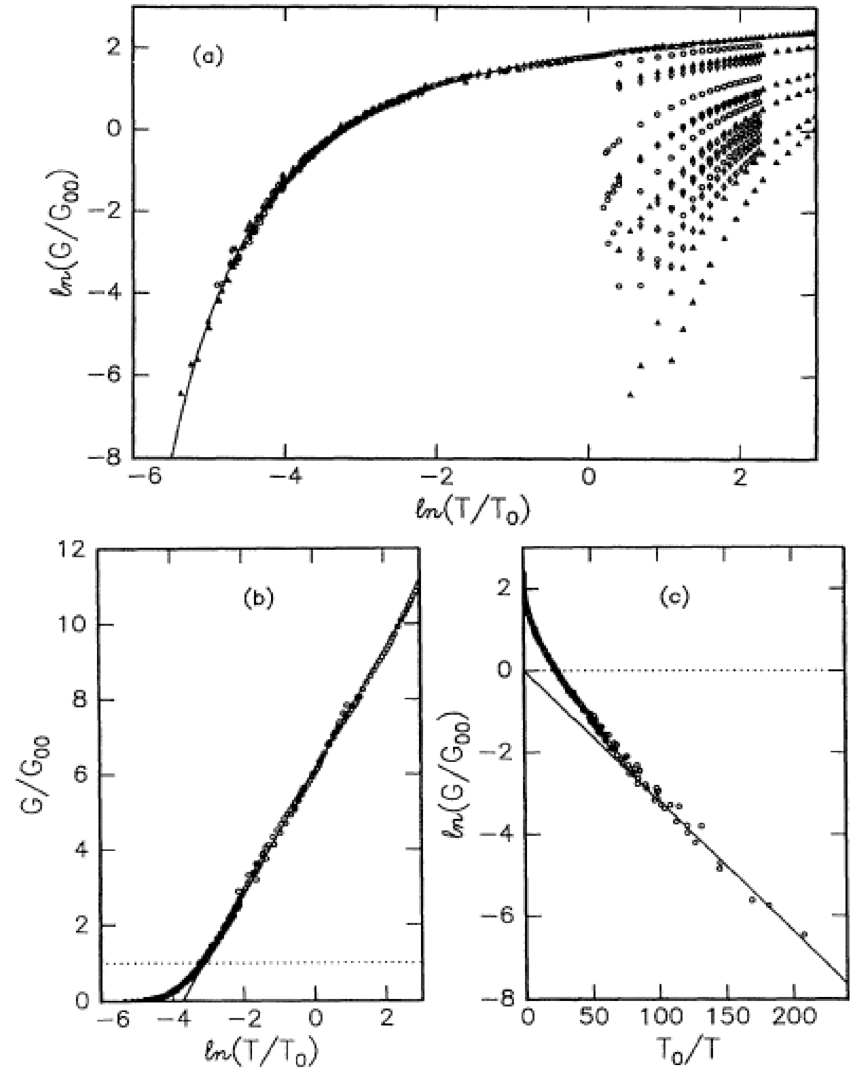


To compare: Disordered semiconductor systems: From metal to insulator with lowering T



quasi-1D geometry (long wires)

