Boulder School for Condensed Matter and Materials Physics: Non-Equilibrium Quantum Dynamics

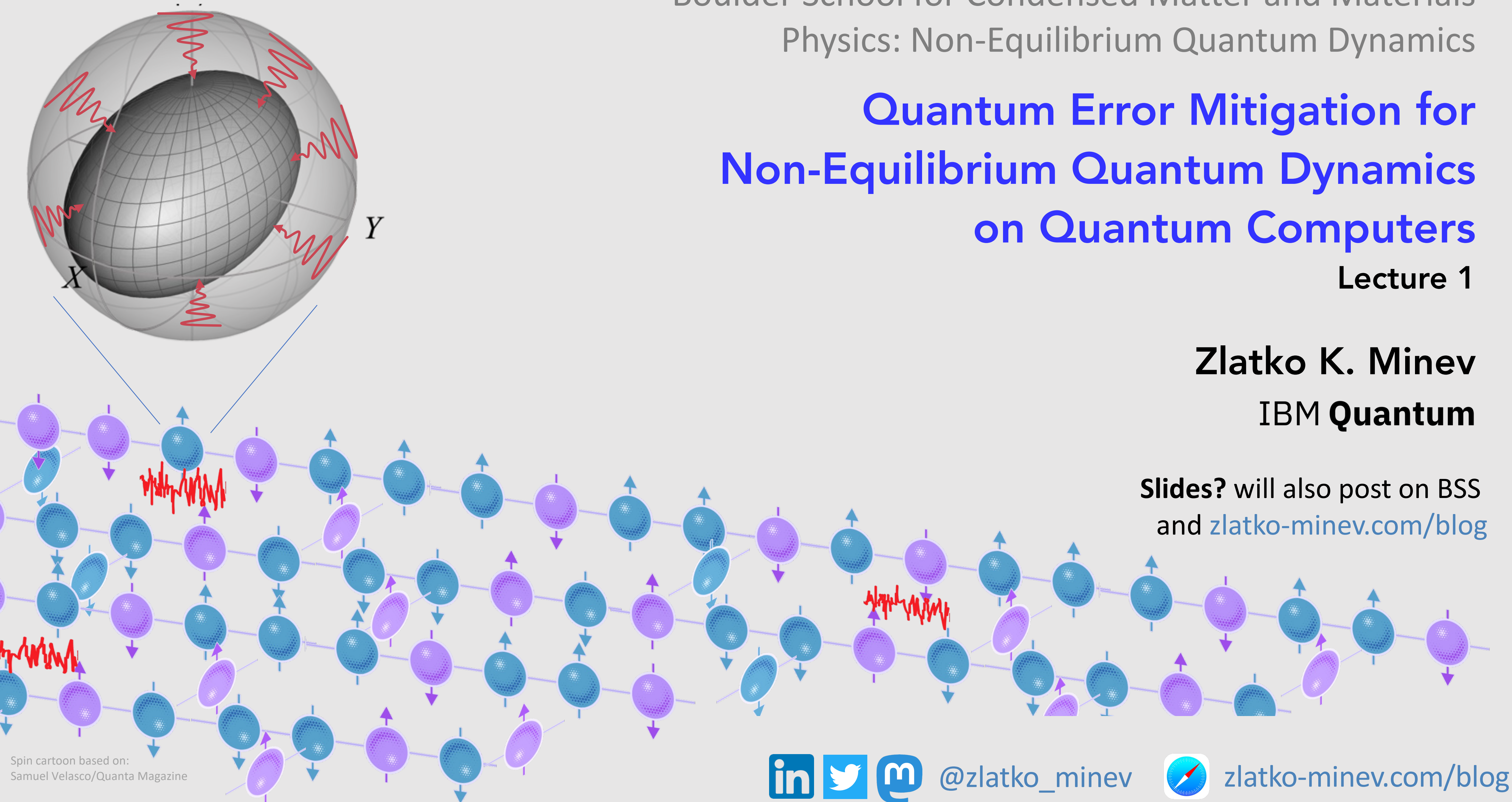
Quantum Error Mitigation for Non-Equilibrium Quantum Dynamics on Quantum Computers

Lecture 1

Zlatko K. Minev

IBM **Quantum**

Slides? will also post on BSS and zlatko-minev.com/blog



Spin cartoon based on:
Samuel Velasco/Quanta Magazine

@zlatko_minev zlatko-minev.com/blog

Where can you find things?

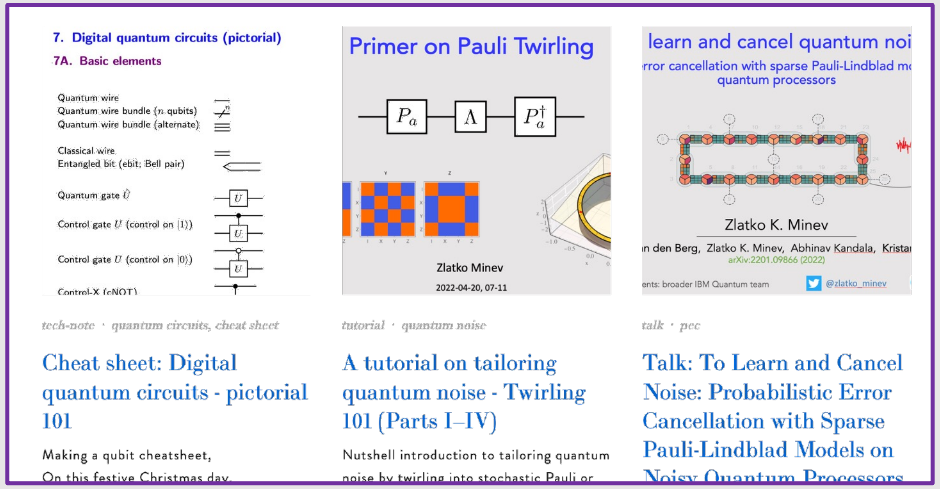
Lecture Slides

Boulder School for Condensed Matter and Materials Physics
boulderschool.yale.edu/2023/boulder-school-2023-lecture-notes

Will also post on zlatko-minev.com/education

Tutorials and additional lecture notes

Twirling, Measurements and Walsh-Hadamard
Cheat sheets, Videos, …
zlatko-minev.com/blog



See also lectures on qiskit.org/learn

Tutorials and additional lecture notes

Latest seminar qiskit.org/events/seminar-series

Have you used a quantum computer?

Image: Freepik

Quantum Error Mitigation for Non-Equilibrium Quantum Dynamics

Lecture 1

Big picture

Why quantum computers?
Status and outlook

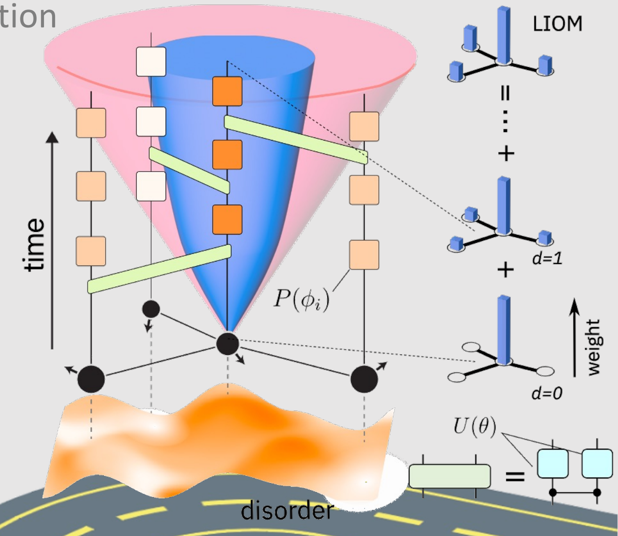
Why error mitigation?
Noise in quantum computers
Overview of error mitigation

Mitigation fundamentals

Probabilistic error cancelation (PEC)
Introduction
One qubit example
General derivation

Next lectures

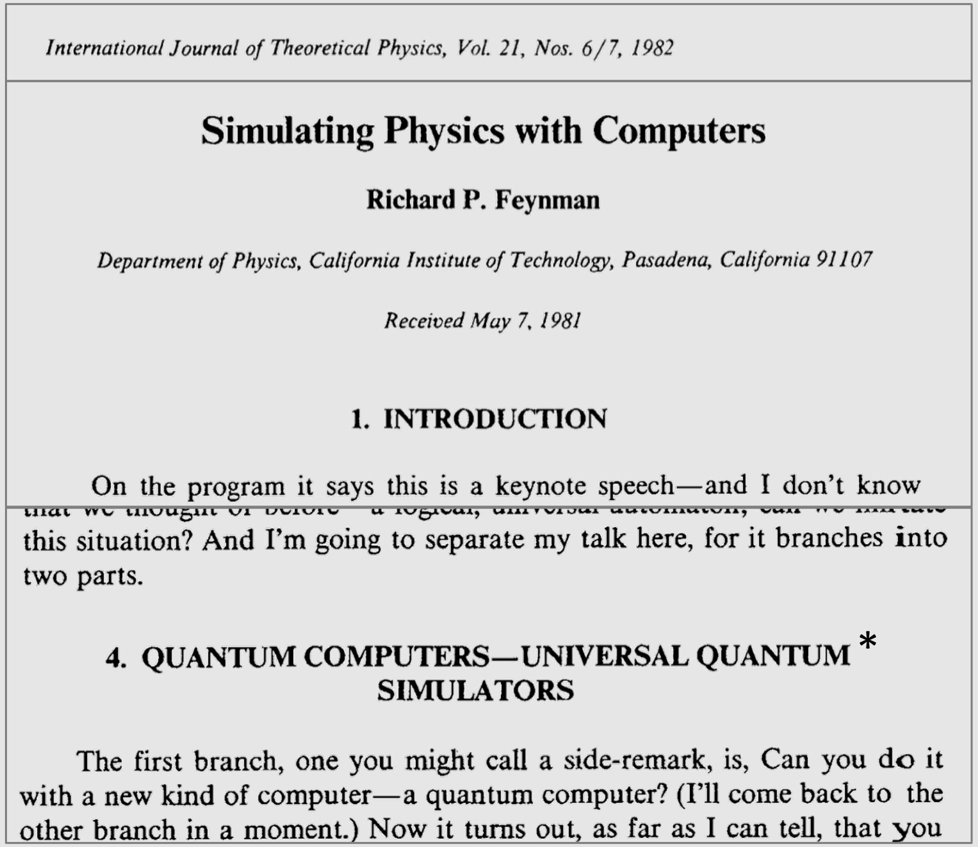
Learning noise
State-of-art PEC experiments
Key techniques: Twirling
T-REX mitigation;
State-of-art experiments at the 120Q+, depth 50+: uncovering local integrals of motion
…



Big picture

“Nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy.”

- R.P. Feynman 1981



International Journal of Theoretical Physics, Vol. 21, Nos. 6/7, 1982

Simulating Physics with Computers

Richard P. Feynman

Department of Physics, California Institute of Technology, Pasadena, California 91107

Received May 7, 1981

1. INTRODUCTION

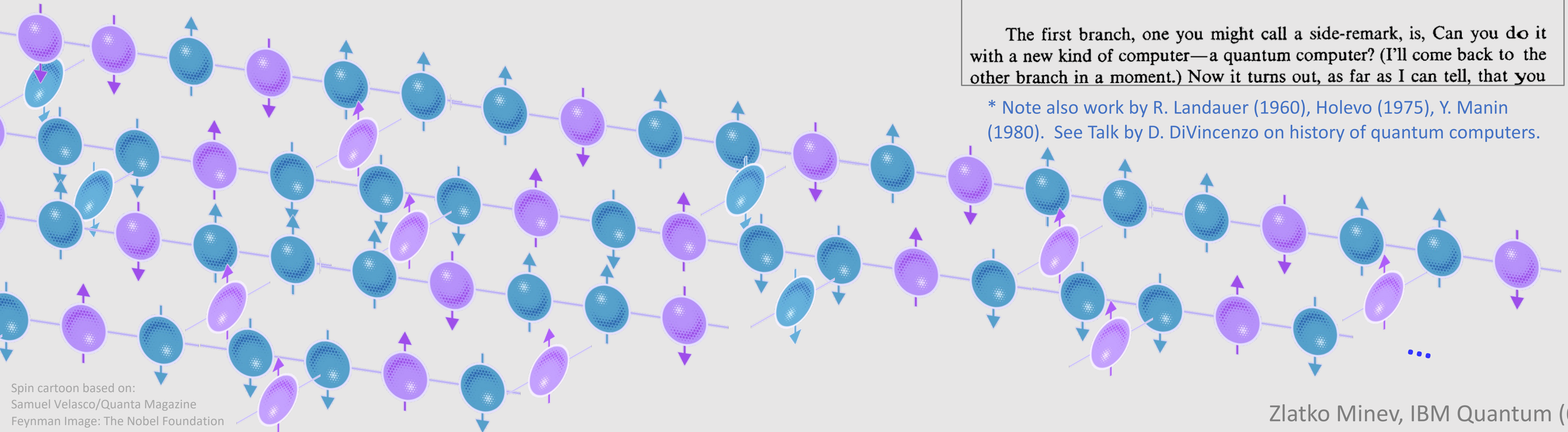
On the program it says this is a keynote speech—and I don't know

this situation? And I'm going to separate my talk here, for it branches into two parts.

4. QUANTUM COMPUTERS—UNIVERSAL QUANTUM SIMULATORS *

The first branch, one you might call a side-remark, is, Can you do it with a new kind of computer—a quantum computer? (I'll come back to the other branch in a moment.) Now it turns out, as far as I can tell, that you

* Note also work by R. Landauer (1960), Holevo (1975), Y. Manin (1980). See Talk by D. DiVincenzo on history of quantum computers.

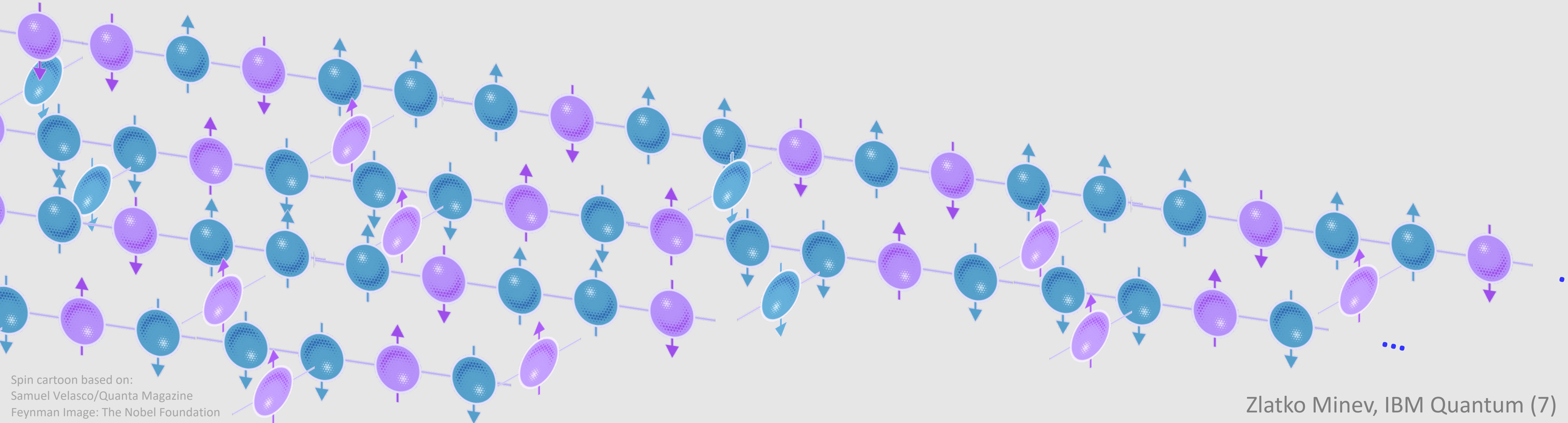


Spin cartoon based on:
Samuel Velasco/Quanta Magazine
Feynman Image: The Nobel Foundation

How is it going for quantum computers?

The last 40 years and looking ahead to 10 more

1980 – . . . – 2010 – 2020 – 2023 in 60 seconds



Spin cartoon based on:
Samuel Velasco/Quanta Magazine
Feynman Image: The Nobel Foundation

I will focus on quantum computers based on **superconducting qubits (**see intro by **Steven Girvin**) since other hardware platforms are covered in the BSS lectures by **Crystal Noel** (next!), **Immanuel Bloch** (this week), **Giulia Semeghini** (earlier), and earlier lectures – and there are a lot of general similarities.

My experience circa 2010

Maybe 1 or 2 qubits working some small fraction of the time in select labs

Hopes for a working qubit in here

Photo with dilution fridge called Sunshine from Michel Devoret's lab at Yale during my Ph.D.

