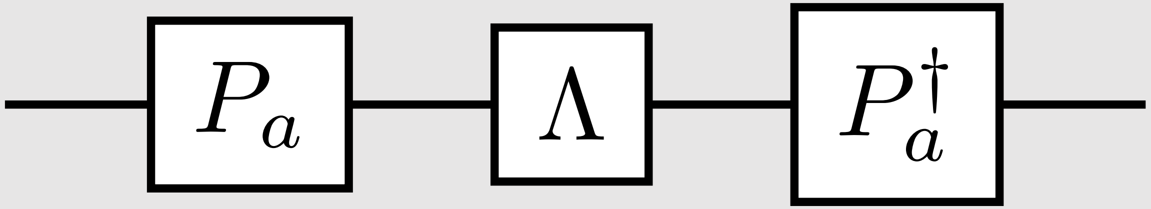
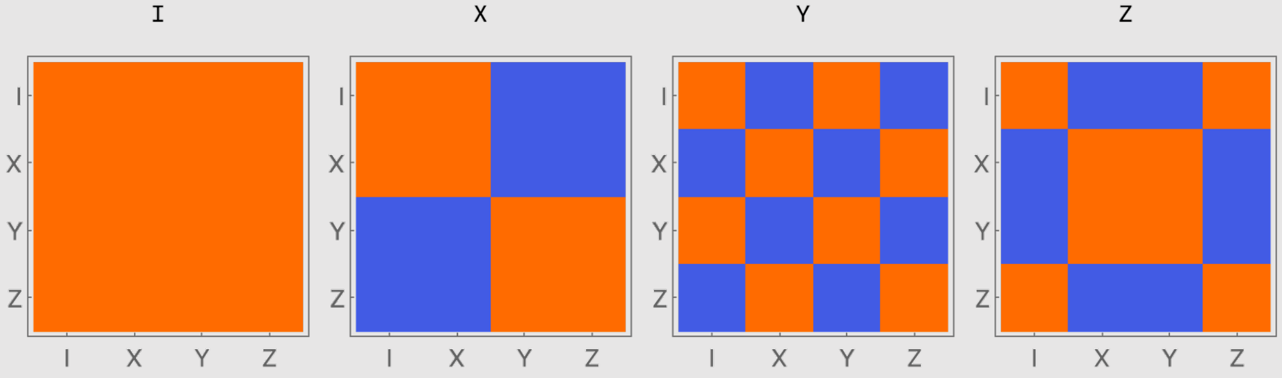
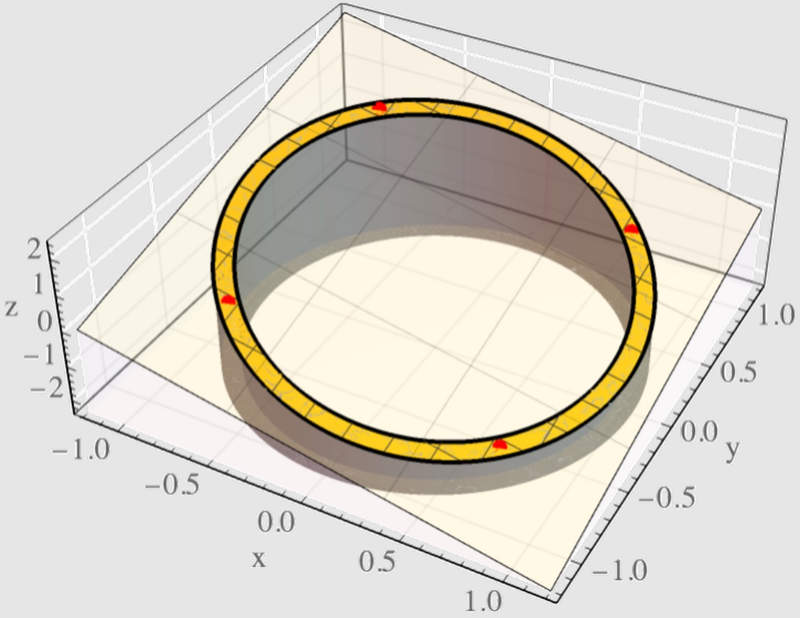
Primer on Pauli Twirling