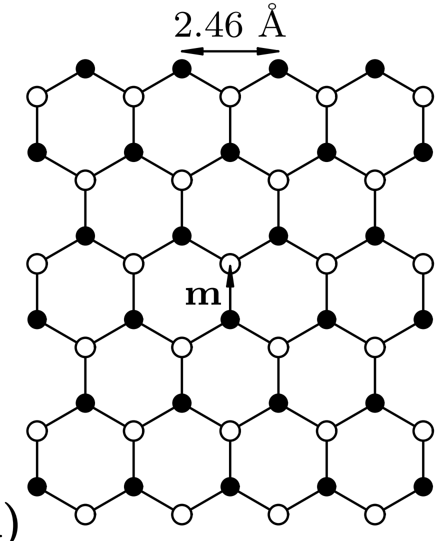
Gershenson et al '97

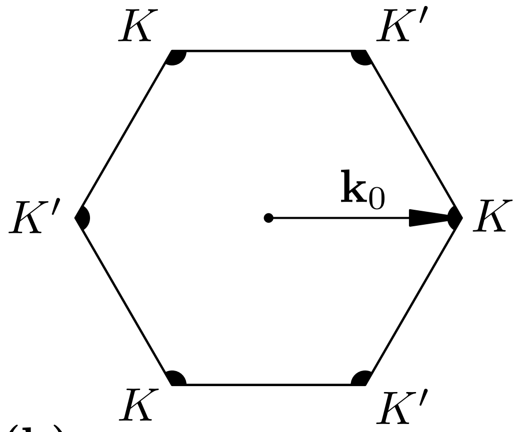


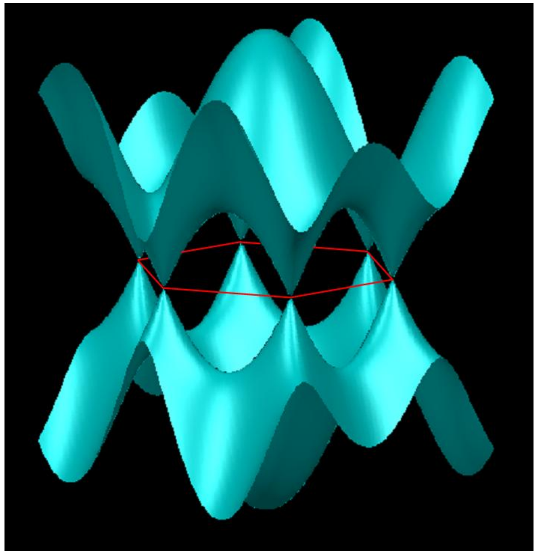
2D geometry

Hsu, Valles '95

Graphene dispersion: 2D massless Dirac fermions







Two sublattices: A and B   **Hamiltonian:** $H = \begin{pmatrix} 0 & t_{\mathrm{k}} \\ t^*_{\mathrm{k}} & 0 \end{pmatrix}$

$t_{\mathrm{k}} = t\left[1 + 2e^{i(\sqrt{3}/2)k_y a}\cos(k_x a/2)\right]$ **Spectrum** $\varepsilon^2_{\mathrm{k}} = |t_{\mathrm{k}}|^2$

The gap vanishes at 2 points, $K, K' = (\pm k_0, 0)$, where $k_0 = 4\pi/3a$.

In the vicinity of K, K': massless Dirac-fermion Hamiltonian:

$$H_K = v_0(k_x\sigma_x + k_y\sigma_y), \qquad H_{K'} = v_0(-k_x\sigma_x + k_y\sigma_y)$$

$v_0 \simeq 10^8$ cm/s – effective "light velocity",   sublattice space $\longrightarrow$ isospin

Graphene: Disordered Dirac-fermion Hamiltonian

Hamiltonian $\longrightarrow$ 4×4 matrix operating in:

AB space of the two sublattices ($\boldsymbol{\sigma}$ Pauli matrices),

$\boldsymbol{K}$–$\boldsymbol{K'}$ space of the valleys ($\boldsymbol{\tau}$ Pauli matrices).

Four-component wave function:

$$\Psi = \{\phi_{AK}, \phi_{BK}, \phi_{BK'}, \phi_{AK'}\}^T$$

Hamiltonian:

$$H = -iv_0\tau_z(\sigma_x\nabla_x + \sigma_y\nabla_y) + V(x, y)$$

Disorder:

$$V(x, y) = \sum_{\mu,\nu=0,x,y,z} \sigma_\mu\tau_\nu V_{\mu\nu}(x, y)$$

Clean graphene: symmetries

Space of valleys K–K': Isospin $\Lambda_x = \sigma_z\tau_x$, $\Lambda_y = \sigma_z\tau_y$, $\Lambda_z = \sigma_0\tau_z$.

Time inversion		Chirality	
T_0 :	$H = \sigma_x\tau_x H^T \sigma_x\tau_x$	C_0 :	$H = -\sigma_z\tau_0 H \sigma_z\tau_0$

Combinations with $\Lambda_{x,y,z}$

T_x :	$H = \sigma_y\tau_0 H^T \sigma_y\tau_0$	C_x :	$H = -\sigma_0\tau_x H \sigma_0\tau_x$
T_y :	$H = \sigma_y\tau_z H^T \sigma_y\tau_z$	C_y :	$H = -\sigma_0\tau_y H \sigma_0\tau_y$
T_z :	$H = \sigma_x\tau_y H^T \sigma_x\tau_y$	C_z :	$H = -\sigma_z\tau_z H \sigma_z\tau_z$

