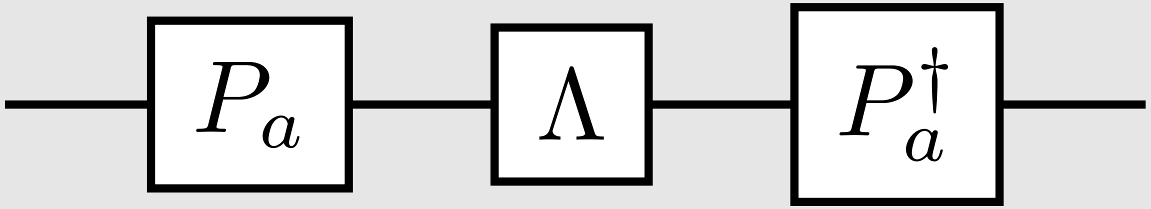
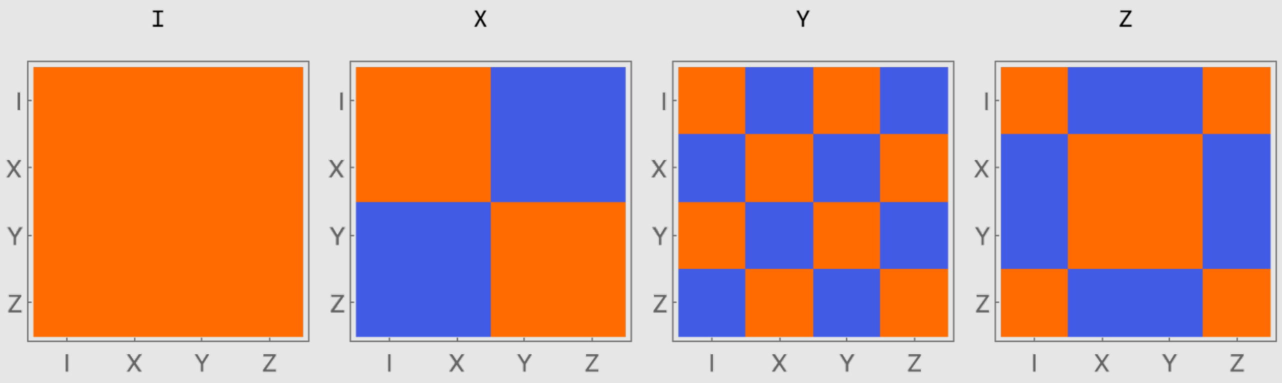
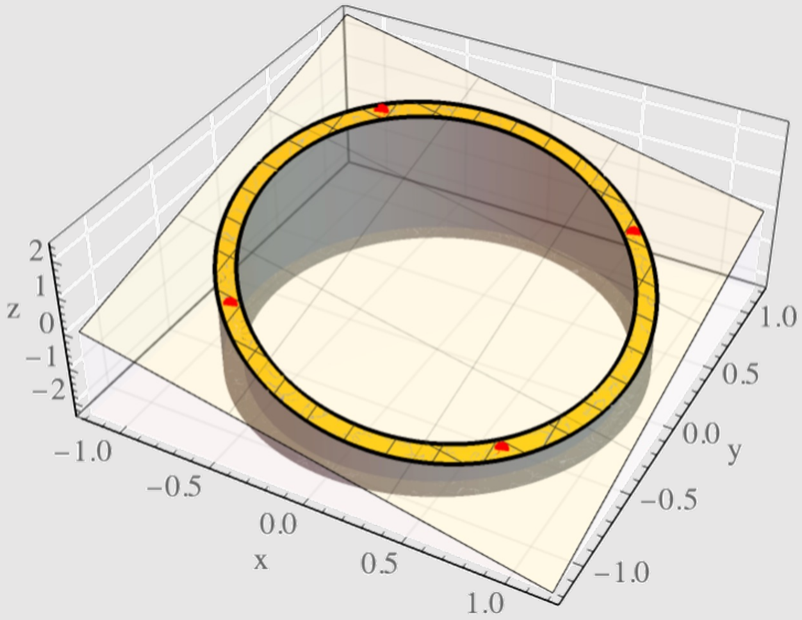
# Primer on Pauli Twirling