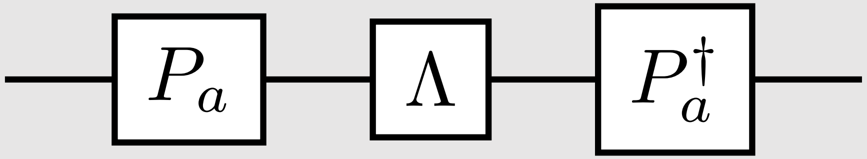


Zlatko Minev

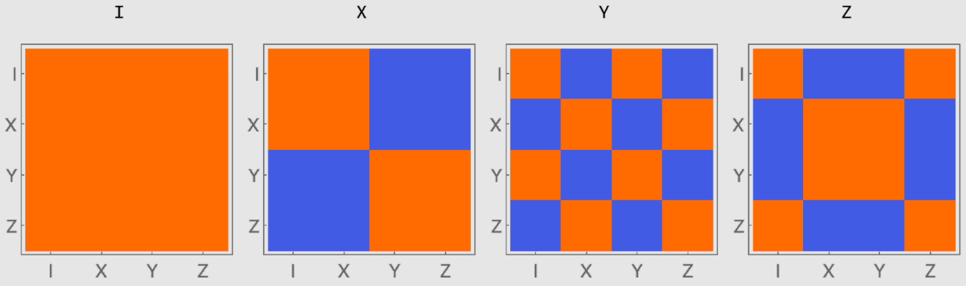
2022-04-20, 07-11

Twirling 101: Overview

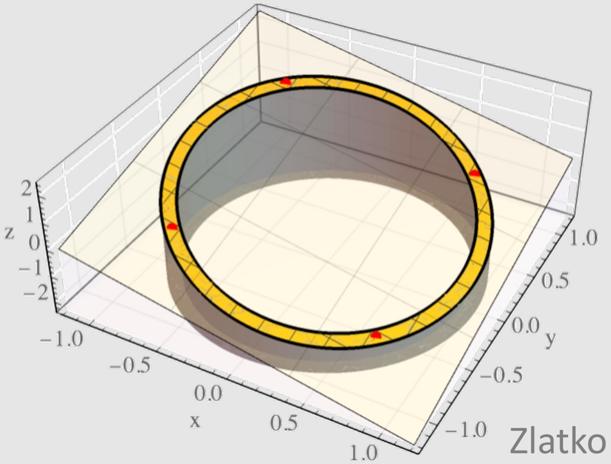
Twirl operationally
- Simple example
- General application
- Summary



Theory of twirling
- Why does twirling work?
- Masking channels



Optional: Advanced
- Why is the Pauli group special for twirling?
- Other twirl groups
- Designs



* pikisuperstar

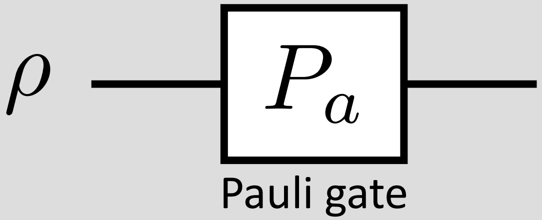
Refresher

More general

Pauli gates & mixed states

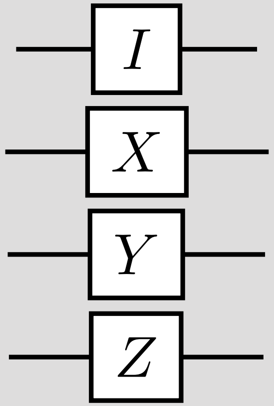
Single-qubit Pauli gate

Pauli gate on a mixed state: conjugation of ρ by Pauli



$$\mathcal{P}_a \rho = P_a \rho P_a^\dagger$$

Single-qubit Pauli set



$$P_a \in \mathbf{P}$$

$$\mathbf{P} := \{I, X, Y, Z\}$$

* by context will use this set as operators or labels

Orthogonal & complete set

$$\langle P_a, P_b \rangle = 2^n \delta_{ab}$$

$$\langle P_a, P_b \rangle = \mathrm{Tr}\left(P_a^\dagger P_b\right)$$

(for all *a*,*b* in the set)