Spatial isotropy $\Rightarrow$ $T_{x,y}$ and $C_{x,y}$ occur simultaneously $\Rightarrow$ $T_\perp$ and $C_\perp$

Conductivity at $\mu = 0$

Drude conductivity (SCBA = self-consistent Born approximation):

$$\sigma = -\frac{8e^2v_0^2}{\pi\hbar}\int\frac{d^2k}{(2\pi)^2}\frac{(1/2\tau)^2}{[(1/2\tau)^2+v_0^2k^2]^2} = \frac{2e^2}{\pi^2\hbar} = \frac{4}{\pi}\frac{e^2}{h}$$

BUT: For generic disorder, the Drude result $\sigma = 4\times e^2/\pi h$ at $\mu = 0$ does not make much sense: Anderson localization will drive $\sigma \to 0$.

Experiment: $\sigma \approx 4\times e^2/h$ independent of T

Can one have non-zero σ (i.e. no localization) in the theory?

Yes, if disorder either

(i) preserves one of chiral symmetries

or

(ii) is of long-range character (does not mix the valleys)

Absence of localization of Dirac fermions in graphene with chiral or long-range disorder

Disorder	Symmetries	Class	Conductivity
Vacancies	C_z, T_0	BDI	$\approx 4e^2/\pi h$
Vacancies + RMF	C_z	AIII	$\approx 4e^2/\pi h$
$\sigma_z \tau_{x,y}$ disorder	C_z, T_z	CII	$\approx 4e^2/\pi h$
Dislocations	C_0, T_0	CI	$4e^2/\pi h$
Dislocations + RMF	C_0	AIII	$4e^2/\pi h$
random v, resonant scatterers	C_0, Λ_z, $T_\perp$	2×DIII	$4e^2/\pi h \times \{1, \log L\}$
Ripples, RMF	C_0, Λ_z	2×AIII	$4e^2/\pi h$
Charged impurities	Λ_z, $T_\perp$	2×AII	$(4e^2/\pi h) \log L$
random Dirac mass: $\sigma_z \tau_{0,z}$	Λ_z, $CT_\perp$	2×D	$4e^2/\pi h$
Charged imp. + RMF/ripples	Λ_z	2×A	$4\sigma_U^*$

C_z-chirality $\longrightarrow$ Gade-Wegner phase

C_0-chirality $\equiv$ random gauge fields $\longrightarrow$ Wess-Zumino-Witten term

Λ_z-symmetry $\equiv$ decoupled valleys $\longrightarrow$ $\theta = \pi$ topological term

2D Dirac fermions: σ-models with topological term

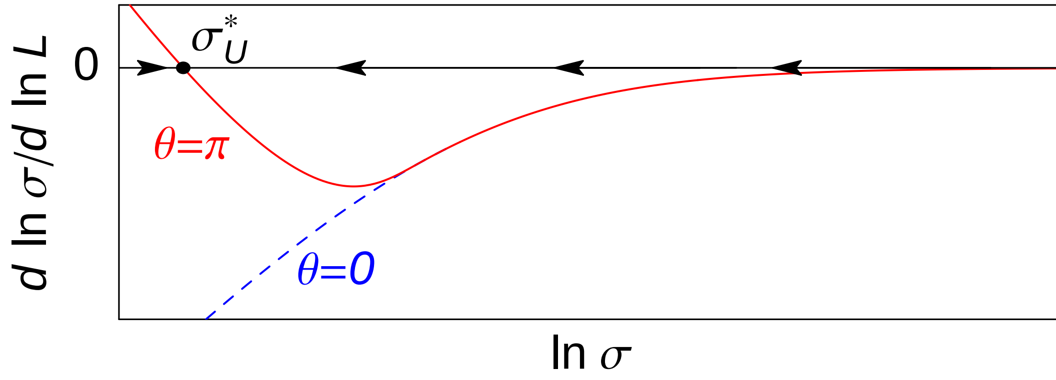
- Generic disorder (broken TRS) $\Longrightarrow$ class A (unitary)

$$S[Q] = \frac{1}{8}\,\mathrm{Str}\left[-\sigma_{xx}(\nabla Q)^2 + Q\nabla_x Q\nabla_y Q\right] \;=\; -\frac{\sigma_{xx}}{8}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