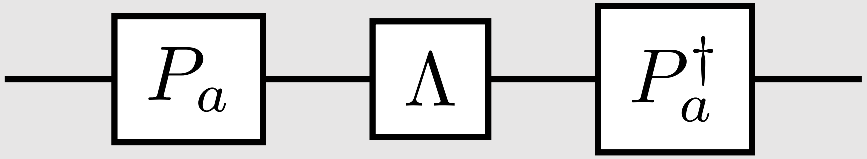

Zlatko Minev

2022-04-20, 07-11

# Twirling 101: Overview

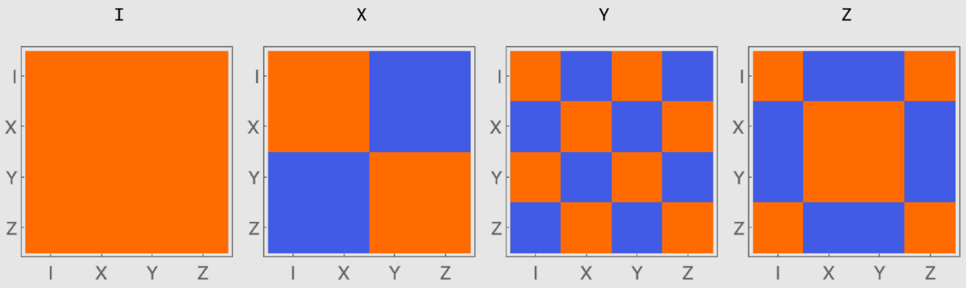
Twirl operationally

- Simple example
- General application
- Summary



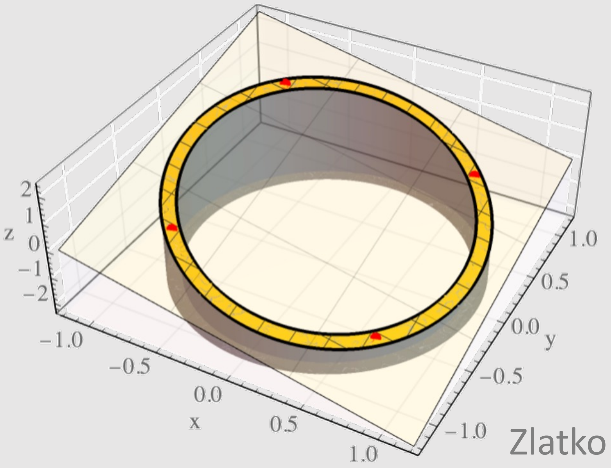

Theory of twirling

- Why does twirling work?
- Masking channels




Optional: Advanced

- Why is the Pauli group special for twirling?
- Other twirl groups
- Designs



# Designs

# Unitary $t$-design

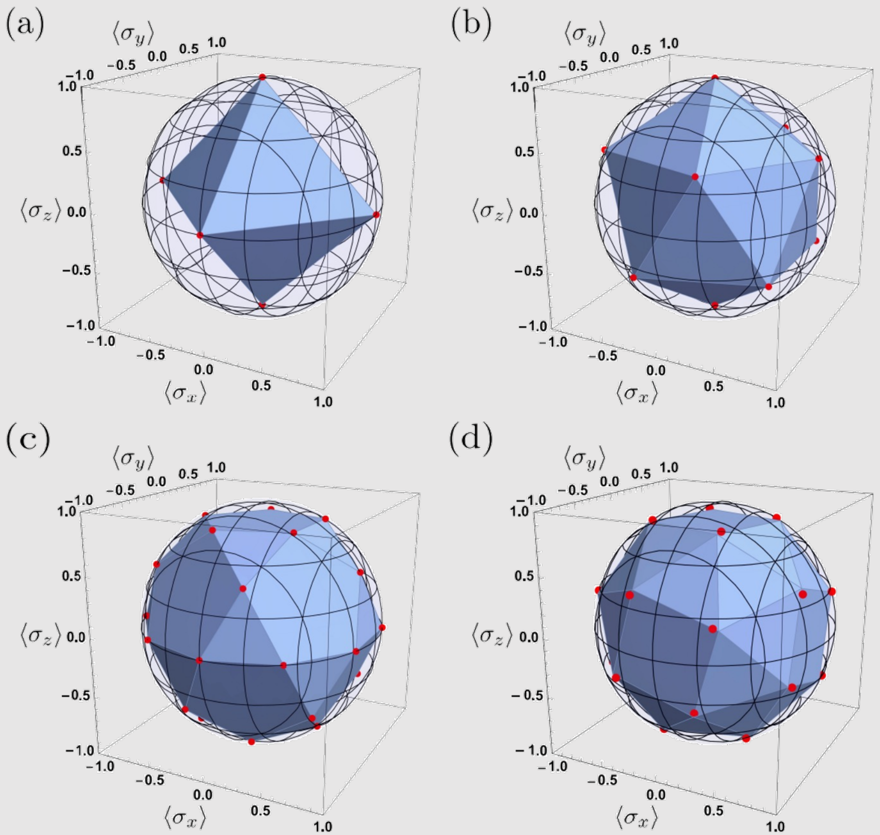
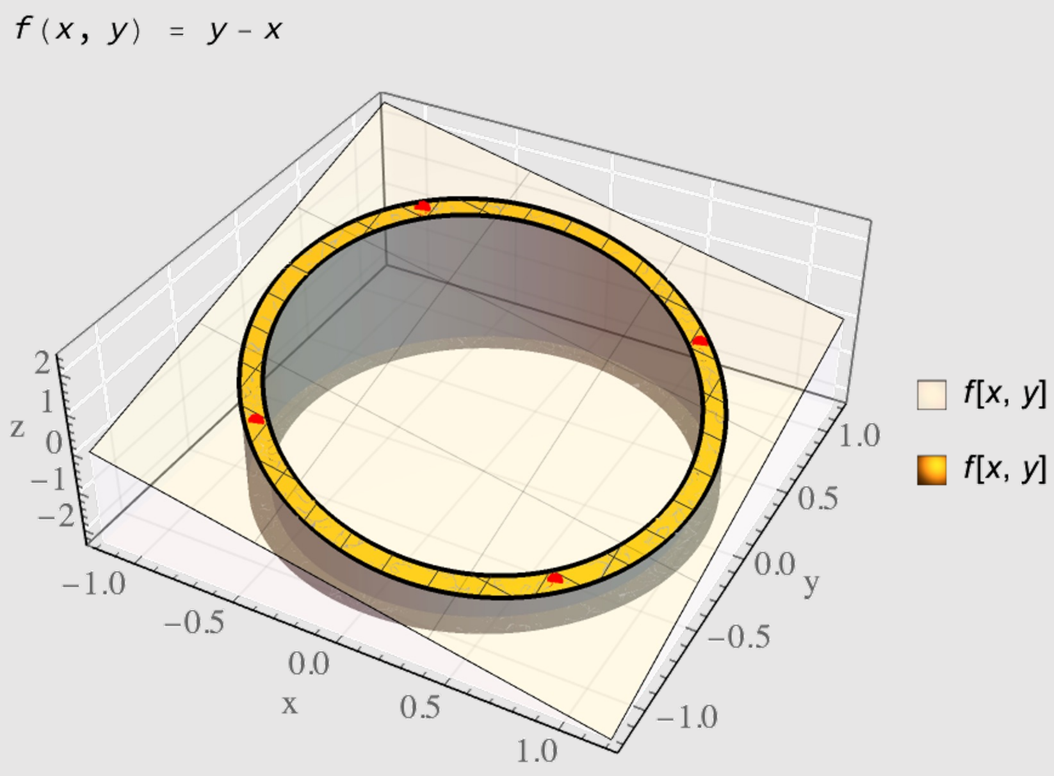



image arxiv: 2004.08402

**Where does this get all used?**

- Average channel fidelity metrics
- RB fidelity
- Random unitaries / matrices
- Random twirling

# Classical Design




Suppose we have some polynomial *f*(*x,y*) in *d=2* variables, and we would like to compute its average over the unit sphere $S_d$.