This year 2023

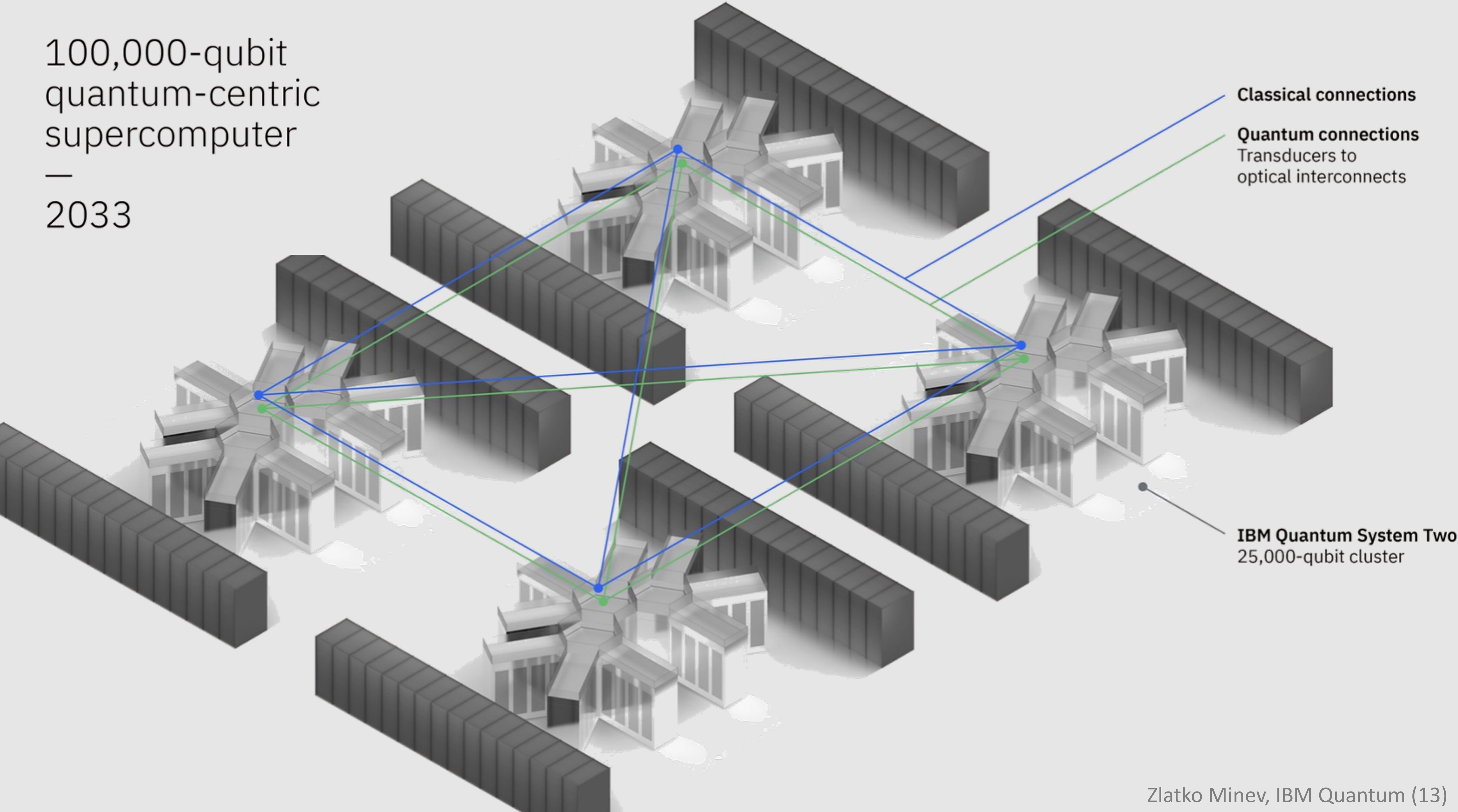
A 127-qubit quantum computer installed in the the lobby cafeteria of a research building dutifully executing jobs almost all the time.

at Cleveland Clinic

Credit: Connie Zhou for IBM

Image: Freepik

100,000-qubit quantum-centric supercomputer
—
2033



Back to today …

Non-equilibrium quantum simulation

Example task:

Simulate the out-of-equilibrium quantum dynamics of a 2D spin chain lattice to find the evolution of the global and local magnetization.

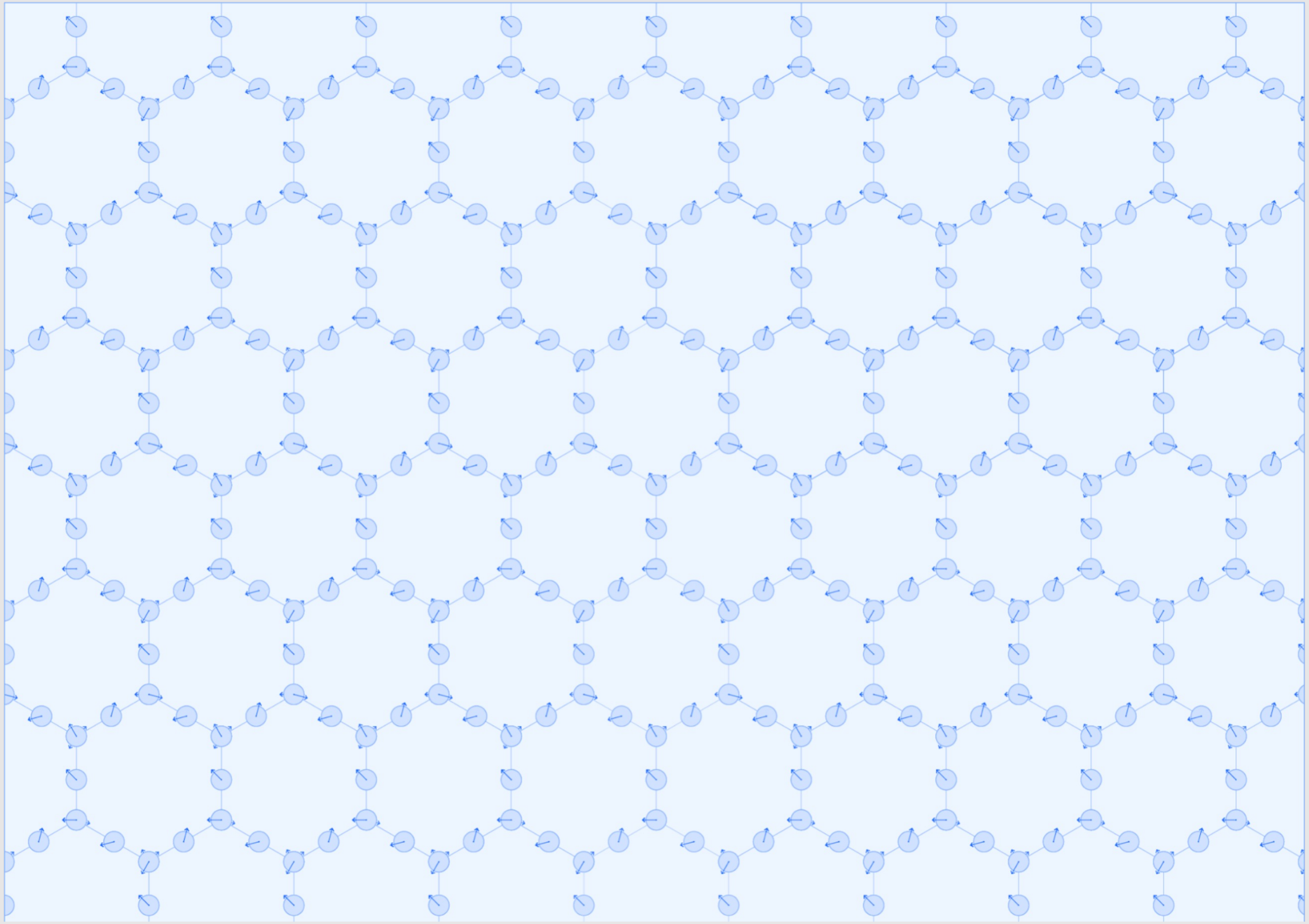
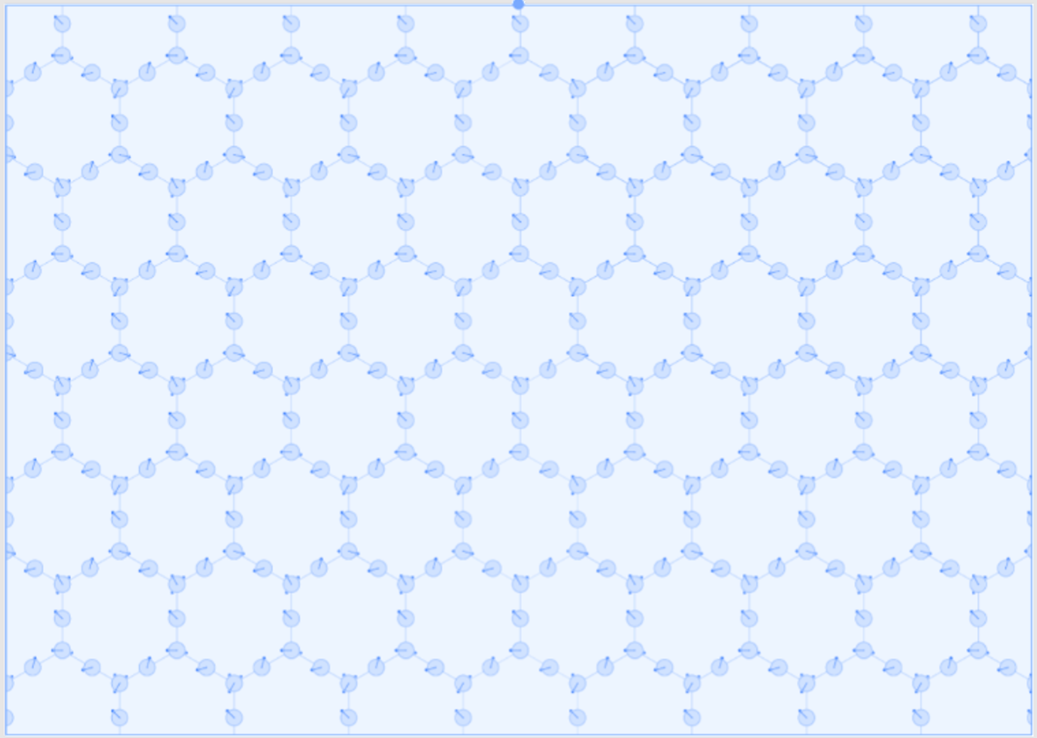


Image: IBM Quantum

Quantum simulation on a quantum computer

Map to quantum circuit



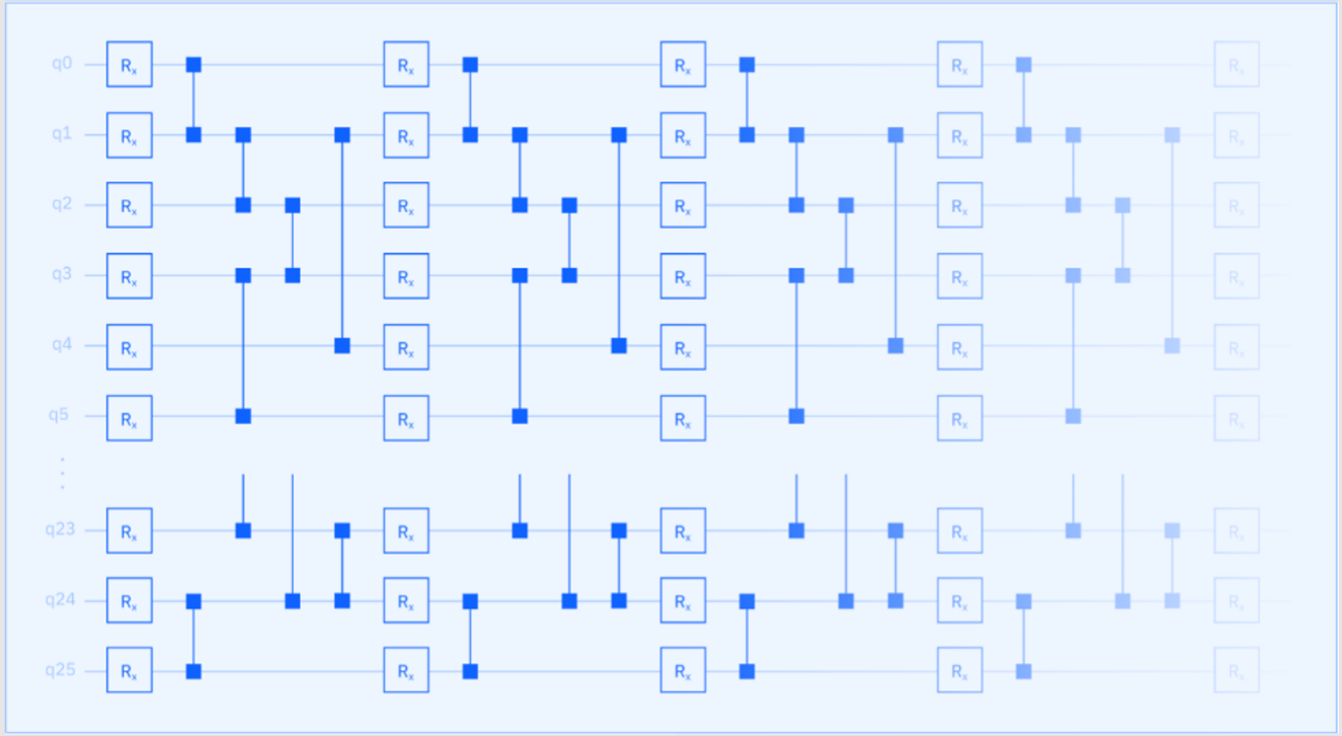
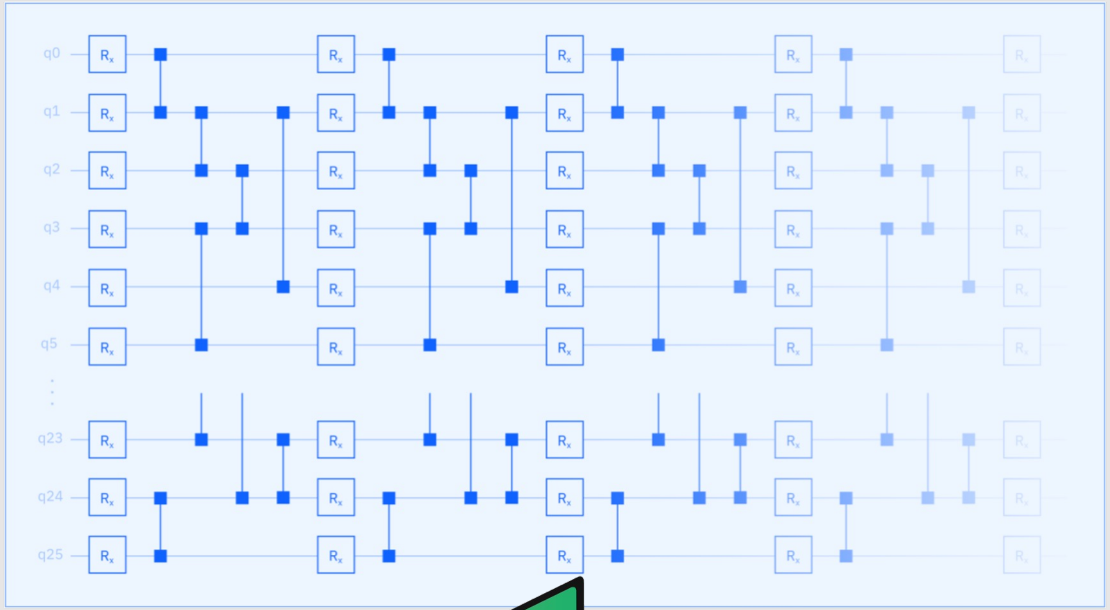


Image: IBM Quantum

Quantum simulation on a quantum computer

Execute on a real quantum computer device and obtain results as classical data



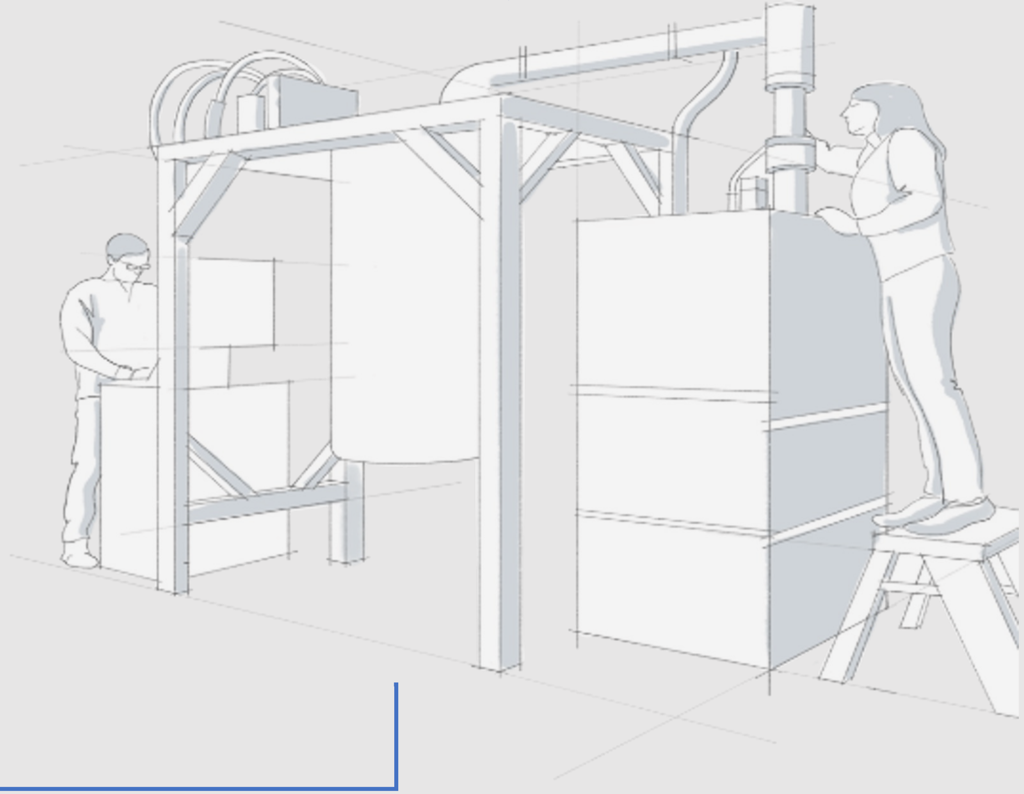


Image: IBM Quantum

Biggest challenge?