topol. invariant $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}$

$\Longrightarrow$ Quantum Hall critical point

$$\sigma \;=\; 4\sigma_U^* \;\simeq\; 4 \times (0.5 \div 0.6)\frac{e^2}{h}$$



- Random potential (preserved TRS) $\Longrightarrow$ class AII (symplectic)

$$S[Q] = -\frac{\sigma_{xx}}{16}\,\mathrm{Str}(\nabla Q)^2 + i\pi N[Q]$$

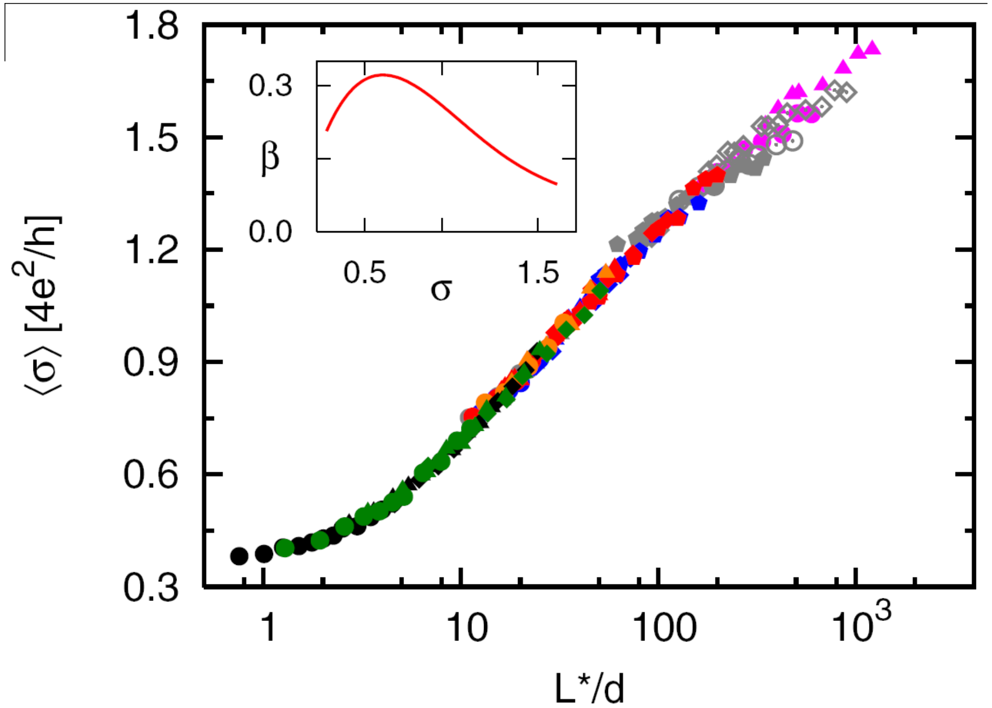
topological invariant: $N[Q] \in \pi_2(\mathcal{M}) = \mathbb{Z}_2 = \{0, 1\}$

Topological protection from localization !

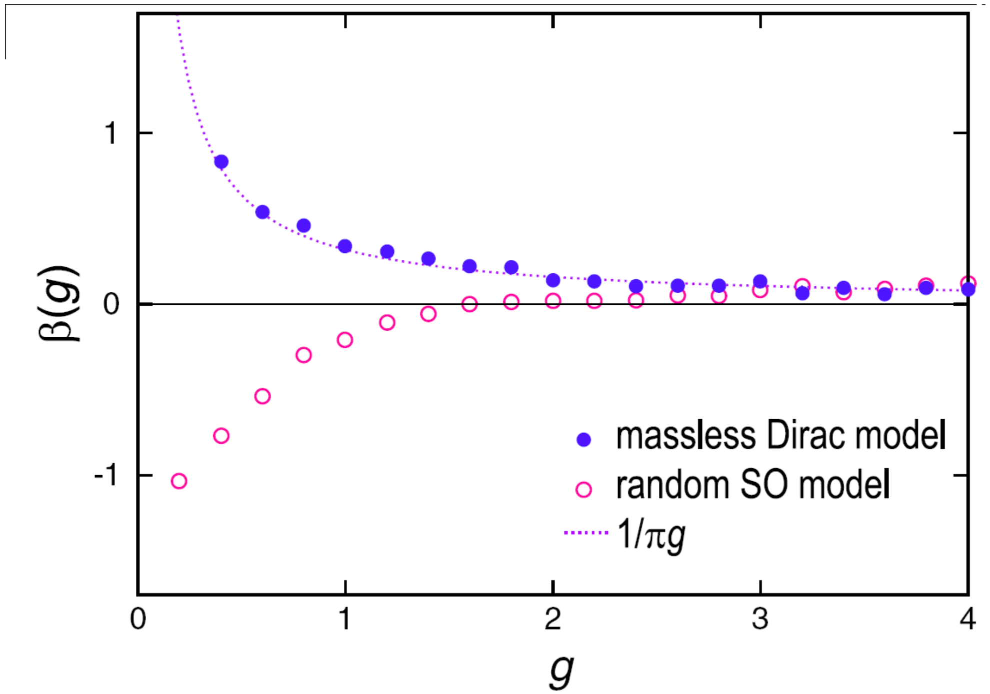
Ostrovsky, Gornyi, ADM, PRL 98, 256801 (2007)