A spherical design is a set of representative points (i.e. d-dimensional unit vectors) on the surface of the sphere such that computing the average of the function only over these points is identical to taking the average over the entire unit sphere.

A set $X = \{x : x \in \mathcal{S}(\mathbb{R}^d)\}$ is a ***spherical* t*-design*** if

$$\frac{1}{|X|}\sum_{x\in X} p_t(x) = \int_{\mathcal{S}(\mathbb{R}^d)} f_t(u)d\mu(u)$$

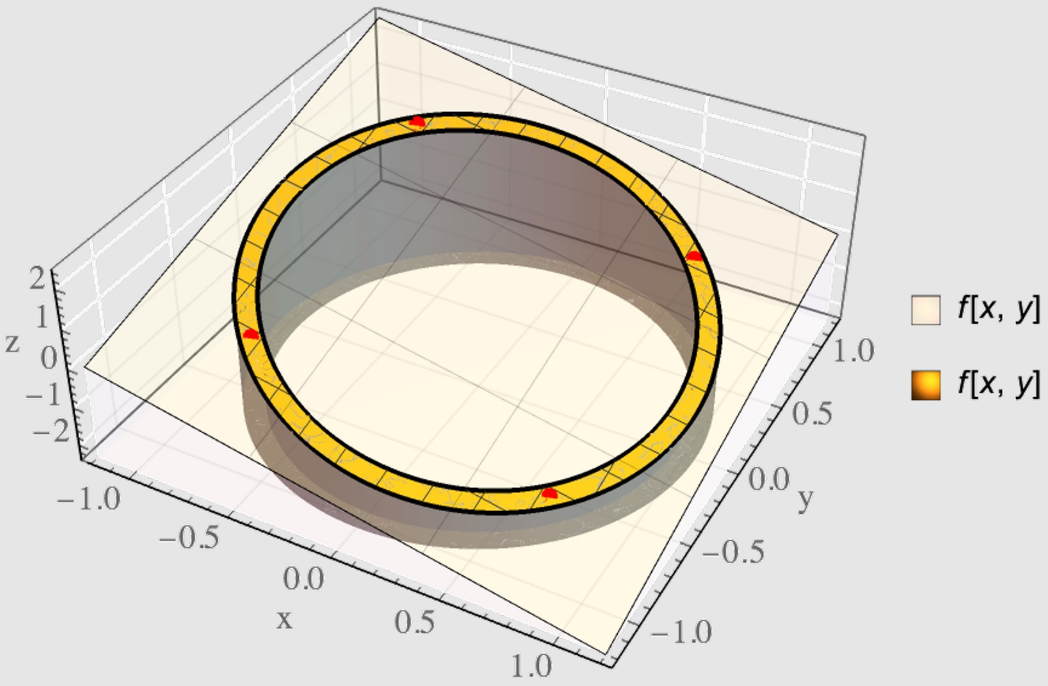
where the polynomial terms are at most degree *t*.

Quantum? We will have polynomials of unitaries.

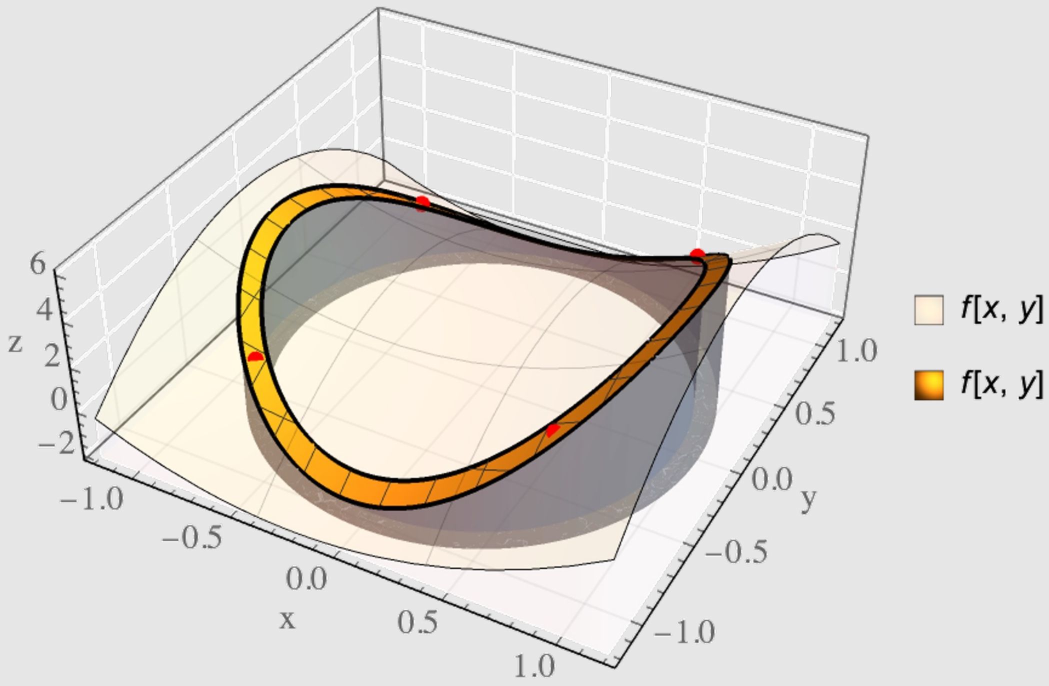
Some statement of symmetries

# Example of a classical design on functions over 2D plane

$f(x, y) = y - x$




$f(x, y) = 2x^2 + x - 3y^2 - 0.2y + 2$




```
N = Integrate[1, {x, y} ∈ Circle[]];
Print["∫ -> ", Integrate[f[x, y], {x, y} ∈ Circle[]] / N]

N = Length[ptsf];
Print["Σ -> ", Sum[x⟦2⟧, {x, ptsf}] / N]
```