Example decomposition
of a qubit mixed state in terms of Paulis

$$\rho = \frac{1}{2}\left(I + r_X X + r_Y Y + r_Z Z\right)$$

Pauli decomposition of a mixed state (holds for qubits)

$$\rho = \sum_{a \in \mathbf{P}} r_a P_a$$

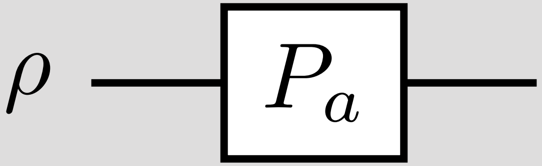
linear vector decomposition onto orthogonal basis

$$r_a = \frac{\langle P_a, \rho \rangle}{\langle P_a, P_a \rangle}$$

Inner product of Hermitian operators $r_a \in \mathbb{R}$

Action of single-qubit Pauli gate on mixed state basis

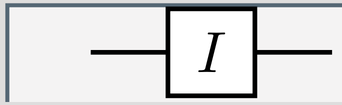
Pauli gate on a mixed state: conjugation of ρ by Pauli



$$\mathcal{P}_a\rho = P_a\rho P_a^\dagger$$

Pauli gate

Conjugation map

		Density matrix component			
	$\mathcal{P}_a\left(P_b\right)$	I	X	Y	Z
	$I\cdot I$	I	X	Y	Z

Basis element by element
linear map

$$\mathcal{P}_a\left(\rho\right) = \sum_b r_b \mathcal{P}_a\left(P_b\right)$$

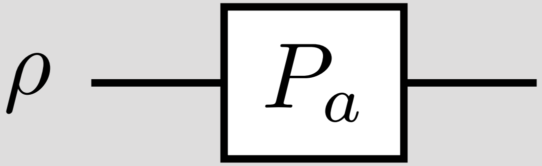
$$= \sum_b r_b P_a P_b P_a$$

(for the experts in the audience, using $Z_2{}^2$ representation)

$$= \left(-1\right)^{\langle a,b\rangle_{\mathrm{Sp}}} r_b P_b$$

Action of single-qubit Pauli gate on mixed state basis

Pauli gate on a mixed state: conjugation of ρ by Pauli



Pauli gate

$$\mathcal{P}_a \rho = P_a \rho P_a^\dagger$$

Conjugation map

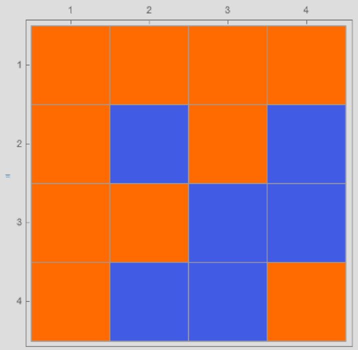
Gate	$\mathcal{P}_a(P_b)$	**Density matrix component**			
		I	X	Y	Z
I	$I \cdot I$	I	X	Y	Z
X	$X \cdot X$	I	X	$-Y$	$-Z$
Y	$Y \cdot Y$	I	$-X$	Y	$-Z$
Z	$Z \cdot Z$	I	$-X$	$-Y$	Z

Walsh-Hadamard transform

$$H_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

*caution: ordering

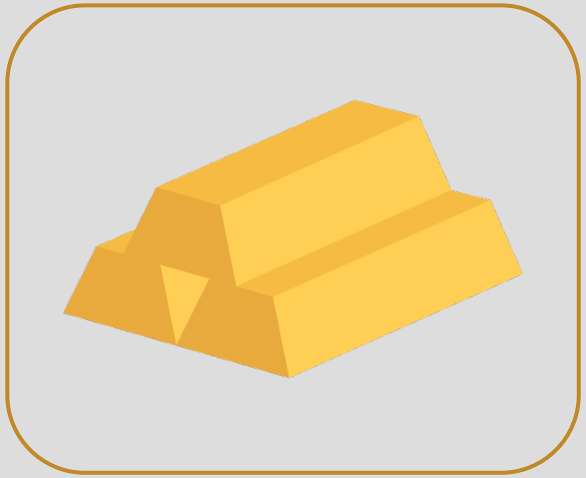
matrix plot



- equivalent to a multidimensional DFT of size 2^n
- +1, -1 eigenvalues
- an orthogonal, symmetric, involutive, linear operation on 2^n real numbers
- note relation of our matrix to symplectic product Z_2^2 representation

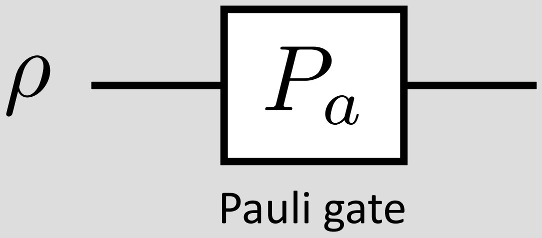
Pauli's are Gold!