Dirac fermions in random potential: numerics

Bardarson, Tworzydło, Brouwer, Beenakker, PRL '07

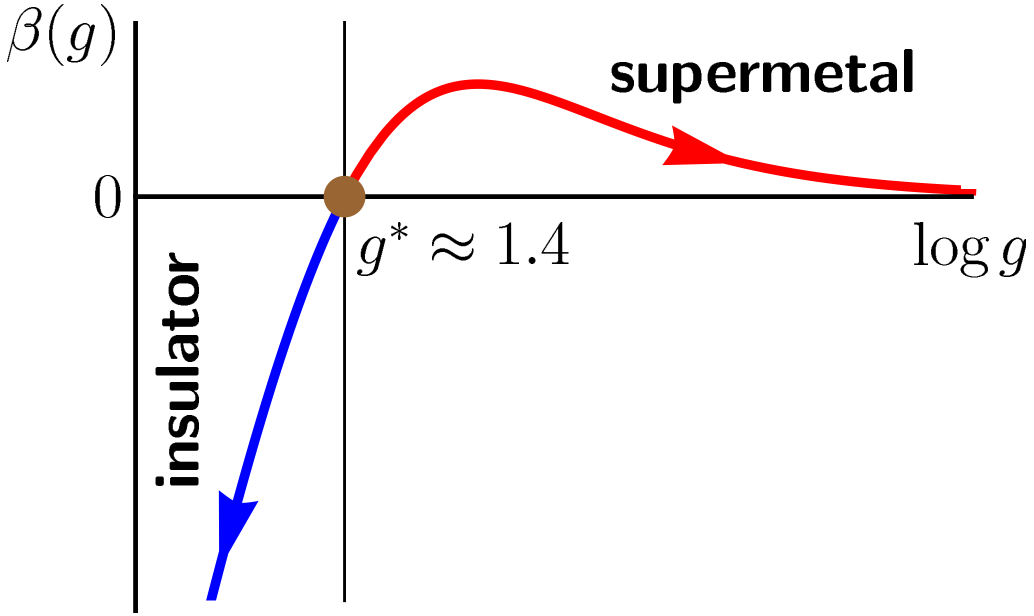


Nomura, Koshino, Ryu, PRL '07

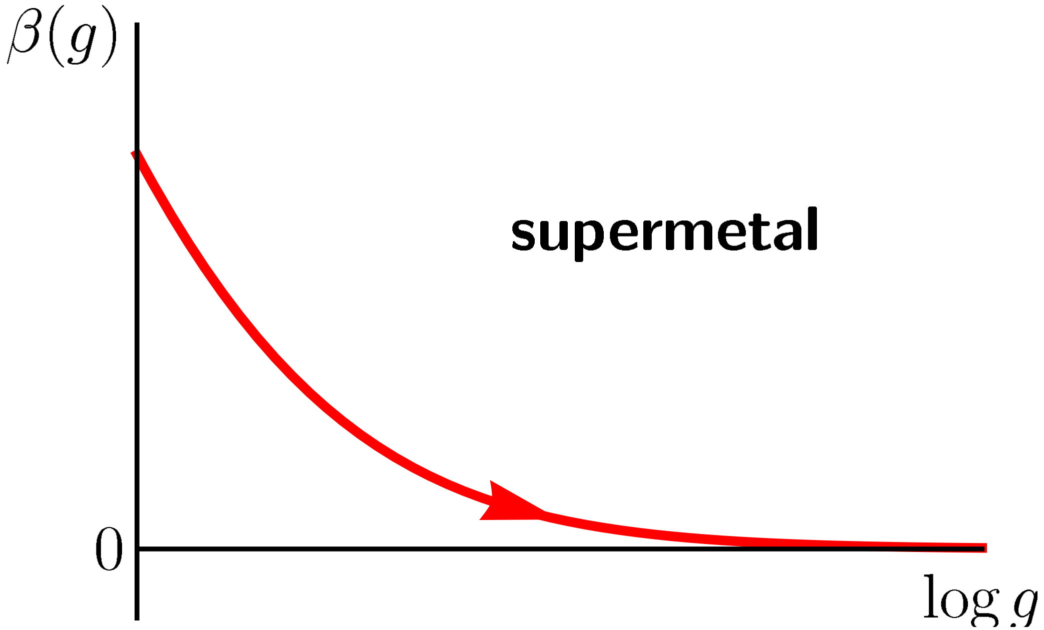


- absence of localization confirmed
- log scaling towards the perfect-metal fixed point $\sigma \to \infty$