∫ -> 0

Σ -> 0

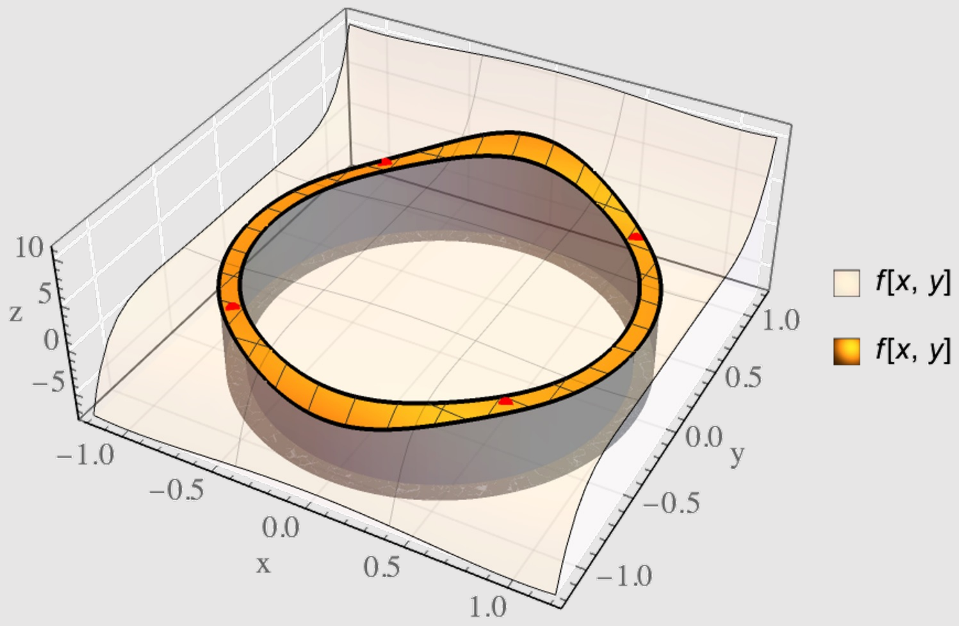
$$\frac{1}{|X|}\sum_{x\in X} p_t(x) = \int_{\mathcal{S}(\mathbb{R}^d)} f_t(u)\,d\mu(u)$$