Please do share

hardware development

decoherence

loss

stability

heat

algo development

error correction overheads

Noise (Errors)

high error rates

scalability

Biggest challenge?

engineering

modularization

importance of N in NISQ

hype

need CS/EE talent

material quality

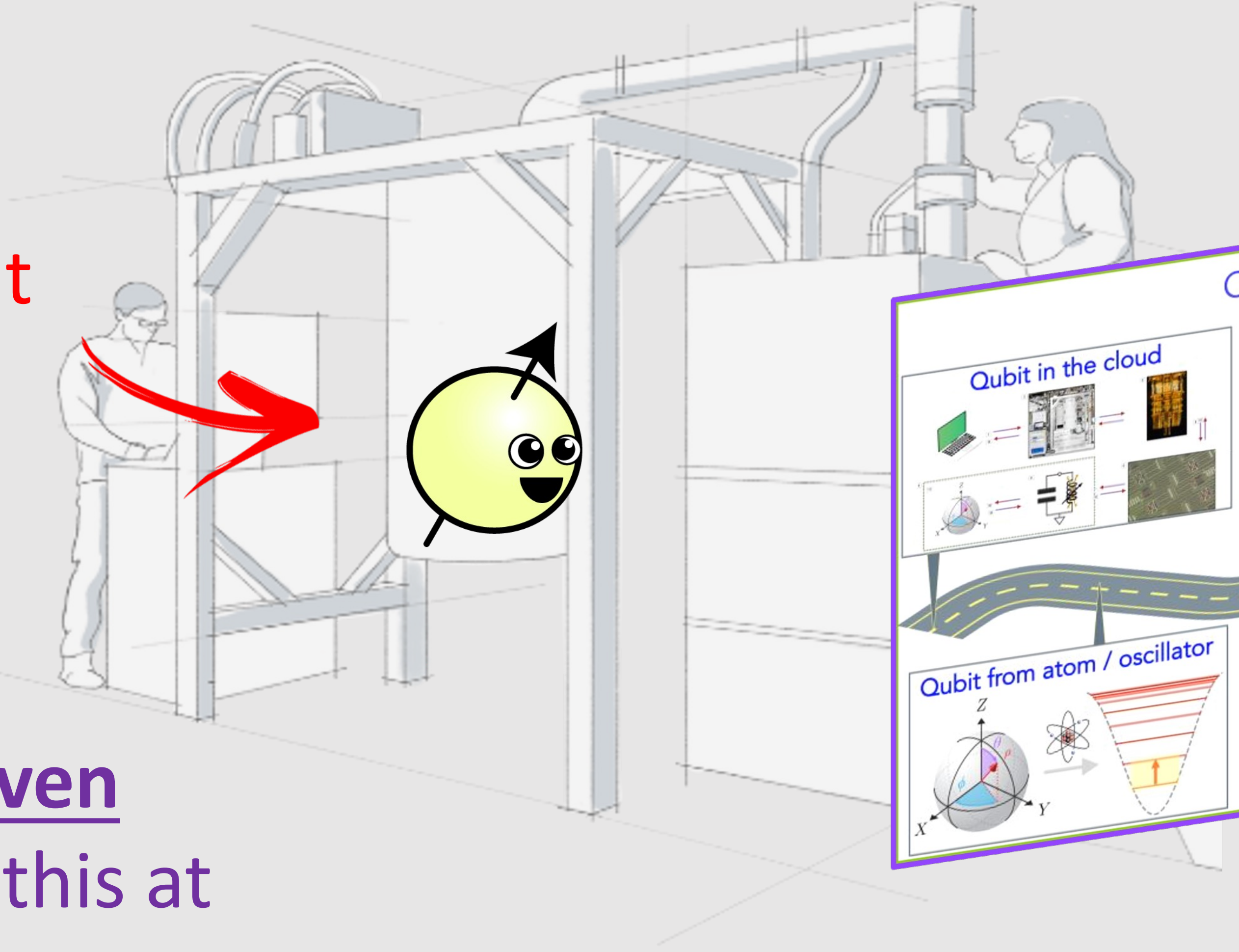
gravity

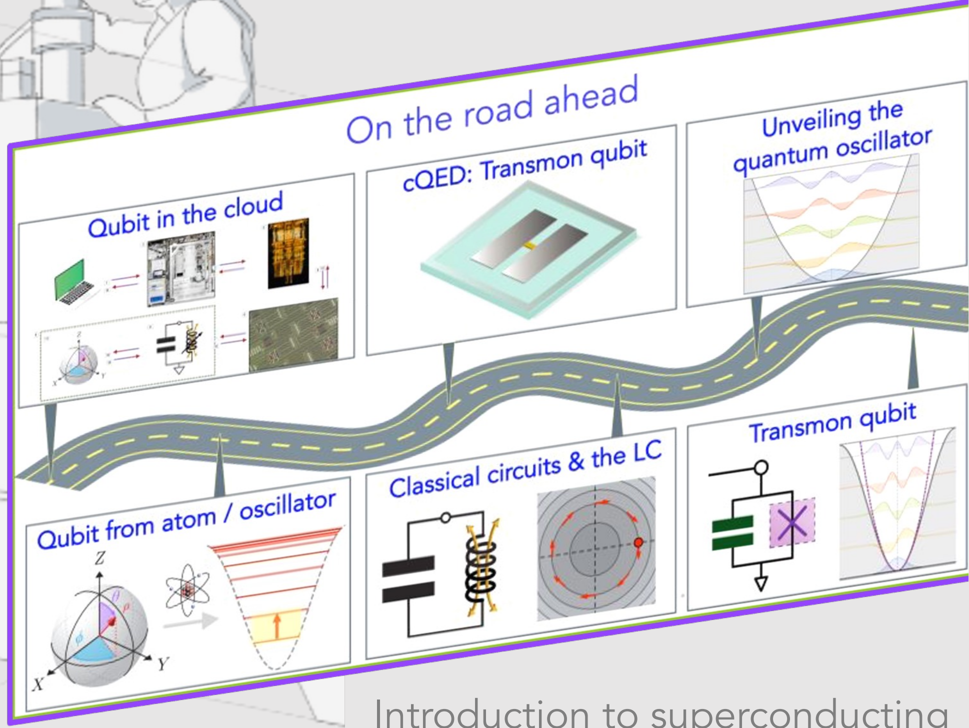
expectations

Biggest challenge

Noise
(Errors)

Hello World with a real experiment!



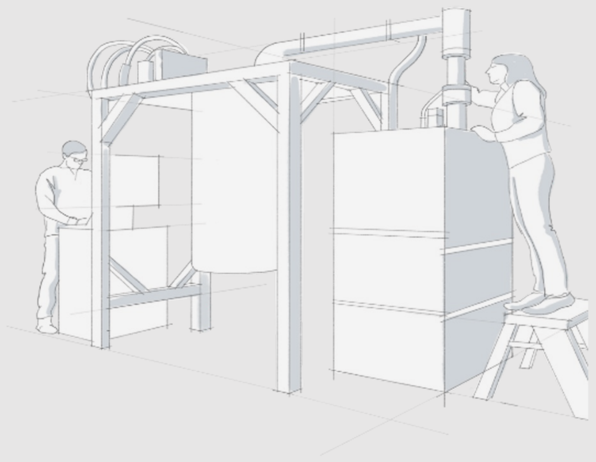


See lectures by **Steven Girvin** right before this at BSS23 for cQED!

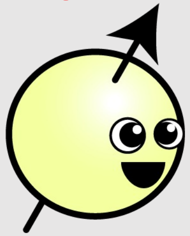
Introduction to superconducting qubits (cQED)
Lecs. 16-21 Minev
QGSS 2020 at qiskit.org/learn

image: qiskit

Hello World! building blocks



A qubit

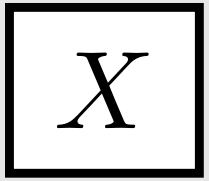


$|1\rangle$

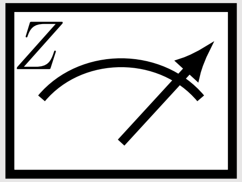
$|0\rangle$

Computational basis states

Operations: qubit gate



Measurements: qubit observable

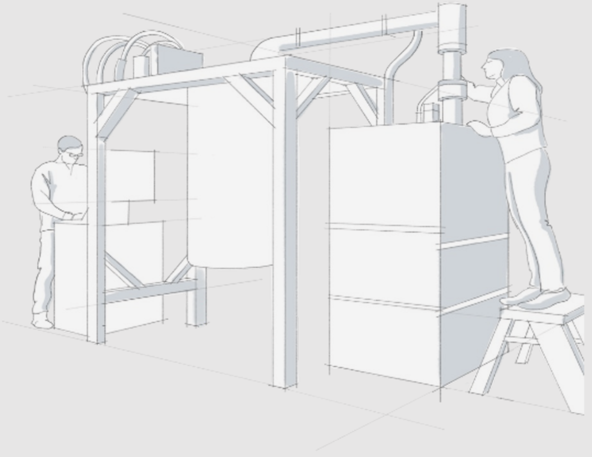


refresher:

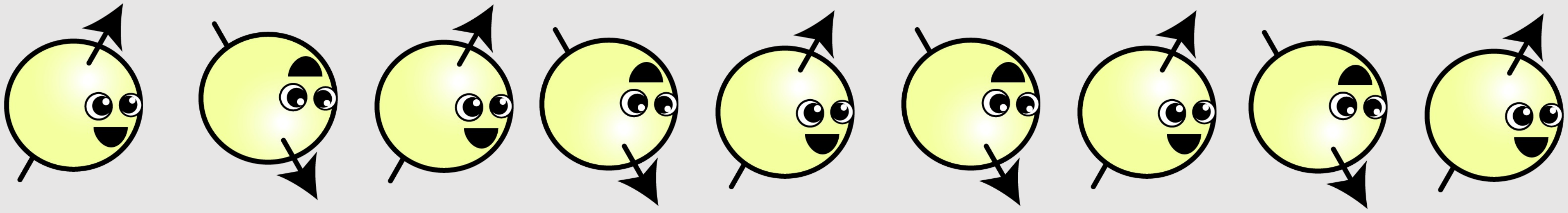
$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

$$Z|0\rangle = +1|0\rangle$$
$$Z|1\rangle = -1|1\rangle$$

Hello World! Even-odd algo: qubit flipper



Task: Classify or report if a classical positive integer *d* is even or odd.



flip spin *d* times, measure polarization

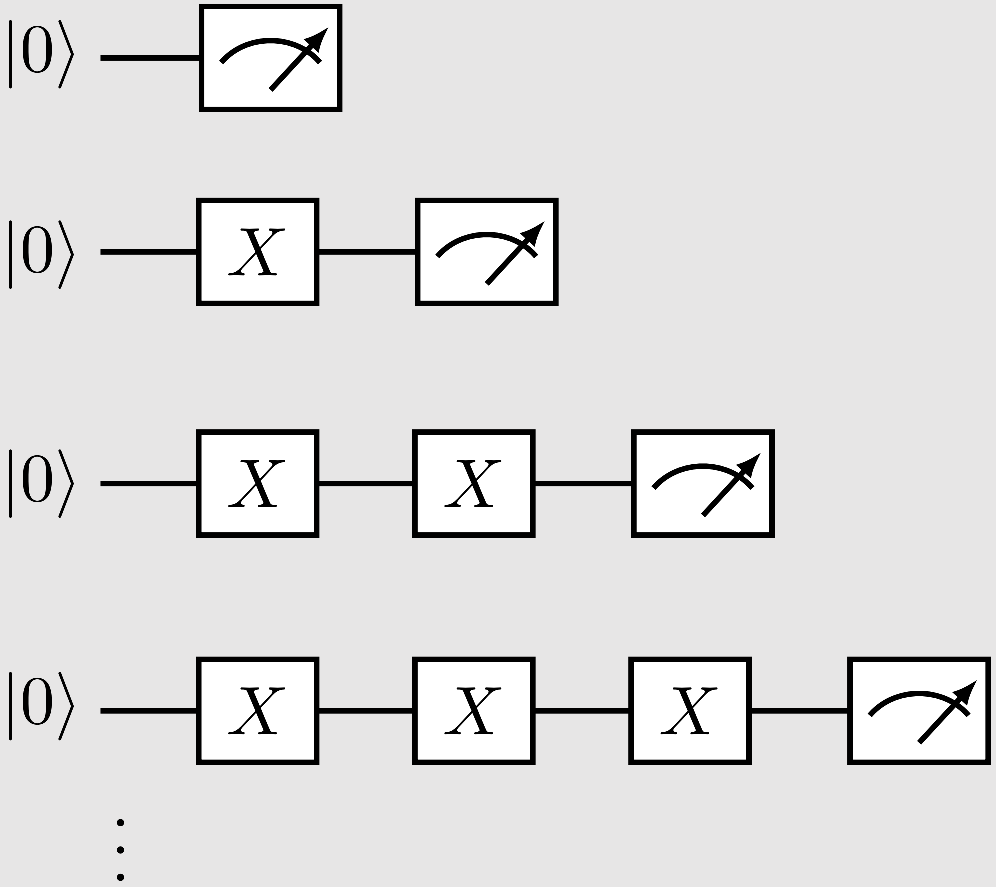
refresher:

$$X|0\rangle = |1\rangle \qquad Z|0\rangle = +1\,|0\rangle$$

$$X|1\rangle = |0\rangle \qquad Z|1\rangle = -1\,|1\rangle$$

Hello World! qubit flipper quantum circuits

depth

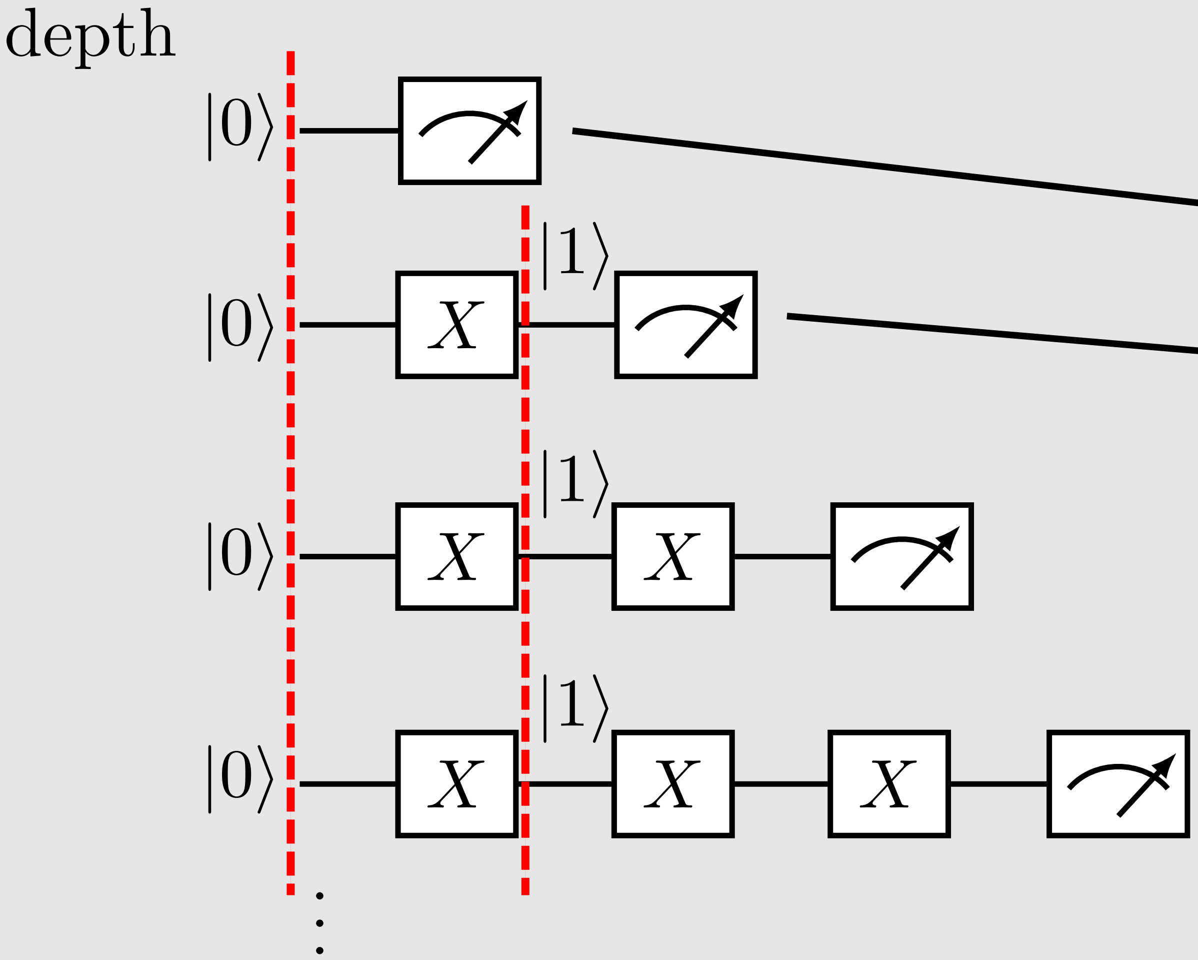


refresher:

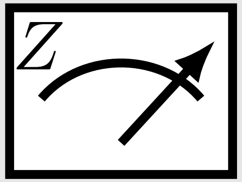
$$X|0\rangle = |1\rangle \qquad Z|0\rangle = +1|0\rangle$$

$$X|1\rangle = |0\rangle \qquad Z|1\rangle = -1|1\rangle$$

Hello World! "debugger" step through



Measurement



$$\langle 0|Z|0\rangle = +1$$

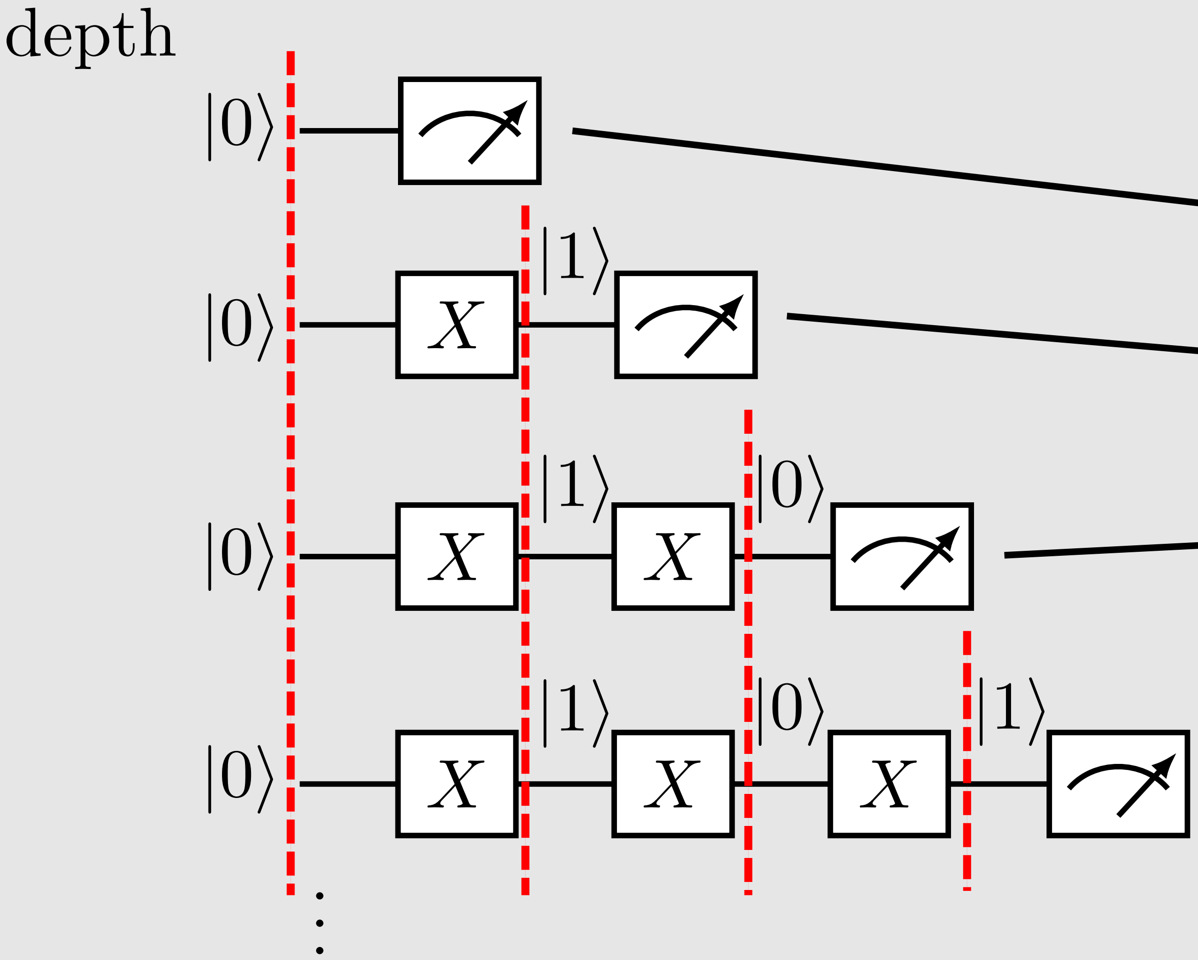
$$\langle 1|Z|1\rangle = -1$$

refresher:

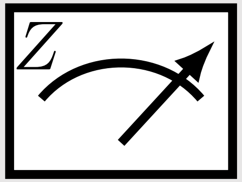
$$X\,|0\rangle = |1\rangle \qquad Z\,|0\rangle = +1\,|0\rangle$$

$$X\,|1\rangle = |0\rangle \qquad Z\,|1\rangle = -1\,|1\rangle$$

Hello World! "debugger" step through



Measurement



$\langle 0|Z|0\rangle = +1$

$\langle 1|Z|1\rangle = -1$

$\langle 0|Z|0\rangle = +1$

$$\langle Z\rangle = (-1)^d$$

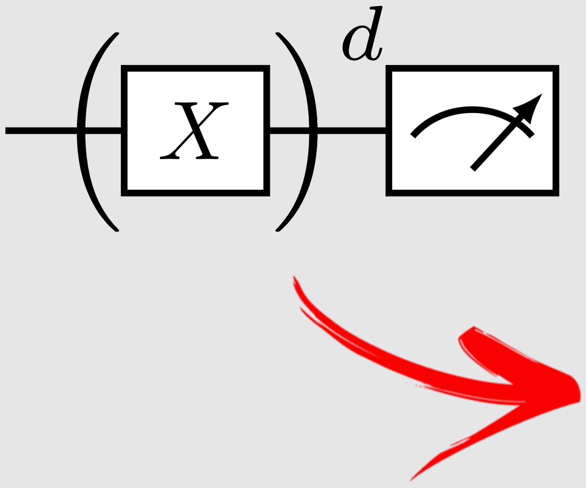
where d is the circuit depth

refresher:

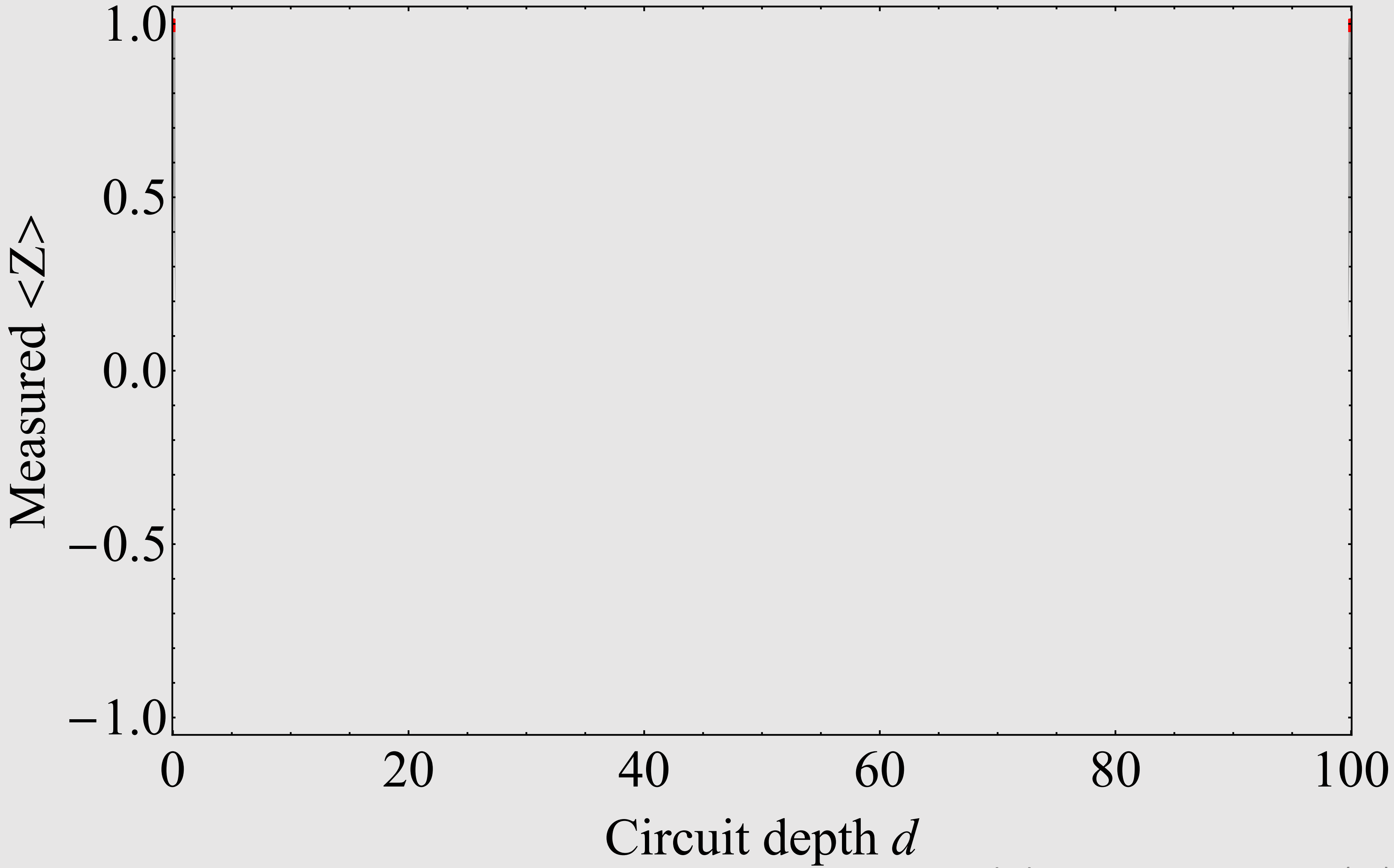
$X|0\rangle = |1\rangle$ $\quad$ $Z|0\rangle = +1|0\rangle$