Schematic beta functions for 2D systems of symplectic class AII



Conventional spin-orbit systems

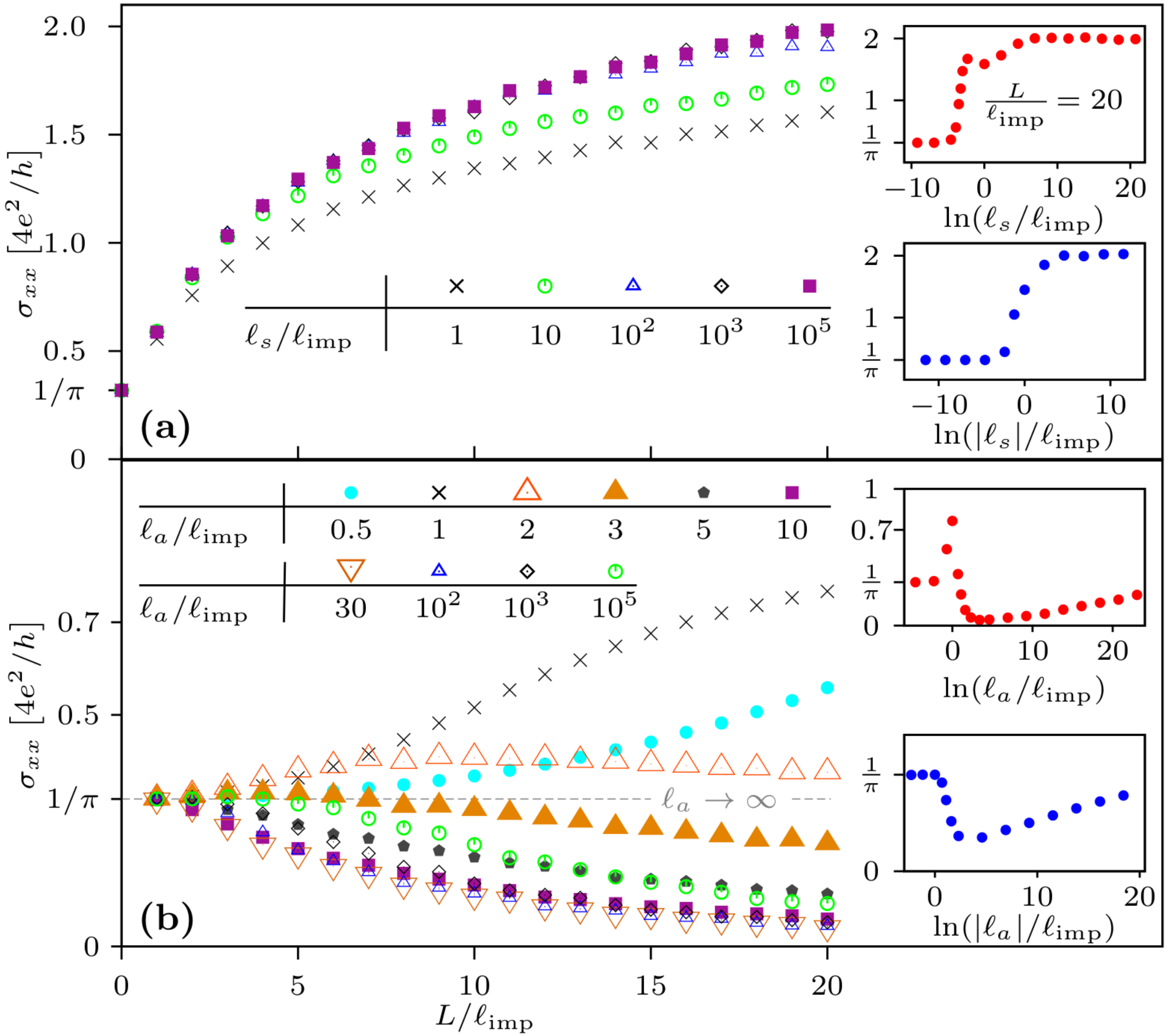


Dirac fermions
(topological protection)

surface of 3D top. insulator
or
graphene without valley mixing

Graphene with rare strong scatterers