∫ -> 1.5

Σ -> 1.5

# When does it fail? Higher designs?

$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$

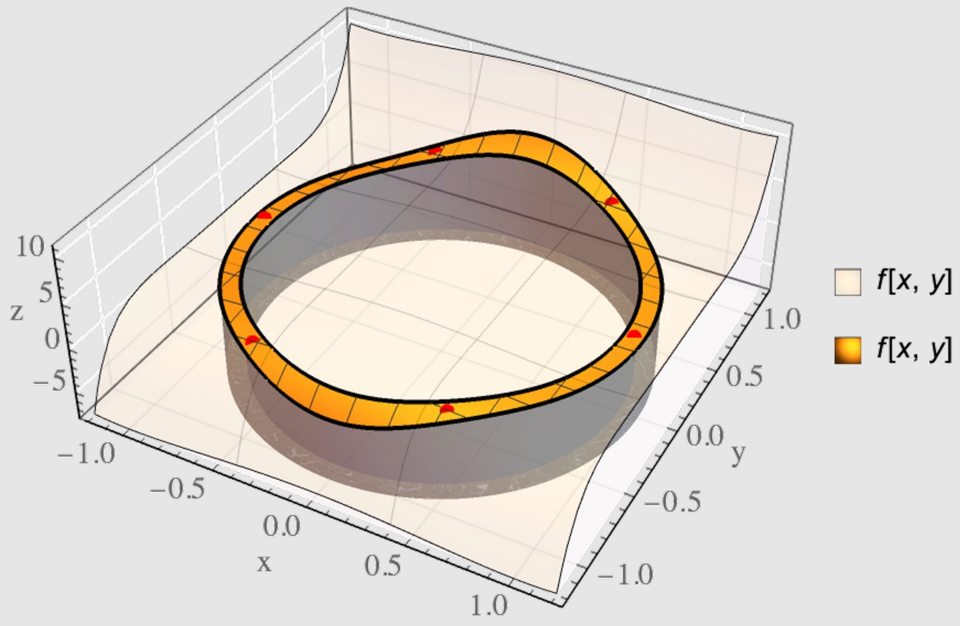



$\int \to \frac{11}{8}$

$\Sigma \to \frac{5}{4}$

Fails with only 4 points in the design set
not $t$>2 design

$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$



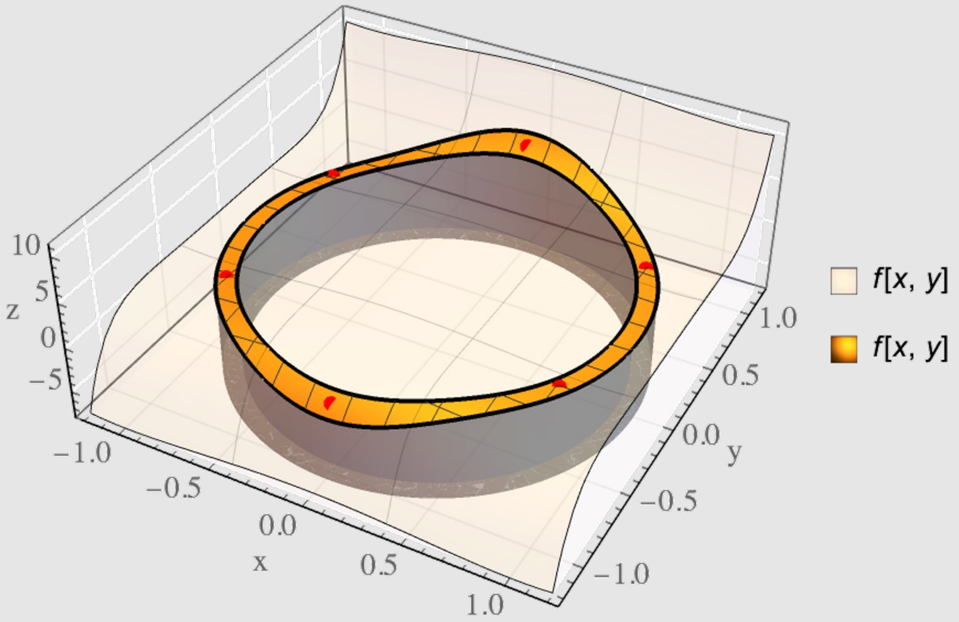

$\int \to \frac{11}{8}$

$\Sigma \to \frac{11}{8}$

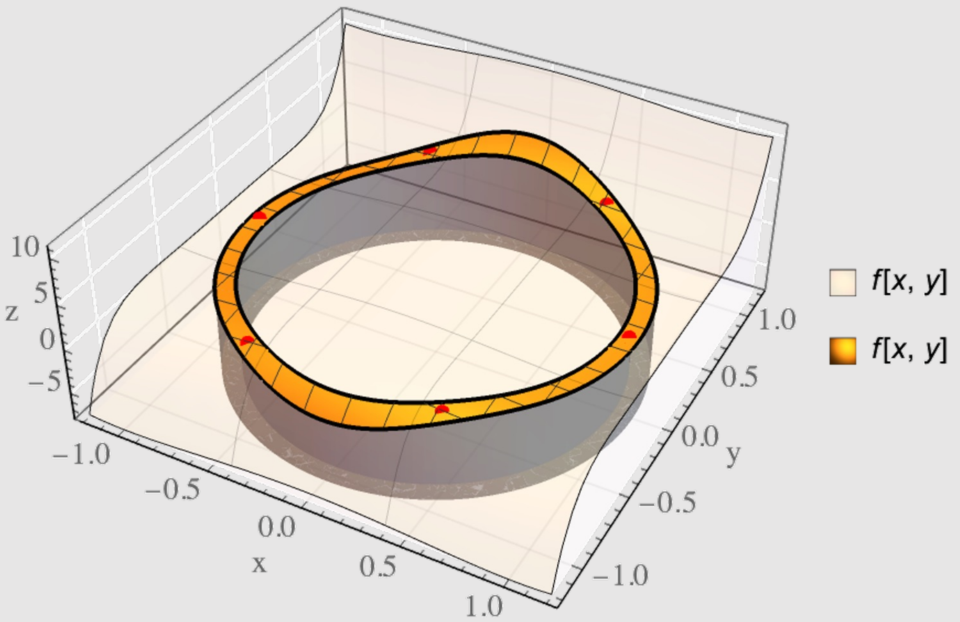
Succeed when we expand design set
to include more

# Rotations?

$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$




$f(x, y) = x^4 - 2x^2 + 3y^7 - y^3 + 2$




Can also rotate points in set so long as they preserve certain properties

```
N = 6; Δθ = 2π/N;
v0 = RotationMatrix[(√2 + 0.1) * Δθ].(1, 0);
vs = Table[Flatten[RotationMatrix[Δθ * n].v0], {n, 0, N - 1}];
```

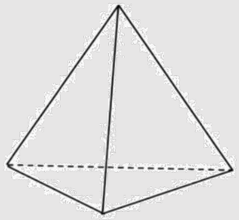
$\int \;\text{->}\; \frac{11}{8}$

$\Sigma \;\text{->}\; \frac{11}{8}$

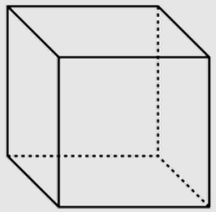
Succeed when we expand design set to include more