$X|1\rangle = |0\rangle$ $\quad$ $Z|1\rangle = -1|1\rangle$

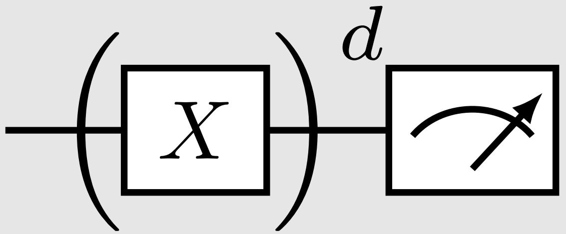
Hello World! Ideal expectation results

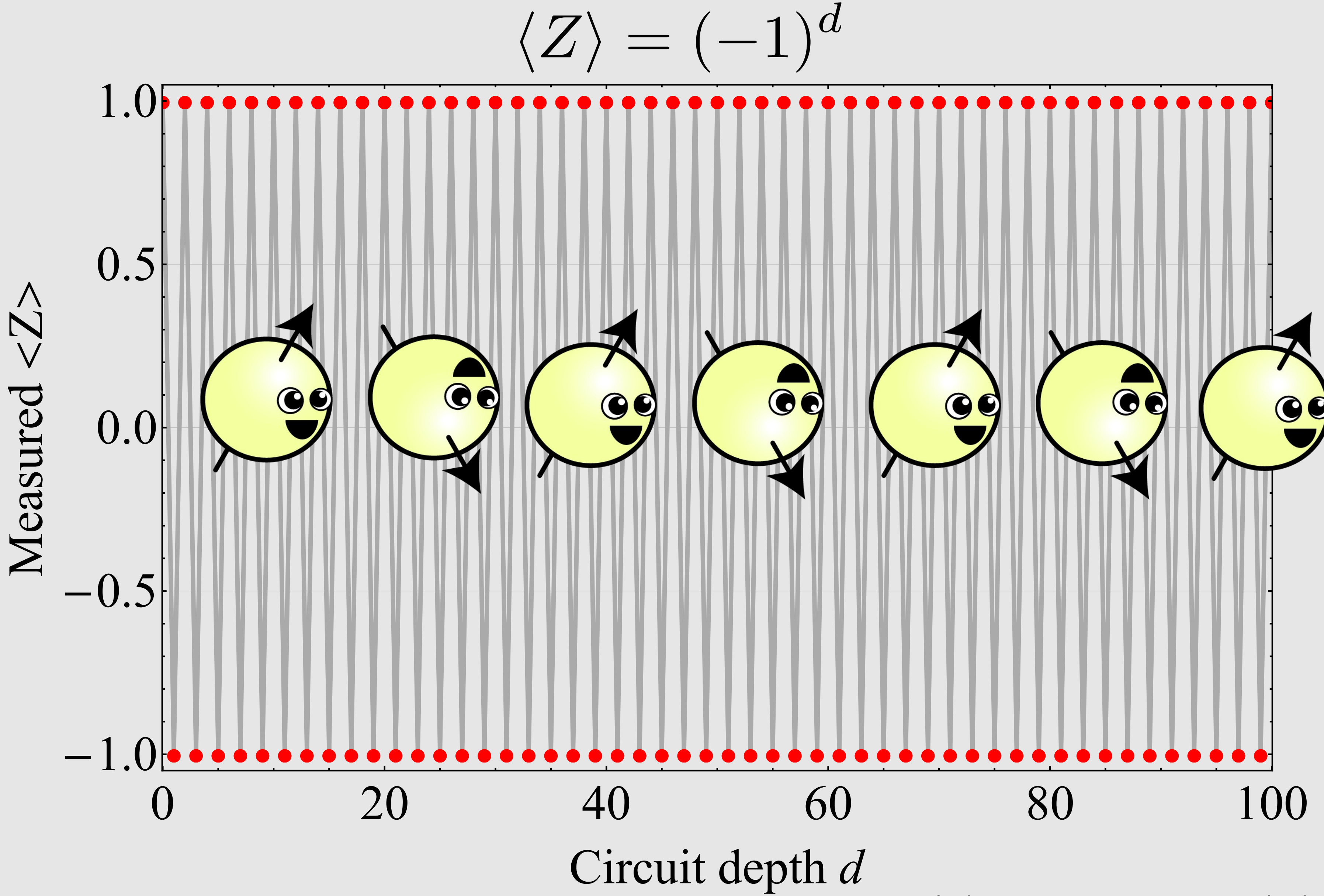


$\langle Z \rangle = (-1)^d$

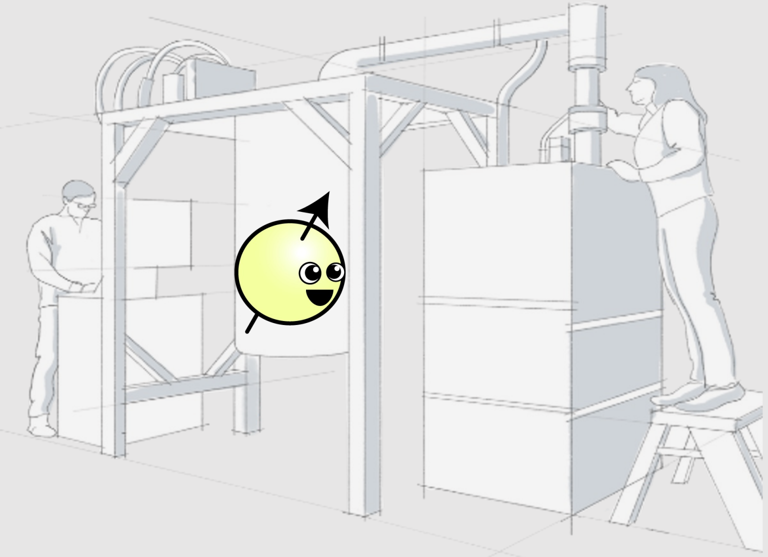


Hello World! Ideal expectation results

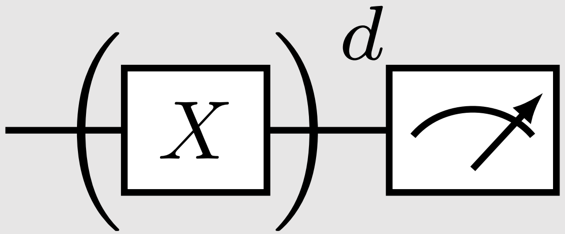


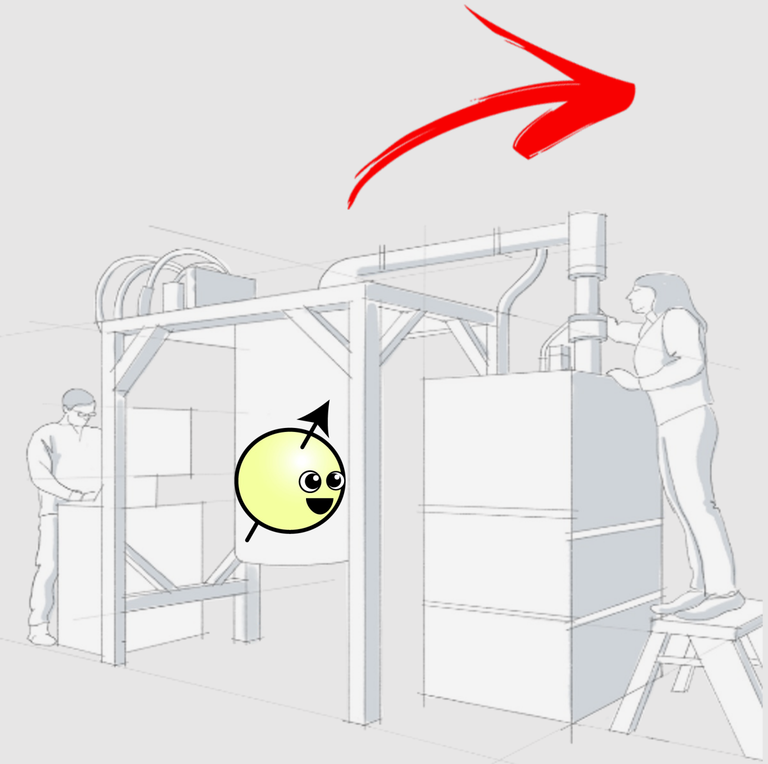


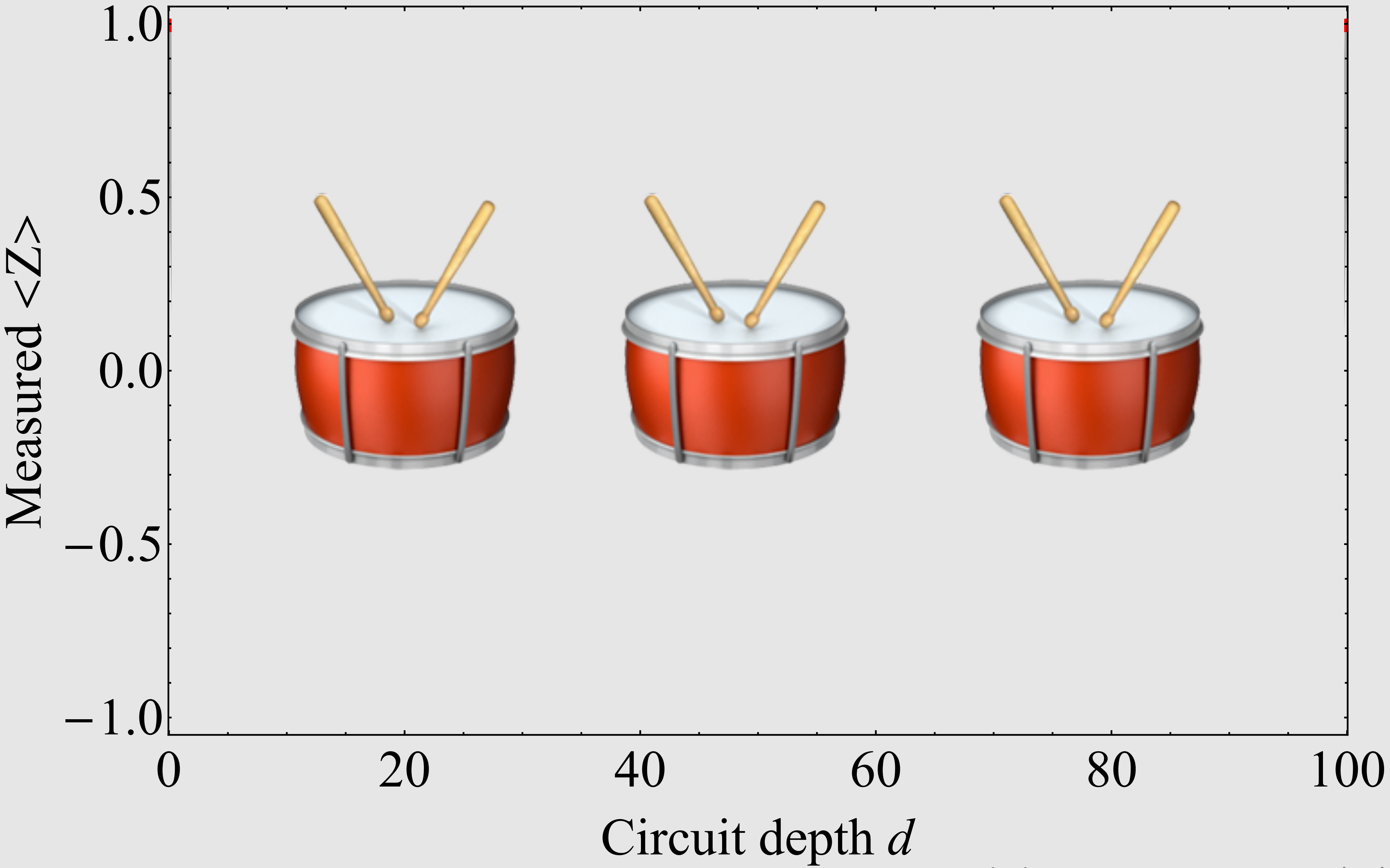
Let's run on a real device!



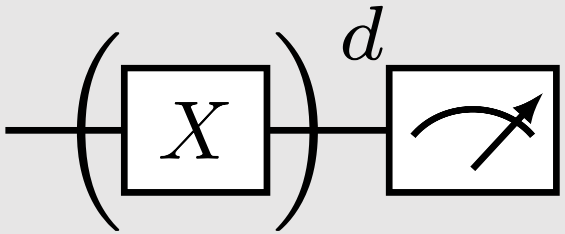
Hello World! Running ….