Titov et al, PRL'10; Ostrovsky et al, PRL'10; Gattenlöhner et al, arXiv'13



no valley mixing ("scalar impurities")

scatttering length ℓ_s

$\ell_s \to \infty \Rightarrow$ class DIII (WZ)

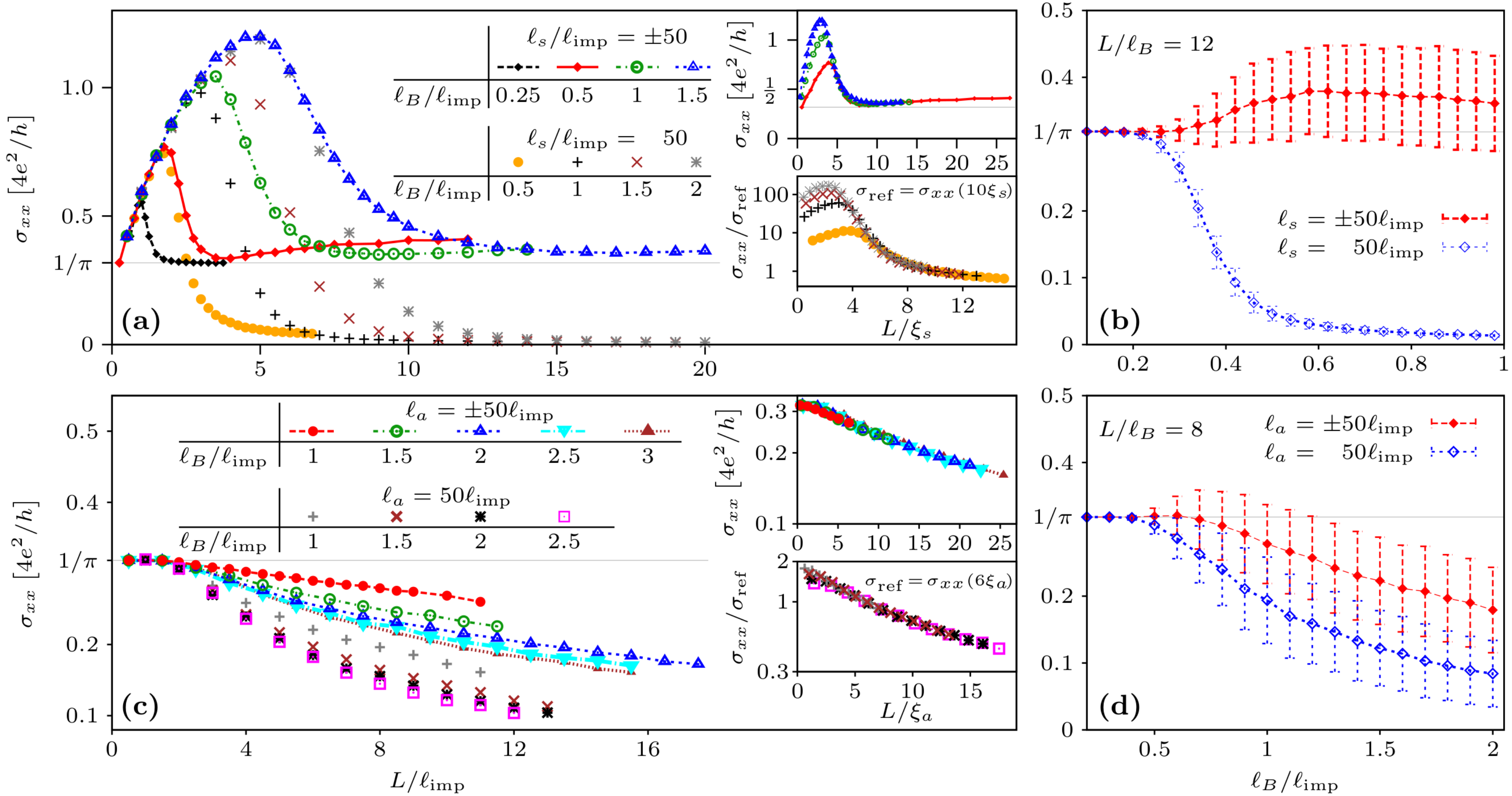
finite $\ell_s \Rightarrow$ class AII (θ)