$$P_a =$$



Action of single-qubit Pauli gate on mixed state basis

Pauli gate on a mixed state: conjugation of ρ by Pauli



$$\mathcal{P}_a \rho = P_a \rho P_a^\dagger$$

Conjugation map

Gate	$\mathcal{P}_a\left(P_b\right)$	**Density matrix component**			
		I	X	Y	Z
I	$I \cdot I$	I	X	Y	Z
X	$X \cdot X$	I	X	$-Y$	$-Z$
Y	$Y \cdot Y$	I	$-X$	Y	$-Z$
Z	$Z \cdot Z$	I	$-X$	$-Y$	Z

Superoperator lens

$\rho \mapsto \vert\rho\rangle\!\rangle$	vec
$P_a \mapsto \vert P_a\rangle\!\rangle$	vec
$P_a \cdot P_a^\dagger \mapsto \mathcal{P}_a$	op
$P_a \rho P_a^\dagger \mapsto \mathcal{P}_a \vert\rho\rangle\!\rangle$	

Key vectorization identity (row stacking)

$$\operatorname{vec}\left(A_0 B A_1^\intercal\right) = \left(A_0 \otimes A_1\right) \operatorname{vec}\left(B\right)$$

$$\operatorname{vec}\left(P_a \rho P_a^\dagger\right) = \left(P_a \otimes P_a^*\right) \operatorname{vec}\left(\rho\right)$$

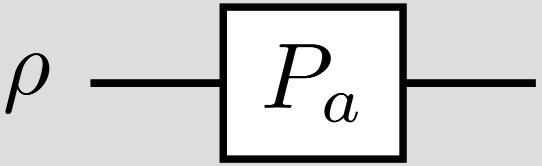
Use as basis elements of Op(*H*) and Op(Op(*H*))

$$\operatorname{Tr}\left(P_a^\dagger \cdot\right) = \langle\!\langle P_a | \cdot$$

$$P_a \operatorname{Tr}\left(P_b^\dagger \cdot\right) = |P_a\rangle\!\rangle\langle\!\langle P_b | \cdot$$

Superoperator Pauli transfer matrix representation

Pauli gate on a mixed state: conjugation of ρ by Pauli



$$\mathcal{P}_a \rho = P_a \rho P_a^\dagger$$

Pauli superoperator

$$\mathcal{P}_a = \sum_b (-1)^{\langle a,b \rangle_{\mathrm{Sp}}} |P_b\rangle\!\rangle \langle\!\langle P_b|$$

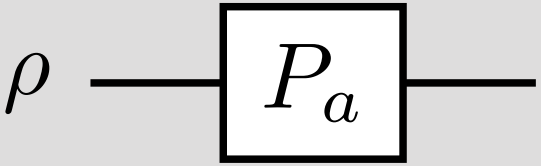
Pauli transfer matrix: chi matrix in the Pauli basis

$$\mathcal{I} : \begin{array}{c|cccc} & I & Z & X & Y \\ \hline I & 1 & & & \\ Z & & 1 & & \\ X & & & 1 & \\ Y & & & & 1 \end{array} \quad \mathcal{Z} : \begin{array}{cccc} I & Z & X & Y \\ \hline 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{array} \quad \mathcal{X} : \begin{array}{cccc} I & Z & X & Y \\ \hline 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{array} \quad \mathcal{Y} : \begin{array}{cccc} I & Z & X & Y \\ \hline 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{array}$$