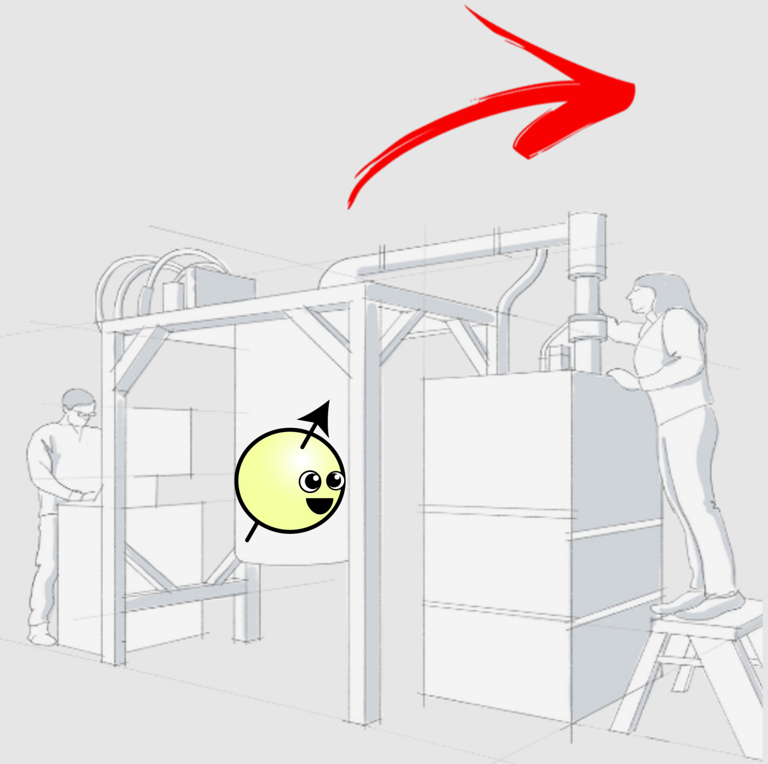


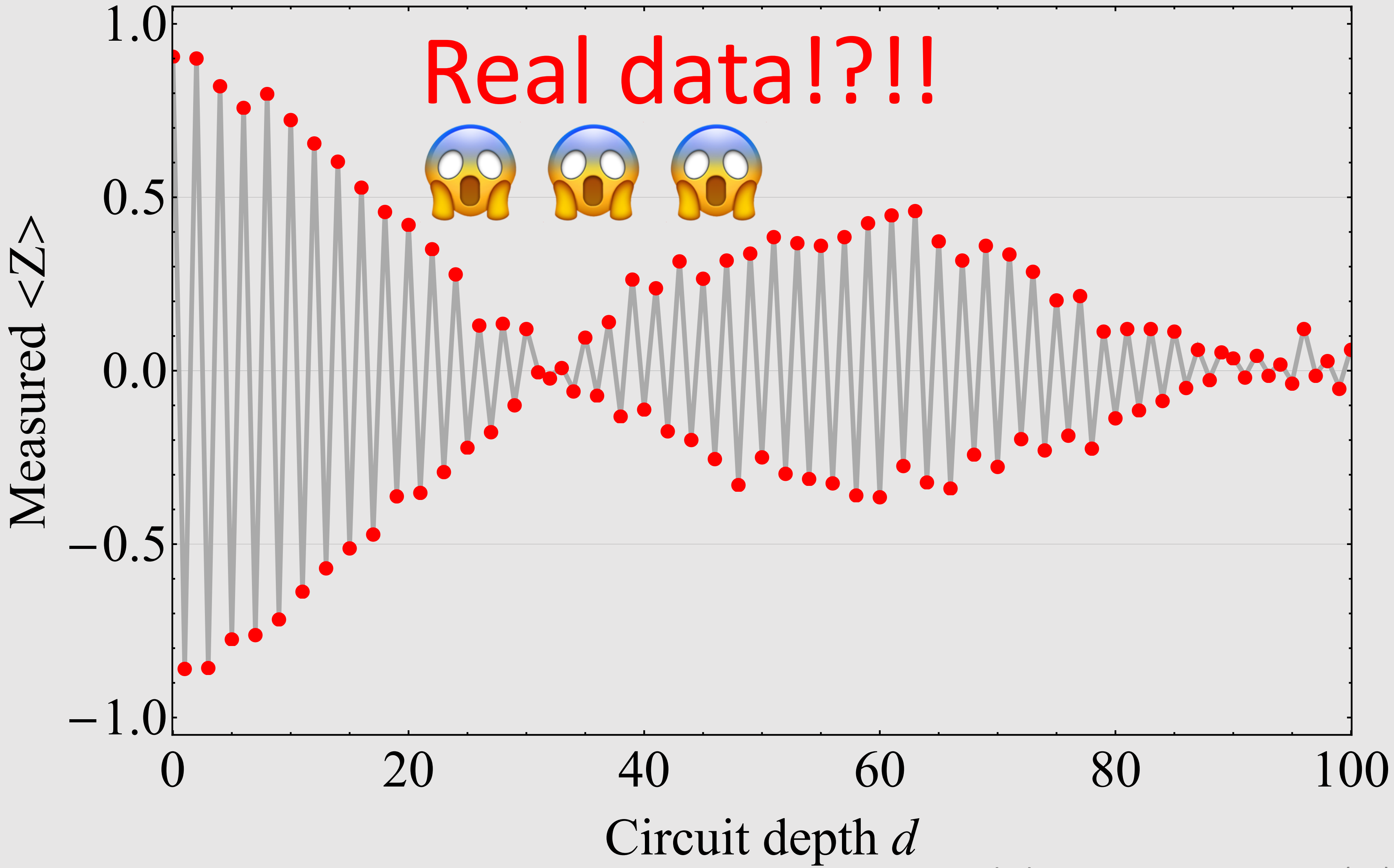




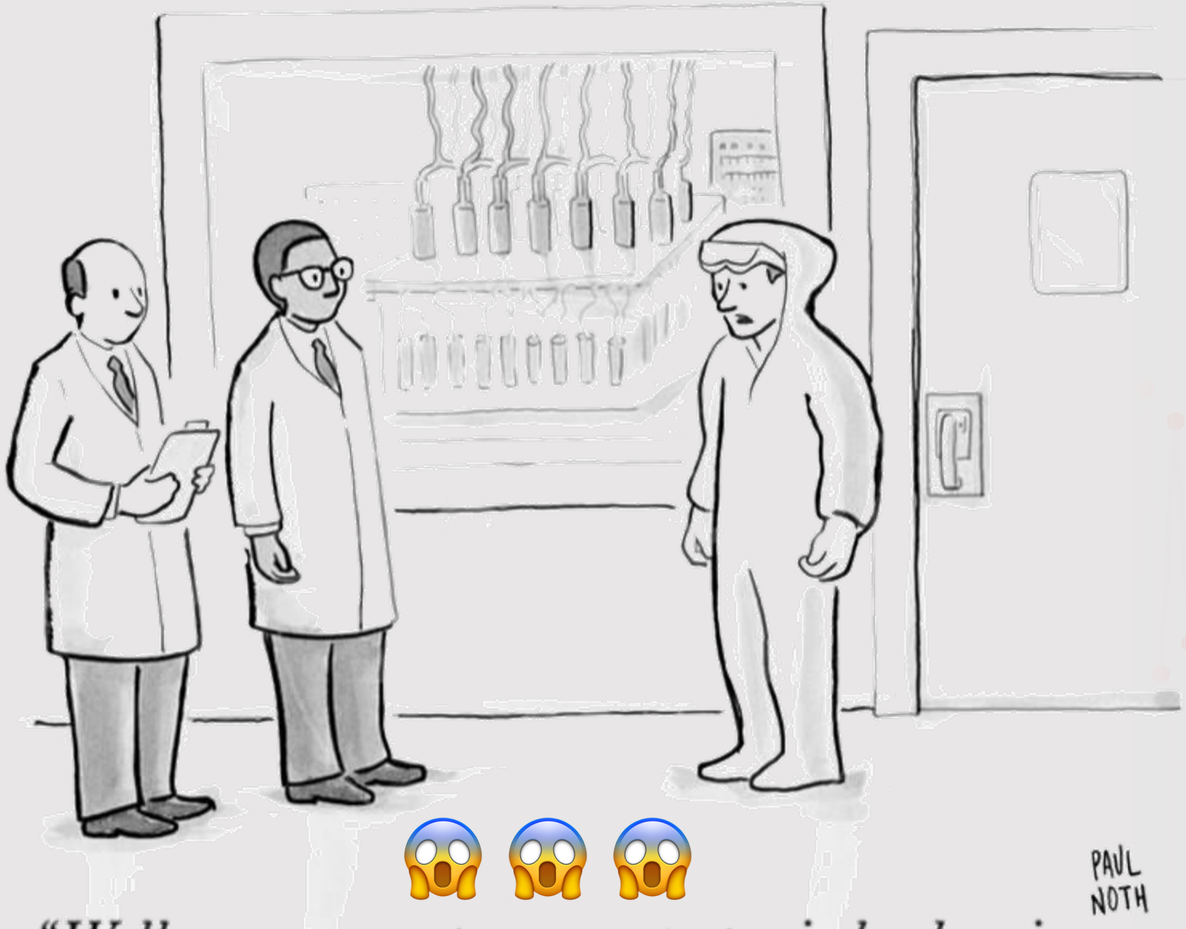
Hello World! Real expectation results



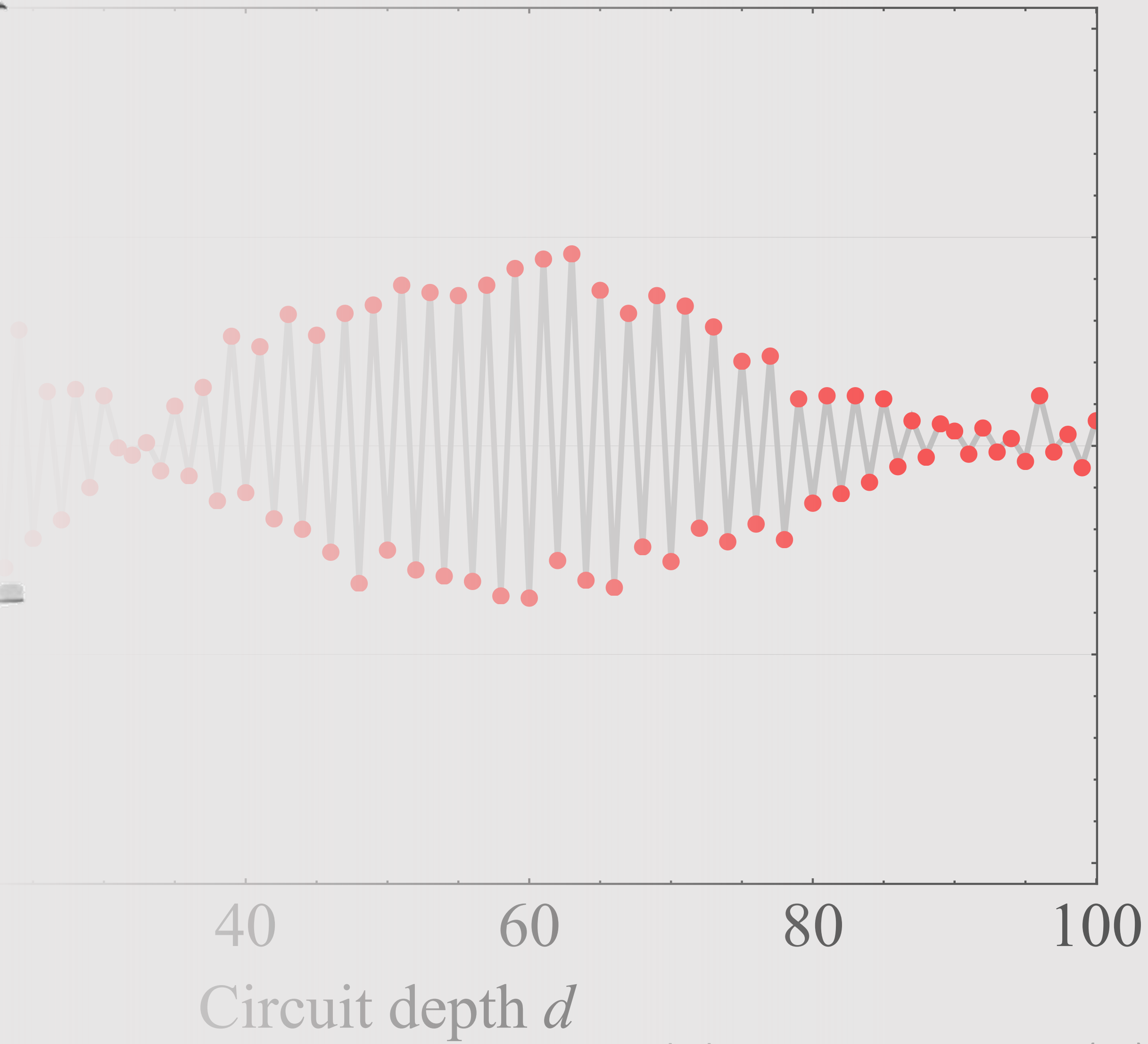




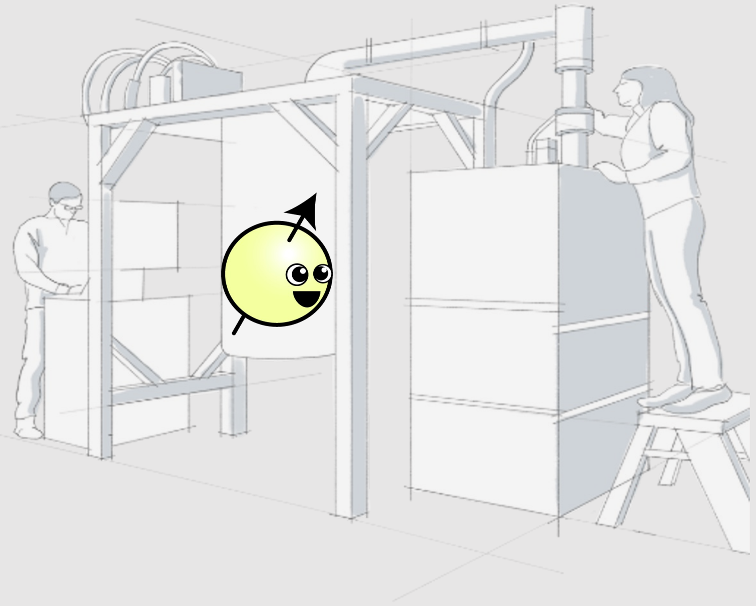
Real & noisy quantum processors: Why study noise?



"Well, your quantum computer is broken in every way possible simultaneously."



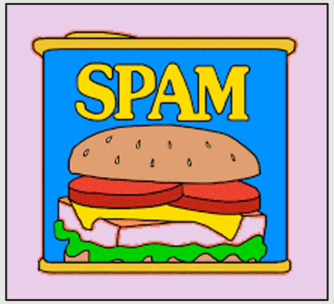
Cartoon: Paul Noth

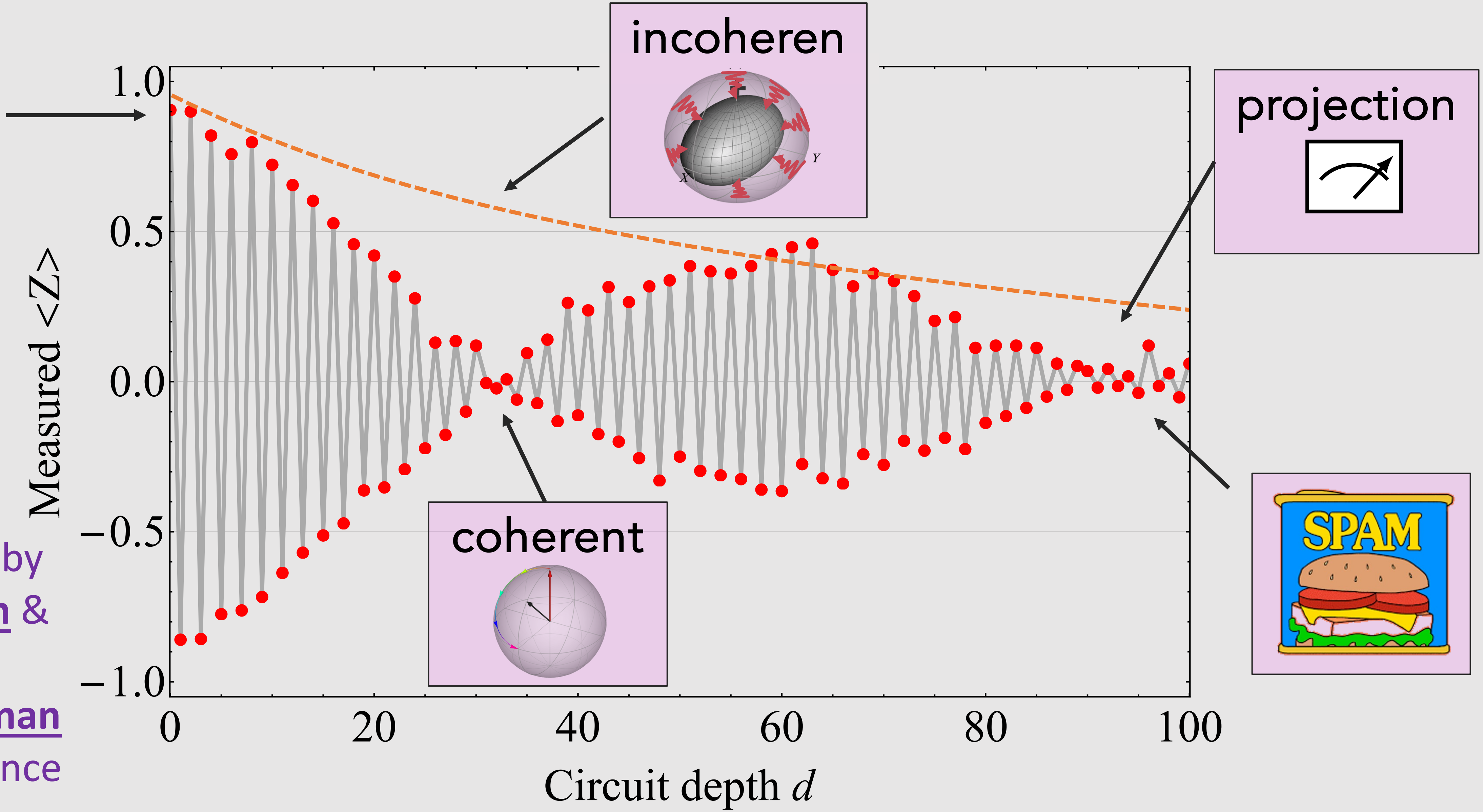


"Quantum phenomena
do *not* occur in a Hilbert space,
they occur in a laboratory."

Asher Peres

Elements of 😱 noise





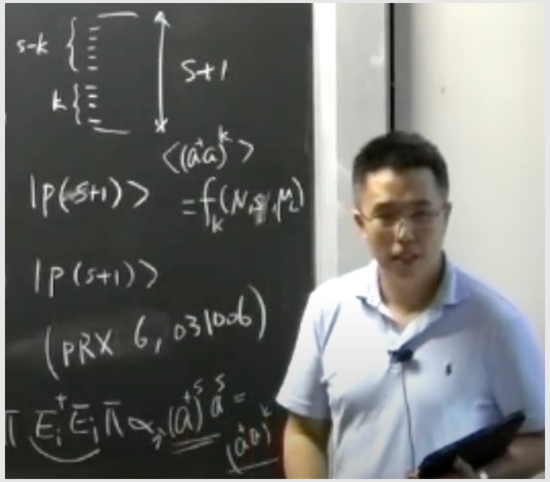
See lectures by **Ehud Altman** & **Sarang Gopalakrishnan** on decoherence at BSS23

How to deal with errors due to noise?

Monitor
Error occurs
Error detect

Quantum error correction

Shor, PRA (1995), ...



See lectures by **Liang Jiang, Victor Albert,** and **Aleksander Kubica** at BSS23 for QEC!

How to deal with errors due to noise?

Monitor
Error occurs
Error detect

Quantum error correction

Shor, PRA (1995), ...

Monitor
Error anticipated
Tell signal detected

Catch and reverse

Minev, Nature (2019), ...

No monitor
Error occurs
Error undetected

Error mitigation

... subject of today

Quantum simulation on a noisy quantum computer

Execute on a real quantum computer device and obtain results as classical data

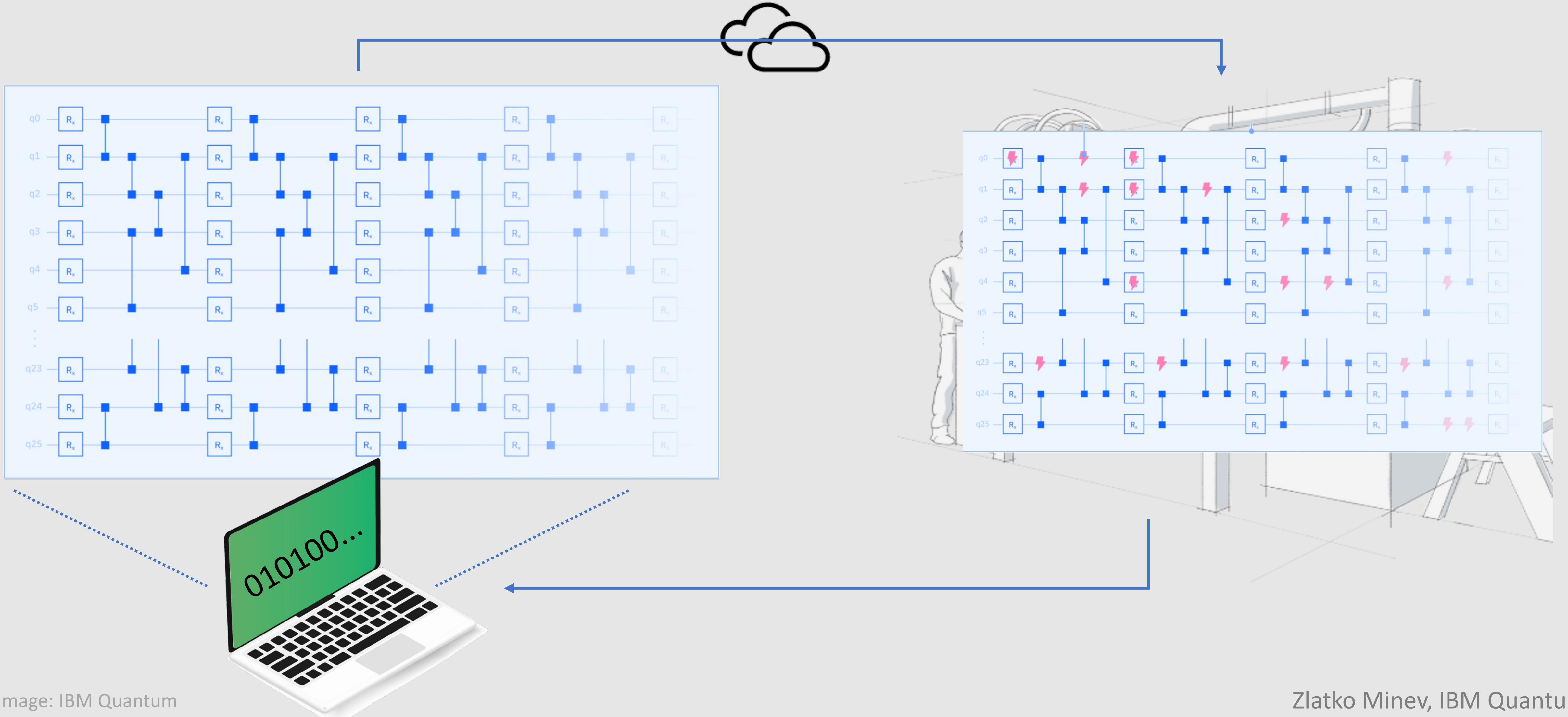


Image: IBM Quantum

Quantum error mitigation overview

Execute on a real quantum computer device and obtain results as classical data

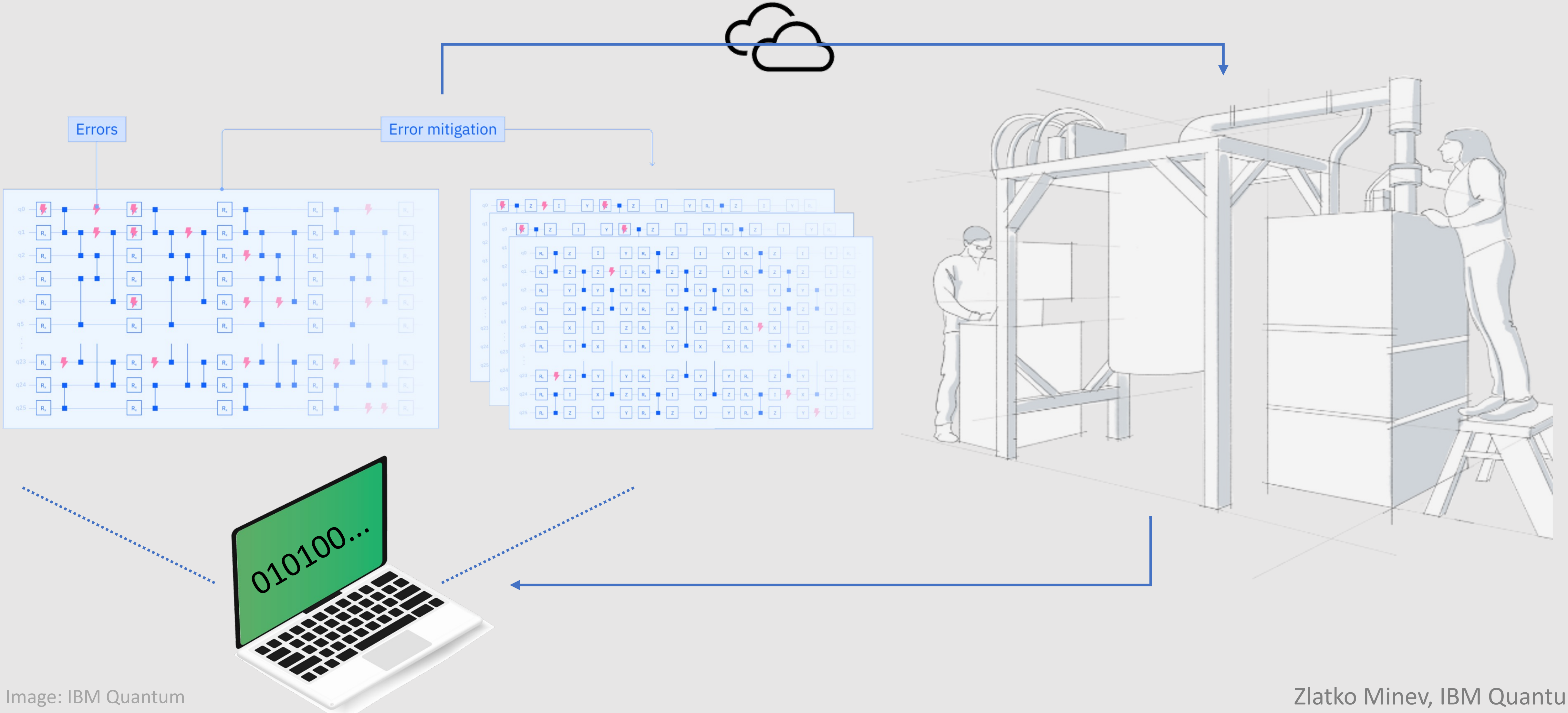
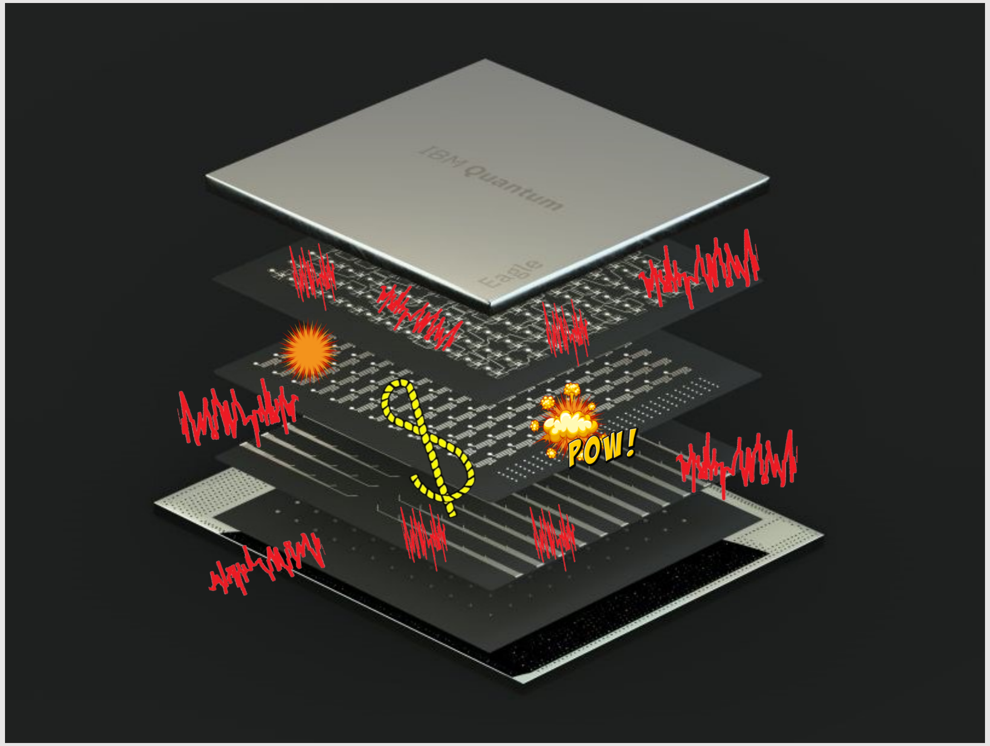