# Higher dimensions?

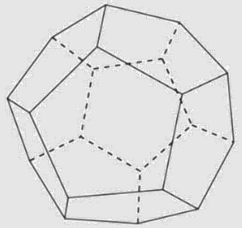
Classical 3- space



2 design



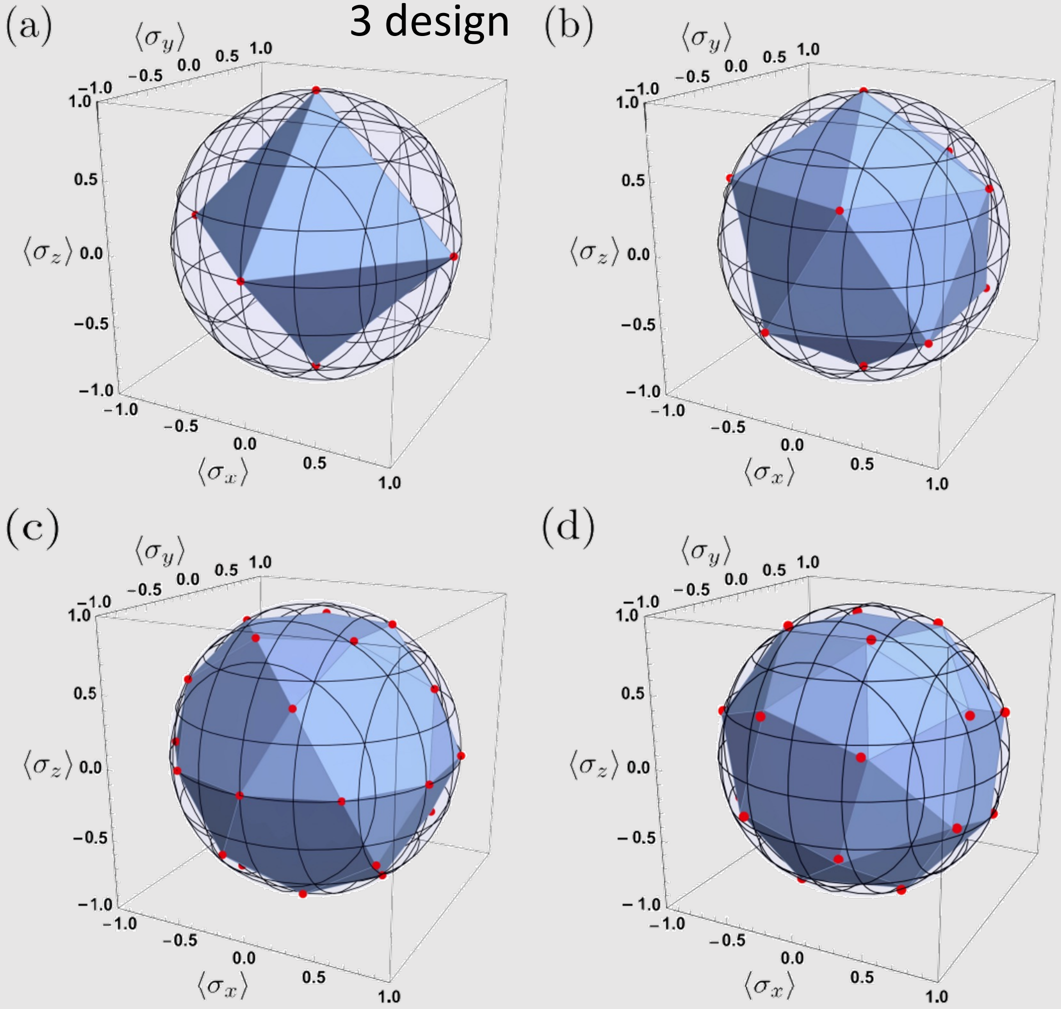
3 design



5 design

Quantum Bloch sphere

3 design




**Where does this get all used?**

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arxiv: 2004.08402

* pikisuperstar

# Refresher

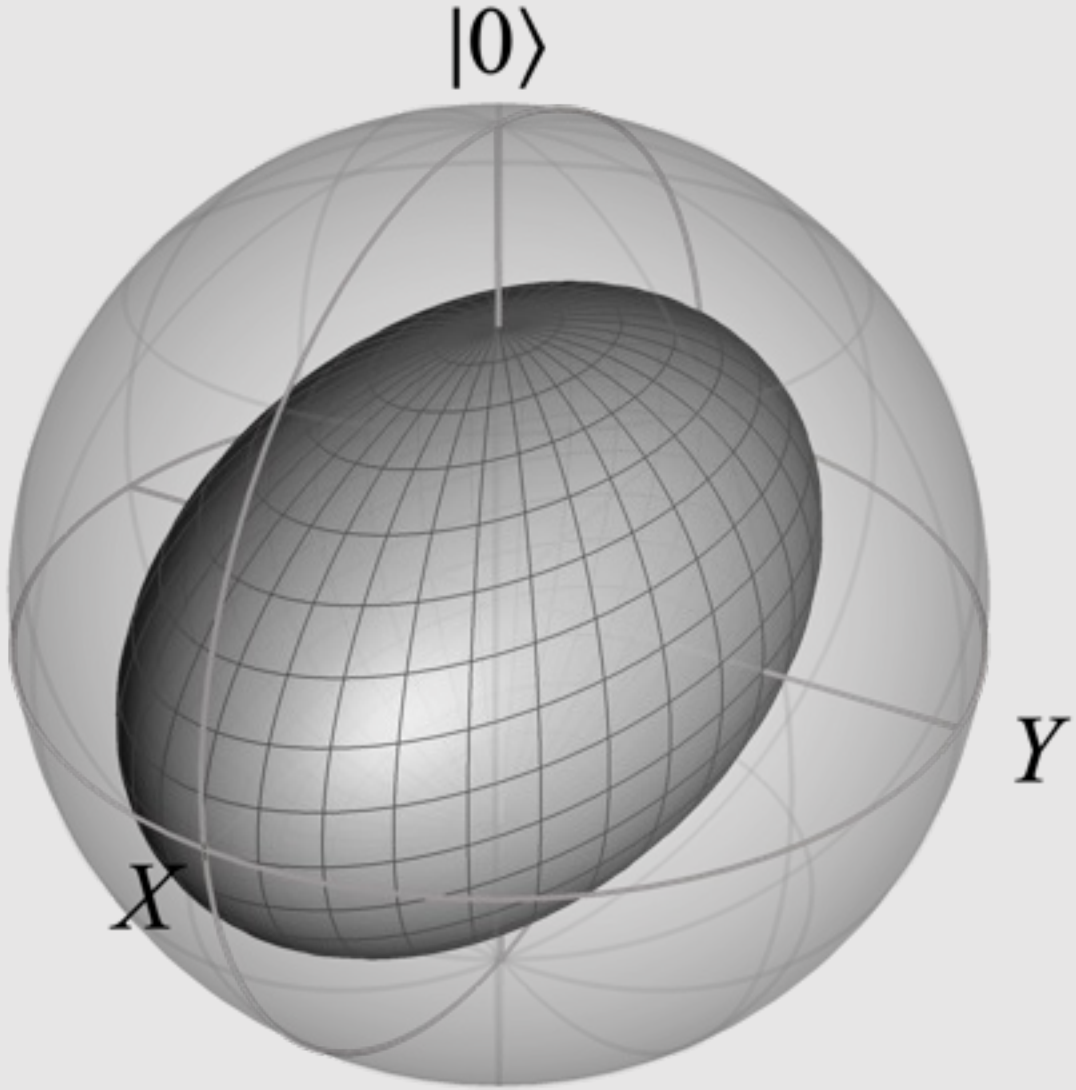
# Modeling noise 101

# How do we model noise?

X bit-flip noise

$$\underbrace{-\boxed{\mathcal{E}}-}$$

| probability | circuit instance |
|---|---|
| $1-p$ | $-\boxed{I}-$ |
| $p$ | $-\boxed{X}-$ |