*caution: ordering is based on binary Z_2^2 notation

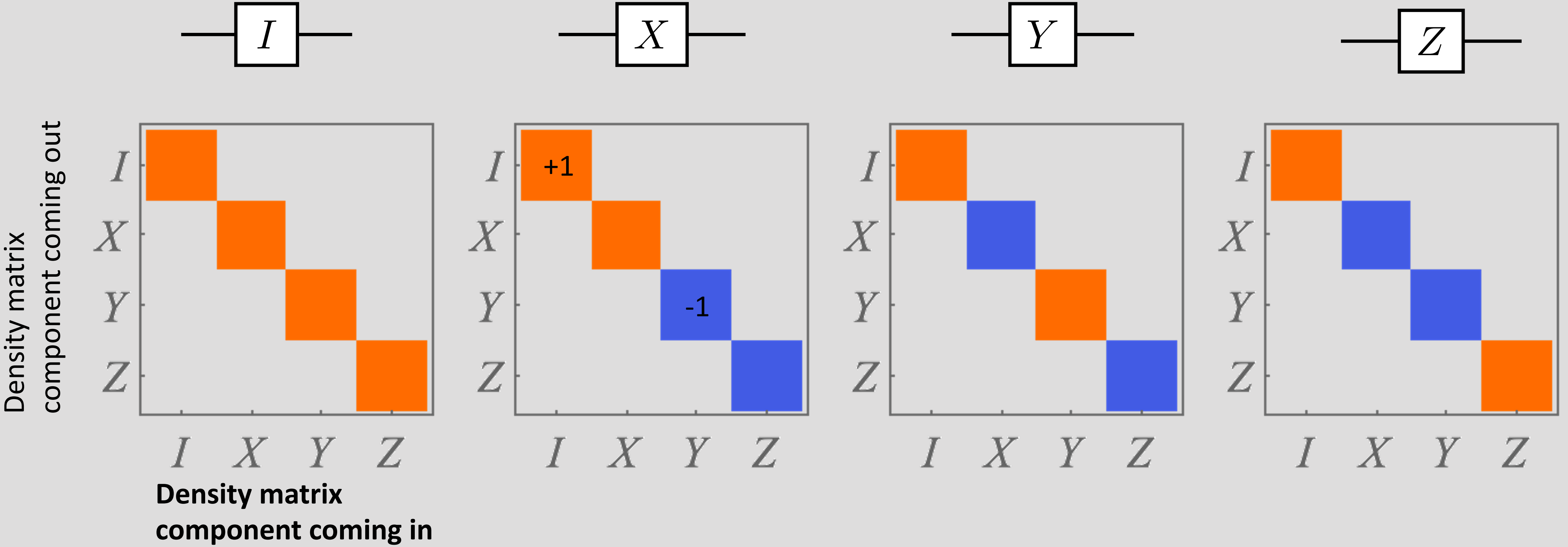
Visualizing the PTM

Pauli gate on a mixed state: conjugation of ρ by Pauli



$$\mathcal{P}_a\rho = P_a \rho P_a^\dagger$$

- diagonals are just columns of the WH matrix



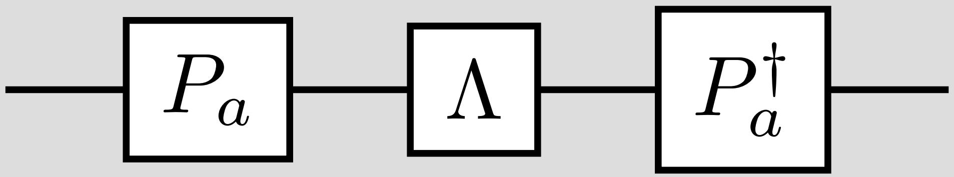
* pininterest

Twirl channel with Pauls

(conjugating a channel by a Pauli gate)

Twirl: acting on super operators

Super-super operators



Algebraic expression of channel sequence:

$$\mathcal{P}_a \Lambda \mathcal{P}_a^\dagger = P_a^\dagger \Lambda (P_a \cdot P_a^\dagger) P_a$$

Vectorize middle channel

Key vectorization identity (row stacking)

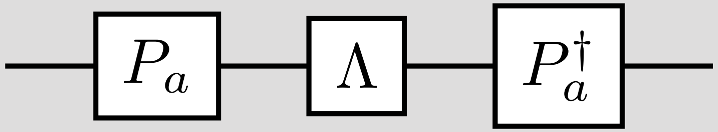
$$\operatorname{vec}(A_0 B A_1^\intercal) = (A_0 \otimes A_1) \operatorname{vec}(B)$$