Image: IBM Quantum

Error mitigation and error correction



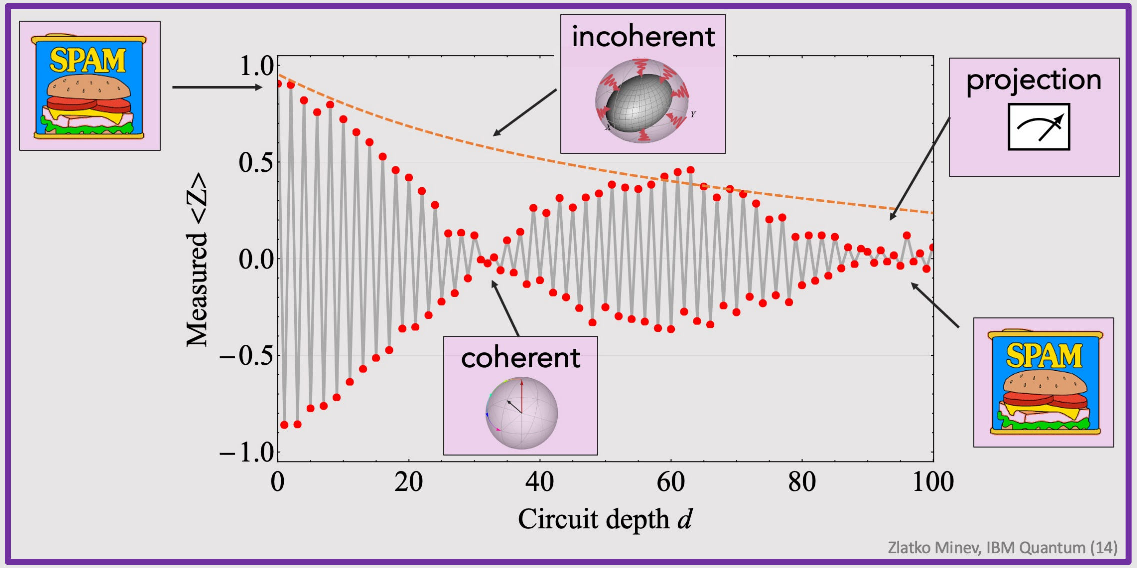
Error mitigation: working with what you have

- **benefit** suppress errors on classical results (expectation values)
- **q-cost** no extra qubits or hardware resources needed
- **c-cost** trades classical resources (post-processing) for lower error
- **limitation** bad asymptotic scaling: high number of samples & circuits

Error correction: protecting quantum information

- **benefit** suppress & correct errors to arbitrarily small level
- **q-cost** very large qubit and hardware overhead
- **c-cost** decoding and encoding can be classically costly
- **challenge** requires fault-tolerant operations and readout

Error mitigation landscape



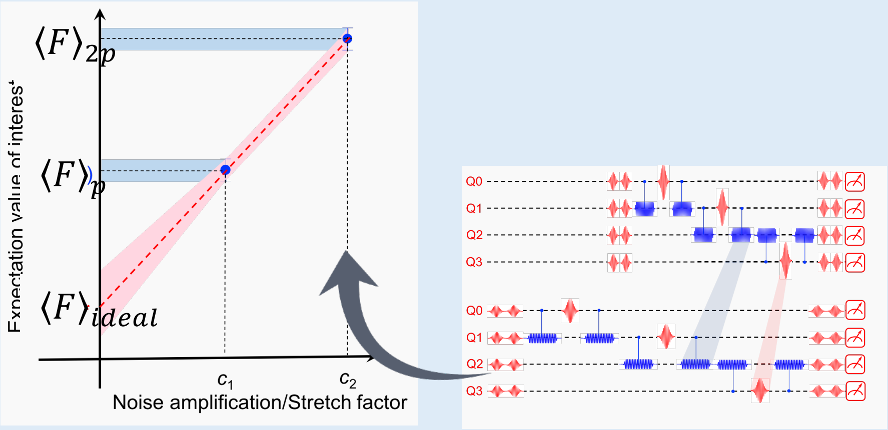
more speed

more information, accuracy

Photos courtesy of Y. Liu

Error mitigation landscape

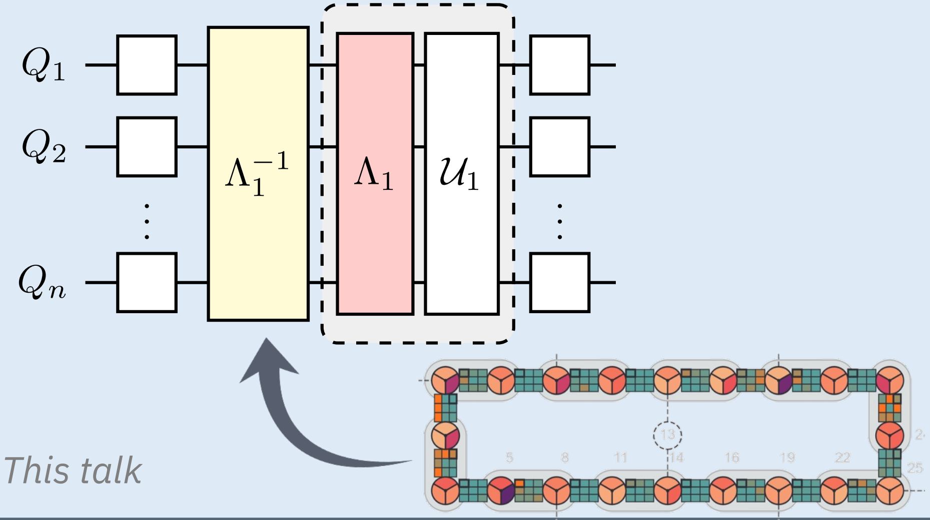
Zero-noise extrapolation (ZNE)



Nature 567, 491 (2019)

...

Probabilistic error cancelation (PEC)

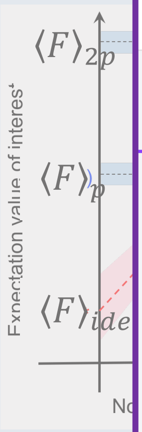


This talk

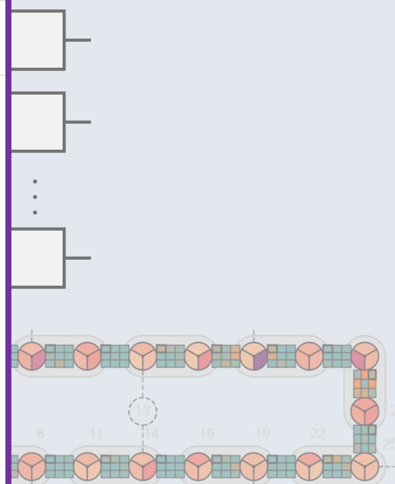
more speed → **more information, accuracy**

Error mitigation landscape

Zero-noise extrapolation (ZNE)



Probabilistic error cancelation (PEC)



Qiskit qiskit.org

Overview Learn Community

English

Search Qiskit Runtime IBM

Get started

- Overview
- Getting Started
- backend.run vs. Qiskit Runtime
- Introduction to primitives

Tutorials

- Get started with Estimator
- Get started with error suppression and error mitigation
- VQE with Estimator
- CHSH with Estimator
- Get started with Sampler

Qiskit Runtime IBM Client documentation > Error supression and error mitigation with Qiskit Runtime

NOTE

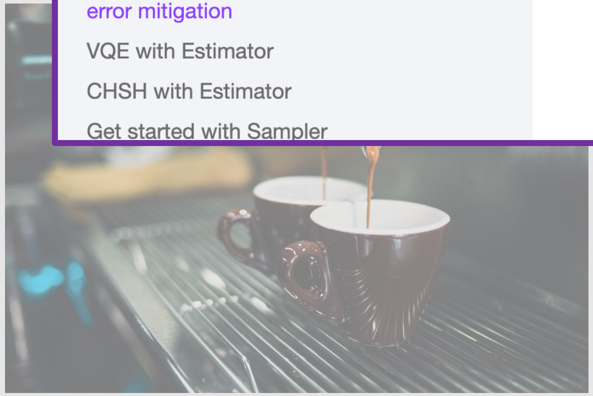
This page was generated from docs/tutorials/Error-Suppression-and-Error-Mitigation.ipynb.

Error supression and error mitigation with Qiskit Runtime

[1]:
```
import datetime
import numpy as np
import matplotlib as mpl
import matplotlib.pyplot as plt
plt.rcParams.update({"text.usetex": True})
plt.rcParams["figure.figsize"] = (6,4)
mpl.rcParams["figure.dpi"] = 200

from qiskit_ibm_runtime import Estimator, Session, QiskitRuntimeService,
Options
from qiskit.quantum_info import SparsePauliOp
from qiskit import QuantumCircuit
```

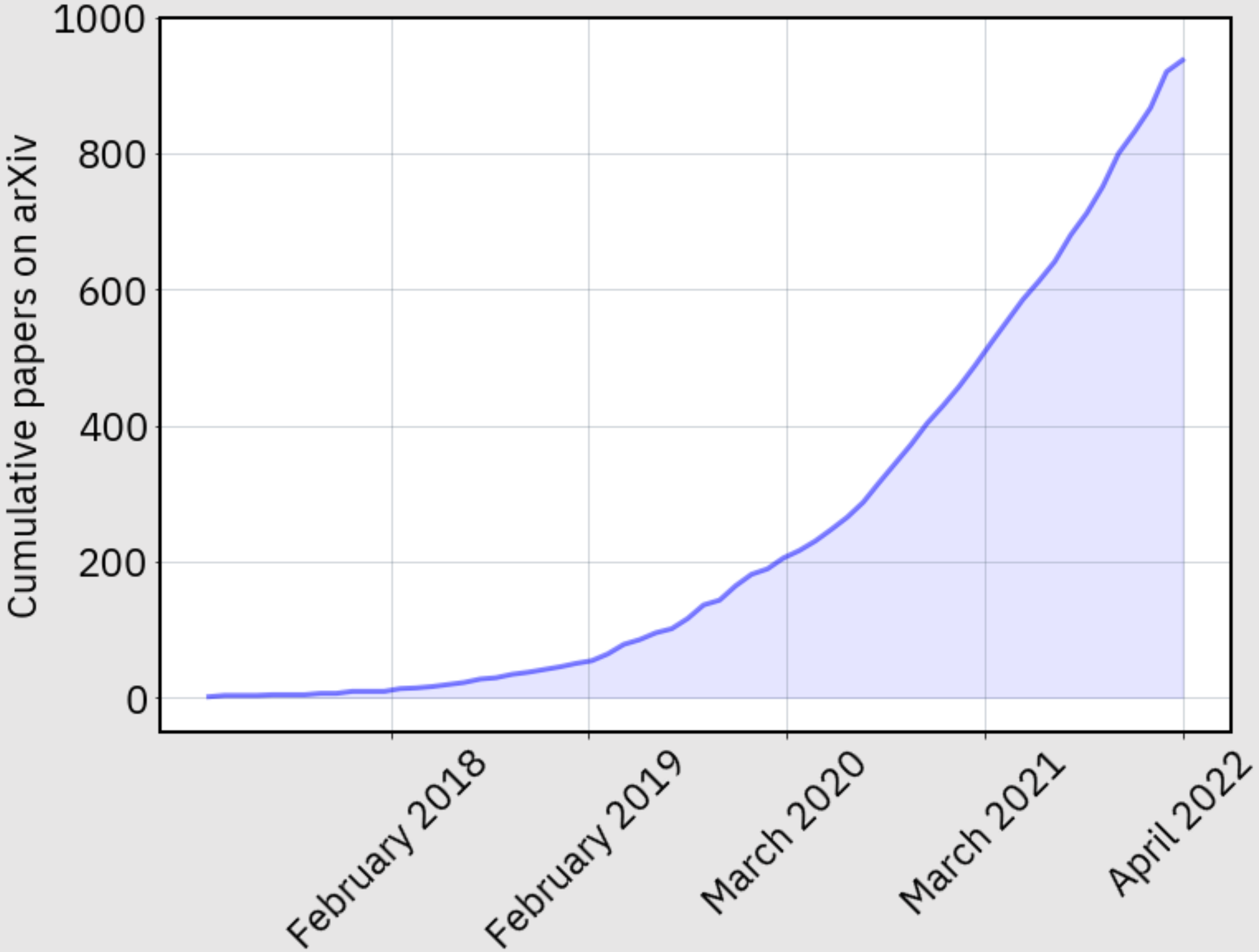
more speed



more information, accuracy

Adoption of error mitigation

Papers involving error mitigation over time



Examples

ARTICLE
https://doi.org/10.1038/s41467-020-14376-z OPEN

Error-mitigated quantum gates exceeding physical fidelities in a trapped-ion system

Shuaining Zhang[1], Yao Lu[1], Kuan Zhang[1,2], Wentao Chen[1], Ying Li[3*], Jing-Ning Zhang[1,4*] & Kihwan Kim[1*]

Matrix product channel: Variation... to mitigate noise and reduce errors

Sergey Filippov,* Boris Sokolov, Ma... Borrelli, Daniel Cavalcanti, Sabrina...
Algorithmiq Ltd, Kanavakat...

Article | Published: 08 May 2023

Probabilistic error cancellation with sparse Pauli–Lindblad models on noisy quantum processors

Ewout van den Berg, Zlatko K. Minev, Abhinav Kandala & Kristan Temme

Nature Physics (2023) | Cite this article

Quantum Error Mitigation

Zhenyu Cai,[1,2,*] Ryan Babbush,[3] ... McClean,[3] and Thomas E. O'Brie...
[1] *Department of Materials, Universit...*
[2] *Quantum Motion, 9 Sterling Way, ...*
[3] *Google Quantum AI, Venice, Califo...*
[4] *NTT Computer and Data Science ...*
[5] *Graduate School of China Academ...*

(Dated: July 3, 2023)

npj | quantum information

ARTICLE OPEN

Fundamental limits of quantum error mitigation

Ryuji Takagi[1], Suguru Endo[2], Shintaro Minagawa[3] and Mile Gu[1,4]

Single-shot error mitiga...

...wout van den Berg, Sergey B...
Dmitri Maslo...
...ntum, IBM T.J. Watson Resea...
Dece...