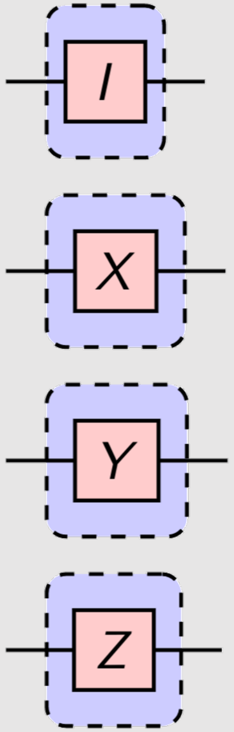


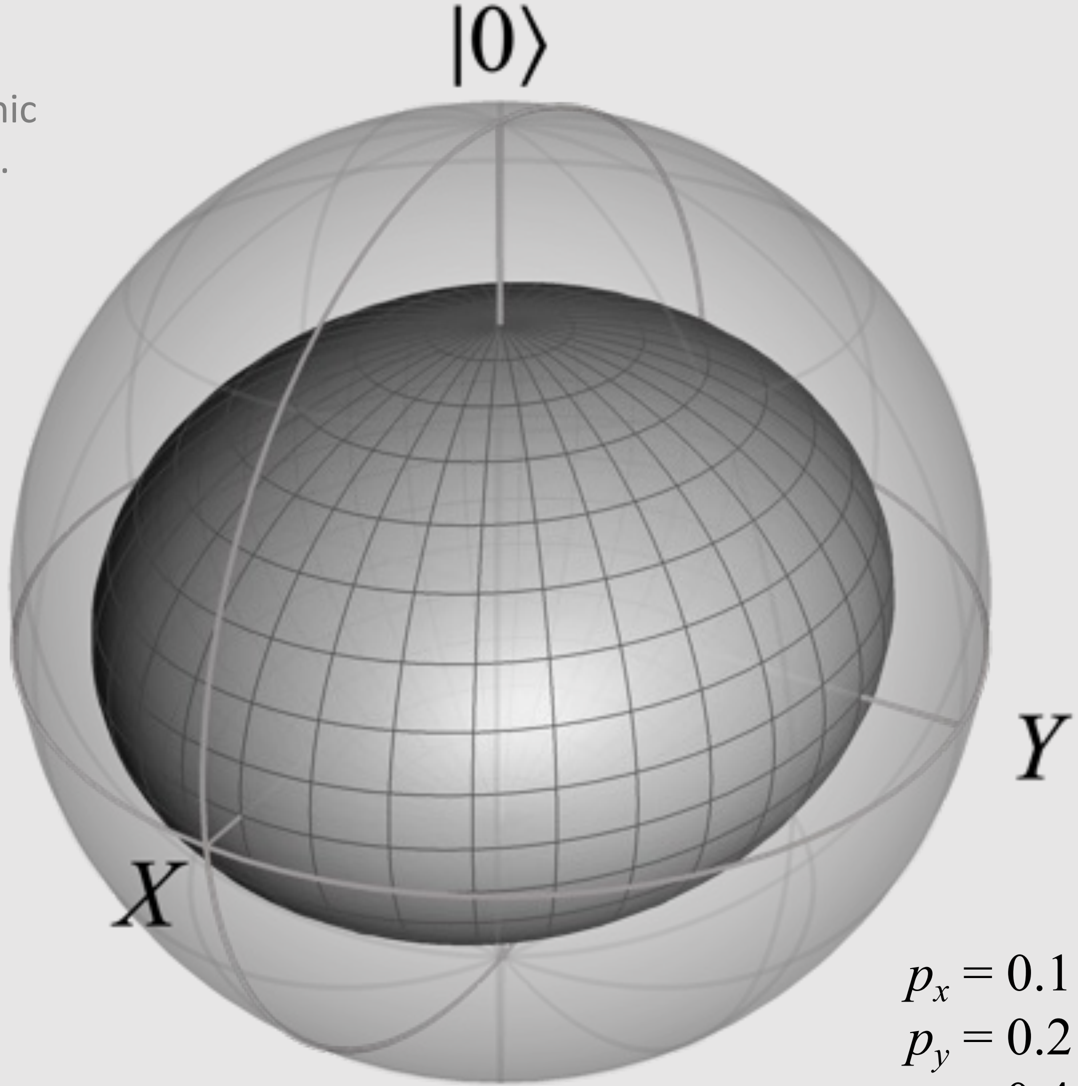
# How do we model noise?

Stochastic Pauli noise


* momentarily i will use calligraphic capital E instead of Greek lambda.

| probability | circuit instance |
| --- | --- |
| $1 - p_X - p_Y - p_Z$ | |
| $p_X$ | |
| $p_Y$ | |
| $p_Z$ | |



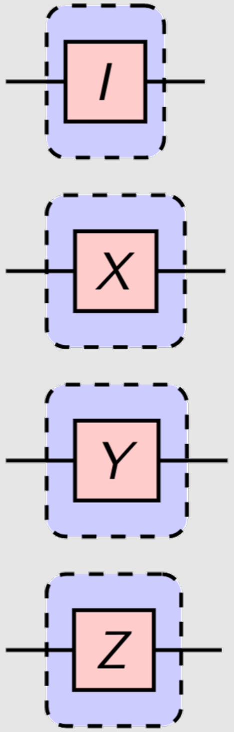




$p_x = 0.1$
$p_y = 0.2$
$p_z = 0.4$

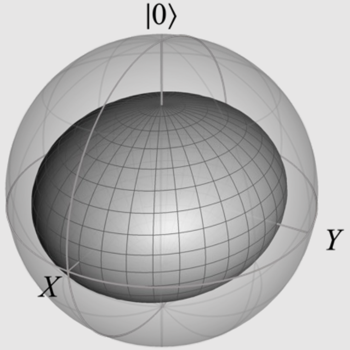
# How do we model noise?

Stochastic Pauli noise


| probability | circuit instance |
|---|---|
| $1 - p_X - p_Y - p_Z$ | |
| $p_X$ | |
| $p_Y$ | |
| $p_Z$ | |




Picture



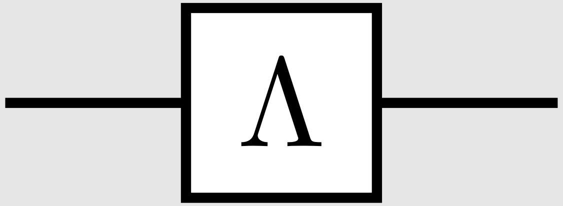

Math

$$\mathcal{E}(\rho) = \chi_I \hat{I}\rho\hat{I} + \chi_X \hat{X}\rho\hat{X} + \chi_X \hat{Y}\rho\hat{Y} + \chi_X \hat{Z}\rho\hat{Z}$$

or

$$\mathcal{E}(\rho) = \sum_{a\in\{I,X,Y,Z\}} p_a \hat{P}_a \rho \hat{P}_a$$

# General noise




**Process matrix representation**

aka. chi matrix (related to Kraus representation)

$$\Lambda(\rho) = \sum_{a,b\in\{I,X,Y,Z\}^{\otimes n}} \chi_{ab}\hat{P}_a\rho\hat{P}_b$$