on basis elements

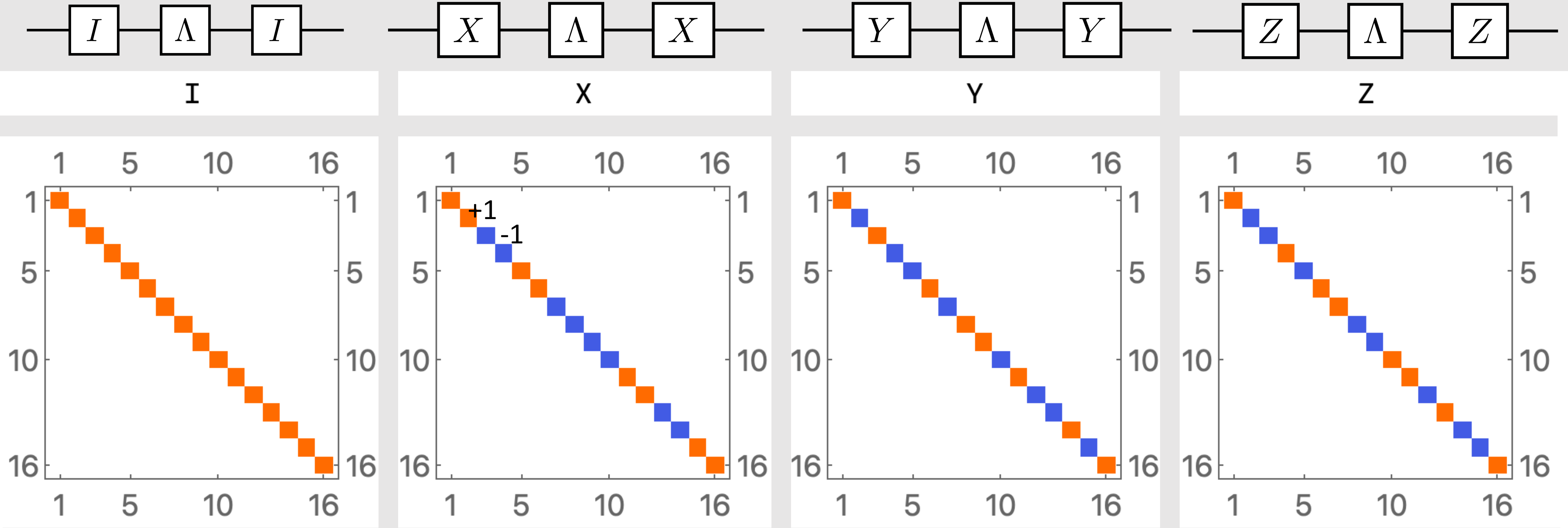
$$|P_a\rangle\rangle\langle\langle P_b| \cdot \mapsto |P_a, P_b\rangle\rangle\rangle\rangle$$

Twirl: acting on super operators

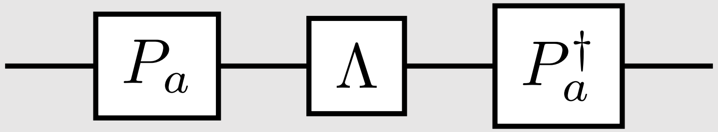
Super-super operators



Single qubits
16 basis elements for superoperators:
II, IX, IY, IZ, XX, XY, ...