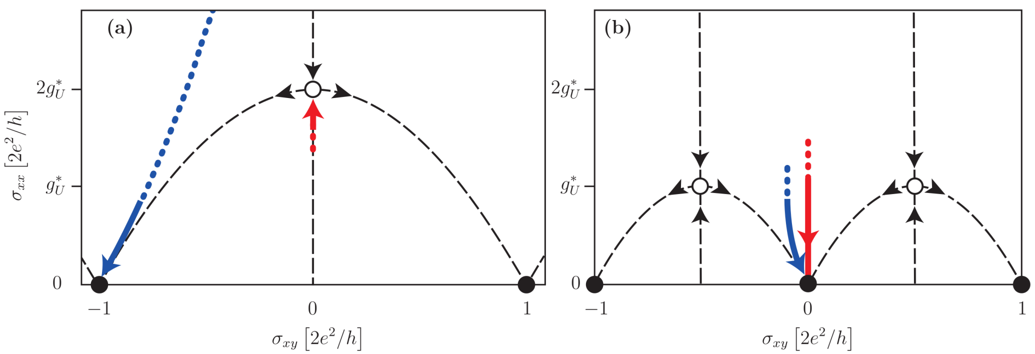
intervalley scattering ("adatoms")

scatttering length ℓ_a

$\ell_a \to \infty \Rightarrow$ class BDI

finite $\ell_a \Rightarrow$ class AI





in magnetic field B:

scalar: DIII → AII → A (weak B)
DIII → AIII → A (strong B)

adatoms: BDI → AI → A (weak B)
BDI → AIII → A (strong B)

Topological Insulators: $\mathbb{Z}$ and $\mathbb{Z}_2$

Topological Insulators

= Bulk insulators with topologically protected delocalized states on their boundary

Theory: Moore, Balents; Kane, Mele; Bernevig, Zhang; Schnyder, Ryu, Furusaki, Ludwig; Kitaev; . . .

Well-known example: Quantum Hall Effect (2D, class A)

QH insulators $\longrightarrow$ $n = \ldots, -2, -1, 0, 1, 2, \ldots$ **edge states**

$\longrightarrow$ $\mathbb{Z}$ **topological insulator**

$\mathbb{Z}_2$ **TIs:** $n = 0$ **or** $n = 1$

Recent experimental realizations: Molenkamp & Hasan groups

2D and 3D systems with strong spin-orbit interaction (class AII)

2D: Quantum Spin Hall Effect

Classification of Topological insulators

How to detect existence of TIs of class p in d dimensions?

$\longrightarrow$ **analyze homotopy groups of the σ-model manifolds $\mathcal{M}_p$:**

$$\begin{cases} \text{TI of type } \mathbb{Z} \iff \pi_d(\mathcal{M}_p) = \mathbb{Z} & \text{Wess-Zumino term} \\ \text{TI of type } \mathbb{Z}_2 \iff \pi_{d-1}(\mathcal{M}_p) = \mathbb{Z}_2 & \theta = \pi \;\; \text{topological term} \end{cases}$$

WZ and $\theta = \pi$ terms make boundary excitations "non-localizable"

TI in d $\iff$ topological protection from localization in $d-1$

alternative approach (Kitaev):
topology in the space of Bloch Hamiltonians

Periodic table of Topological Insulators

	Symmetry classes		Topological insulators				
p	H_p	$\mathcal{M}_p$	d=1	d=2	d=3	d=4	d=5
0	AI	CII	0	0	0	$\mathbb{Z}$	0
1	BDI	AII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
2	BD	DIII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
3	DIII	BD	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
4	AII	BDI	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
5	CII	AI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
6	C	CI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
7	CI	C	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
$0'$	A	AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$1'$	AIII	A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$