**Matrix elements of the Pauli transfer matrix (PTM)**

$$\mathrm{PTM}(\Lambda)_{ab} := \frac{1}{2^n}\mathrm{Tr}\left[P_a^\dagger\Lambda\left(P_b\right)\right] = \frac{1}{2^n}\langle\!\langle P_a|\Lambda|P_b\rangle\!\rangle$$

* I dropped the hats

# Amplitude damping

## 4D-2. Amplitude damping

Photon loss channel. The probability of loosing a photon $0 \leq p \leq 1$ is given by the noise map

$$\mathcal{A}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \,, \quad \text{where } A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} . \tag{4.5}$$

The no-loss and photon loss channels in terms of Paulis are $A_0 = \left(1 + \sqrt{1-p}\right) \hat{I} + \left(1 - \sqrt{1-p}\right) \hat{Z}$ and $A_1 = \sqrt{p} \left( \hat{X} + i \hat{Y} \right)$, resp.

| **Kraus $K_i$** | **Pauli transfer matrix (PTM)** | **Process $\chi$-matrix** |
|---|---|---|
| $K_0 = \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix}$ <br> $K_1 = \begin{pmatrix} 1 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$ | $\begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{array}$ | $\begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} \frac{1}{4}\left(\sqrt{1-p}+1\right)^2 & & & \frac{p}{4} \\ & \frac{p}{4} & -i\frac{p}{4} & \\ & \text{H.c.} & \frac{p}{4} & \\ \text{H.c.} & & & \frac{1}{4}\left(\sqrt{1-p}-1\right)^2 \end{pmatrix} \end{array}$ |

# Amplitude damping

**Applied *m* times**

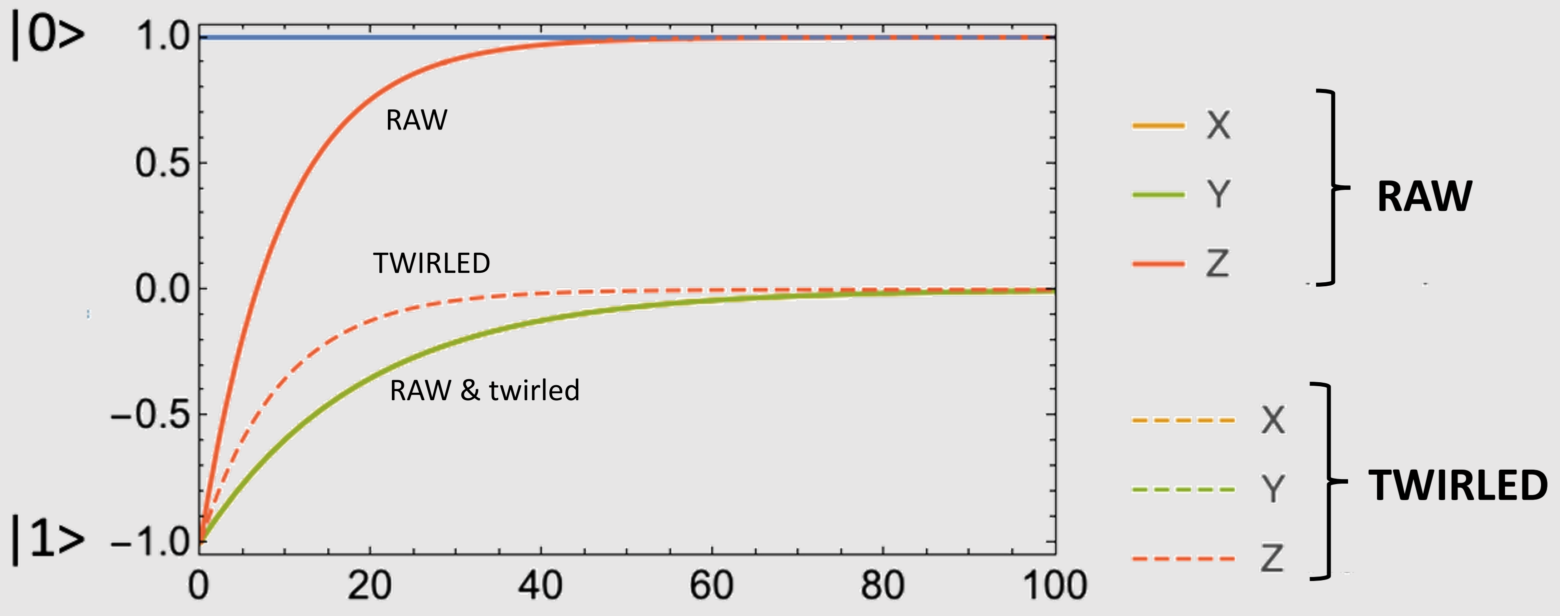
**Pauli transfer matrix (PTM)**

$$\begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{array}$$

**Twirled Pauli transfer matrix (PTM)**

$$\begin{array}{c} \\ I \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{cccc} I & X & Y & Z \end{array} \\ \begin{pmatrix} 1 & & & \\ & \sqrt{1-p} & & \\ & & \sqrt{1-p} & \\ p & & & 1-p \end{pmatrix} \end{array}$$

Starting in -1,-1,-1




AD: $\langle| I \to 1,\ X \to X\,(1-\gamma)^{m/2},\ Y \to Y\,(1-\gamma)^{m/2},\ Z \to 1 + (-1+Z)\,(1-\gamma)^{m} |\rangle$

RC AD: $\langle| I \to 1,\ X \to X\,(1-\gamma)^{m/2},\ Y \to Y\,(1-\gamma)^{m/2},\ Z \to Z\,(1-\gamma)^{m} |\rangle$