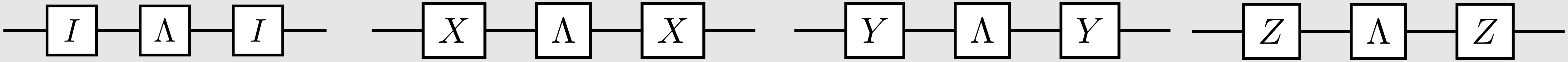


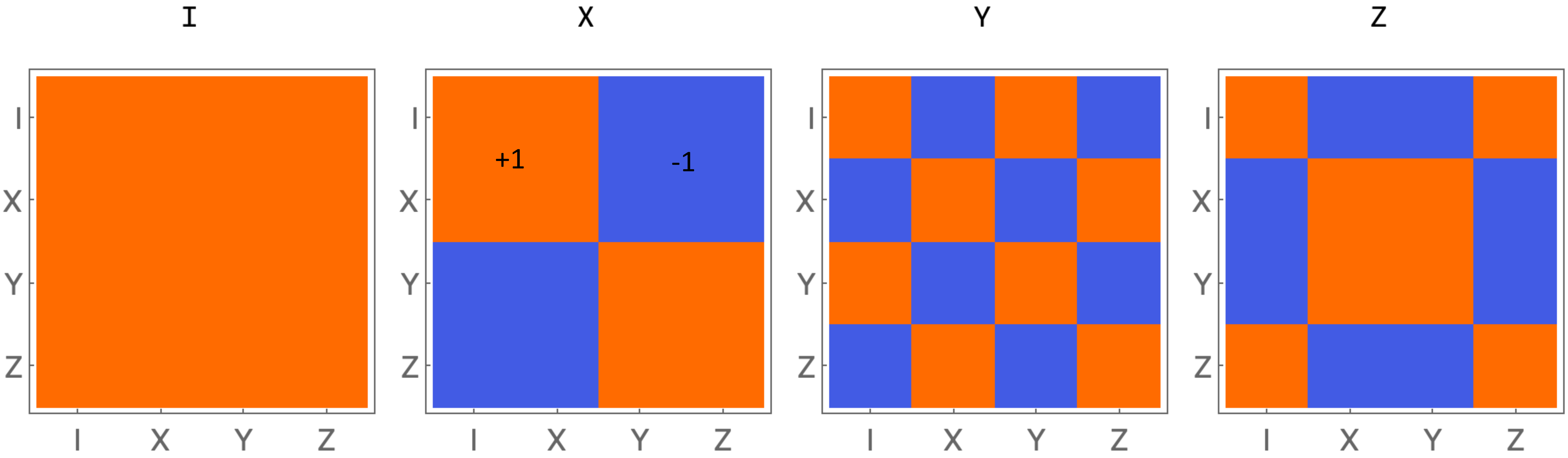
Twirl: acting on super operators



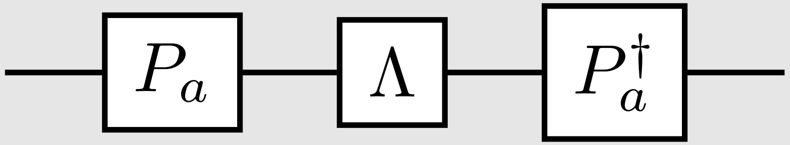
$\mathcal{P}_a \Lambda \mathcal{P}_a^\dagger$

Super super operators

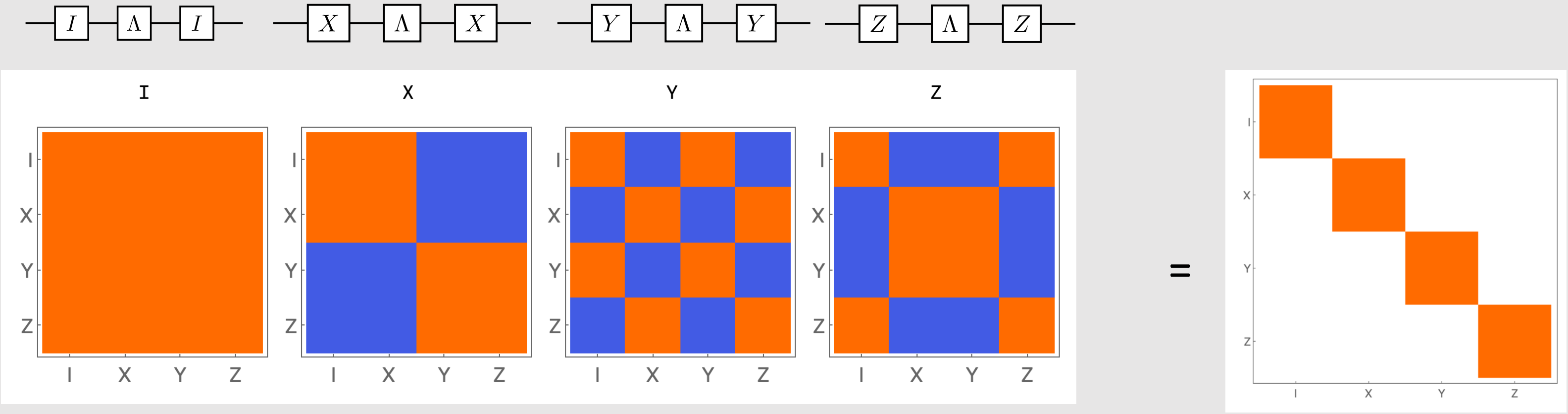




Twirl: acting on super operators



Average over these masks $\mathcal{M}(P_a)$



Directly **MASK** elements of superoperator Lambda in PTM basis

$$\frac{1}{|\Sigma|}\sum_{a\in\Sigma}\mathcal{P}_a\Lambda\mathcal{P}_a^\dagger = \left(\frac{1}{|\Sigma|}\sum_{a\in\Sigma}\mathcal{M}(\mathcal{P}_a)\right)\odot\Lambda = \mathcal{M}\odot\Lambda$$

element-wise product (Hadamard)

Designs