PHYSICAL REVIEW LETTERS **127**, 200505 (2021)

Error Mitigation for Universal Gates on Encoded Qubits

Christophe Piveteau
IBM Quantum, IBM Research—Zurich, 8803 Rüschlikon, Switzerland

Model-free readout-error mitigation for quantum expectation values

Ewout van den Berg, Zlatko K. Minev, and Kristan Temme
Phys. Rev. A **105**, 032620 – Published 30 March 2022

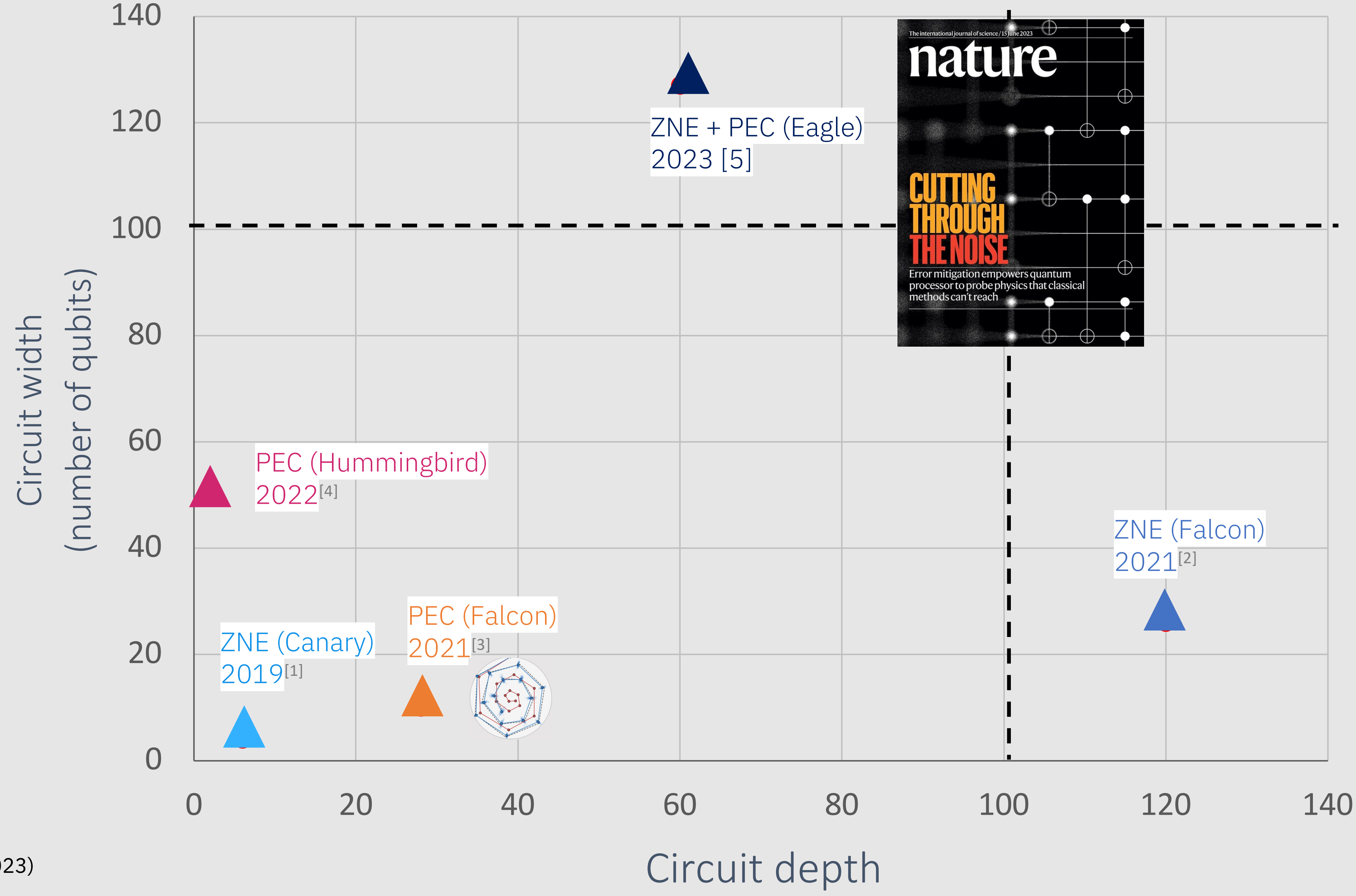
Overview of some key experimental progress in error mitigation:

Error mitigation

No matter what you do you have to chop it to this graph

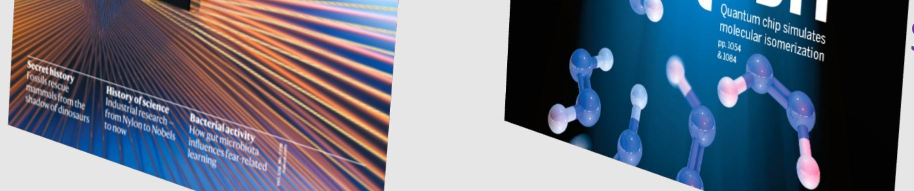
PEC: Probabilistic error cancelation

ZNE: Zero-noise extrapolation

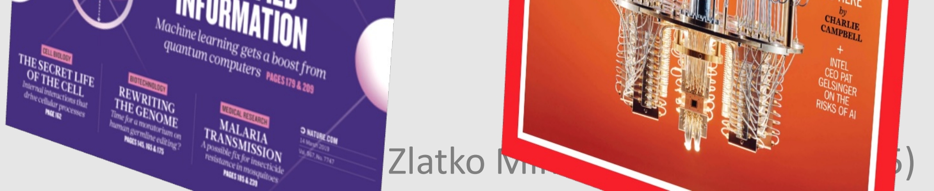


[1] Kandala, Nature (2019)
[2] Kim, Nature Phys. (2023)
[3] van den Berg, Minev, Nature Phys. (2023)
[4] Temme, IBM Research Blog (2022)
[5] Kim, Nature (2023)

Is science with noisy devices of broad interest today?



Some of these ideas covered in lectures at BSS23, see also BSS seminar by **Vedika Khemani**



Zlatko Mi

Deep dive:
Probabilistic error cancellation (PEC)
To learn and cancel quantum noise

Got Slides?

SCAN ME

Paper: Nature Physics (2023)

Ewout van den Berg, Zlatko K. Minev, Abhinav Kandala, Kristan Temme

Cancel quantum noise

Icon: Flaticon

High-level message

Learn
accurate, efficient, scalable

Cancel
noise with noise, practical

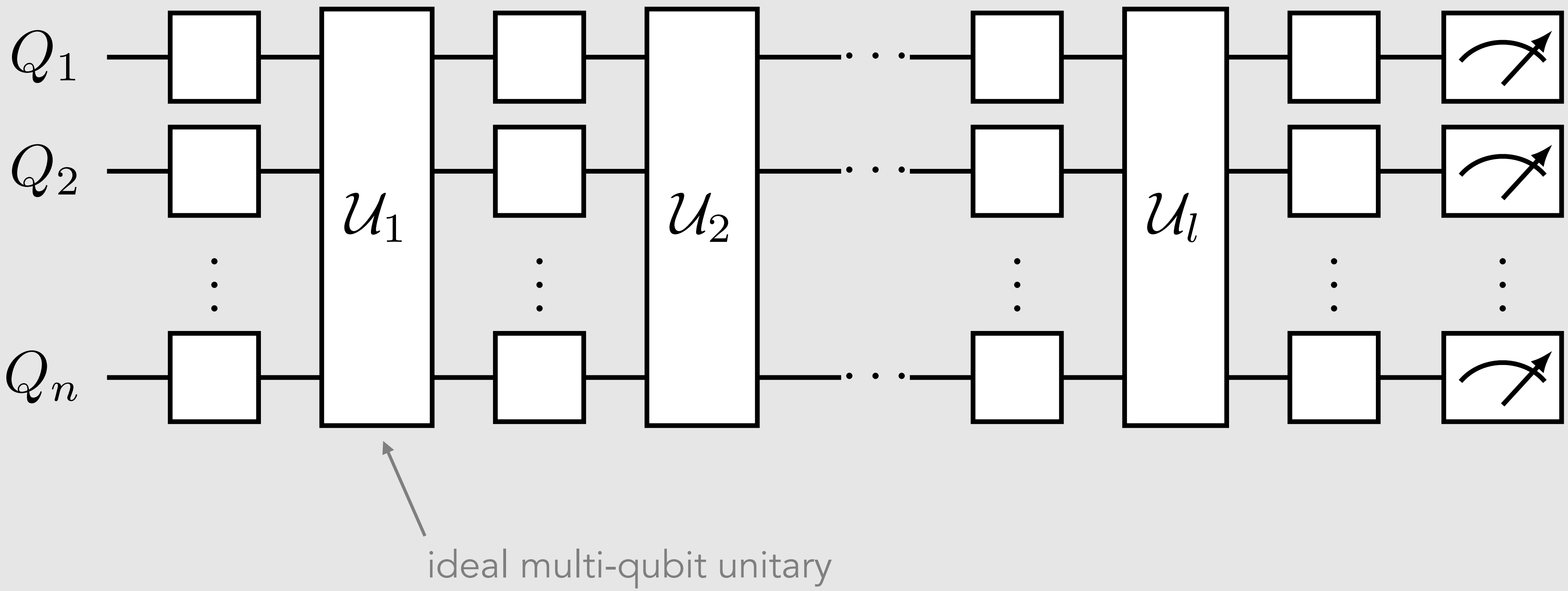
Cost
more noise more cost

Icon: Flaticon

Idea

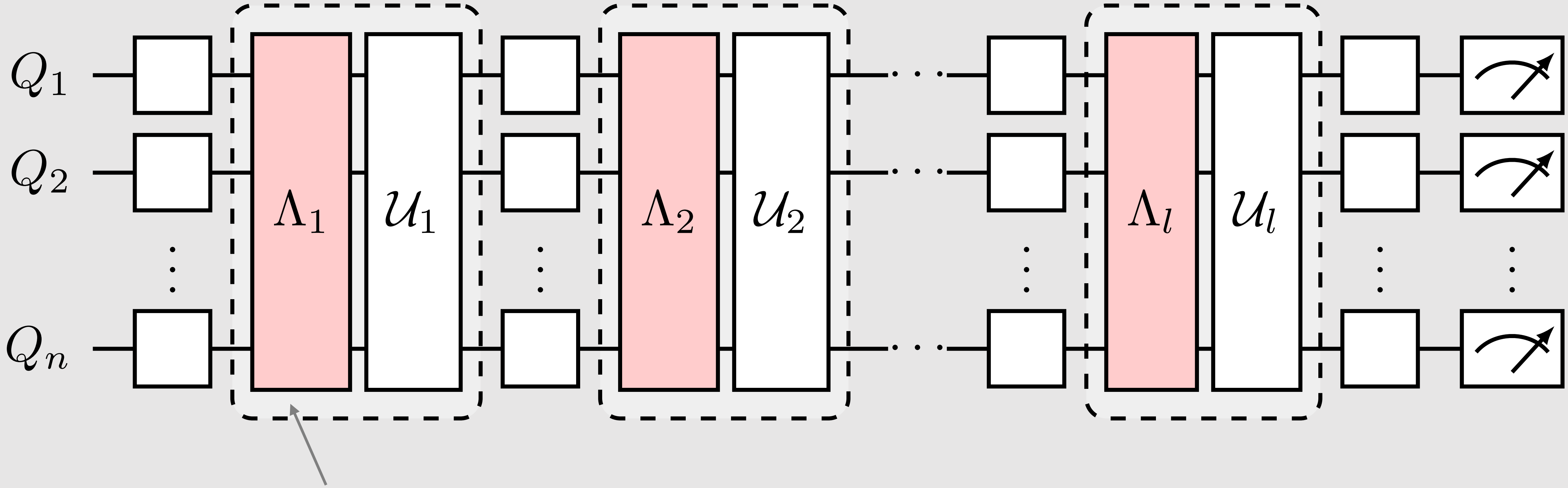
Icon: Flaticon

Ideal (noise-free) quantum circuit



A circuit can be decomposed into a layer construction
Example: Trotterization of Ising model simulation

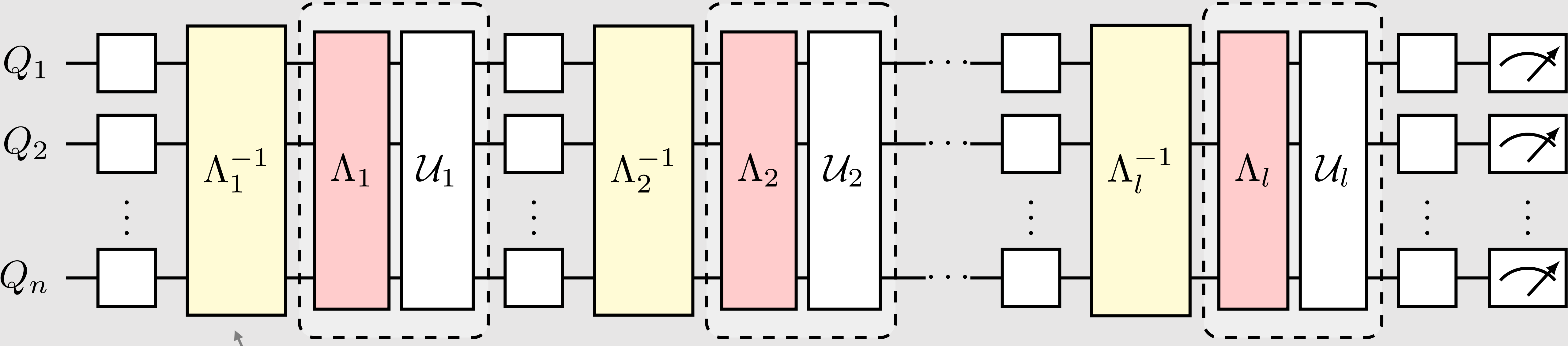
Real (noisy) quantum circuit



multi-qubit noise channel
inseparable from gate

completely positive and trace preserving (CPTP)
representable by a 4^n x 4^n matrix

Why not invert noise?

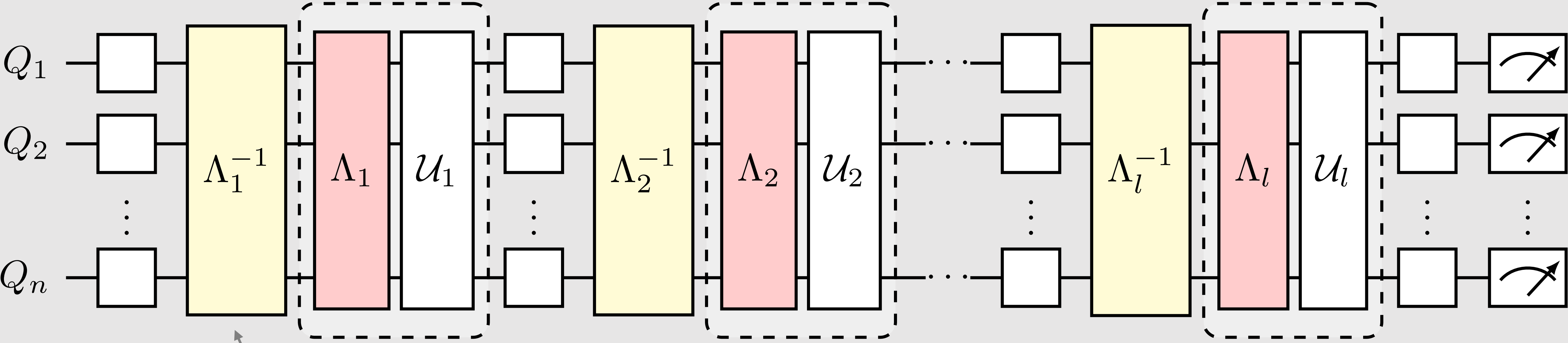


inverse operation of noise channel

Not possible?

unphysical
would need to know lost information due to noise
non CPTP map
has negative eigenvalues
...

Probabilistic error cancelation

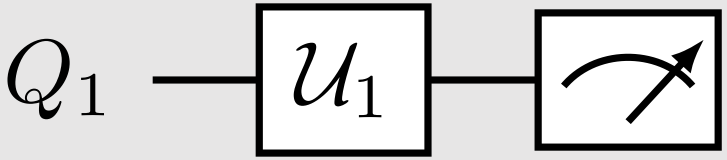


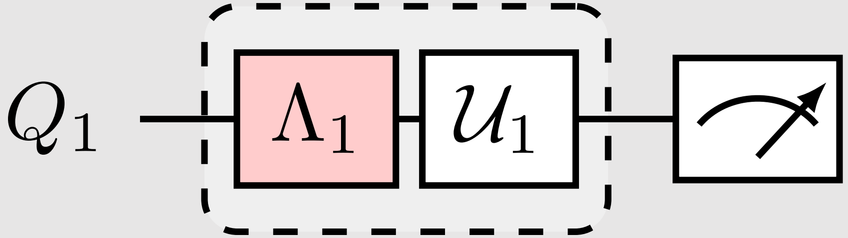
inverse operation of noise channel
implement on average

K. Temme, S. Bravyi, and J. M. Gambetta
PRL 119, 180509 (2017)

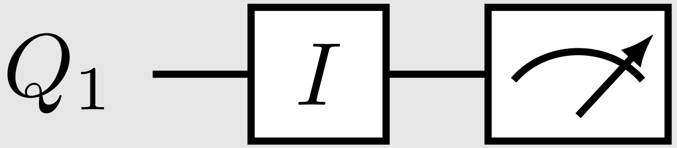
See also S. Endo, S. Benjamin, and Y. Li
Phys. Rev. X 8, 031027 (2018)

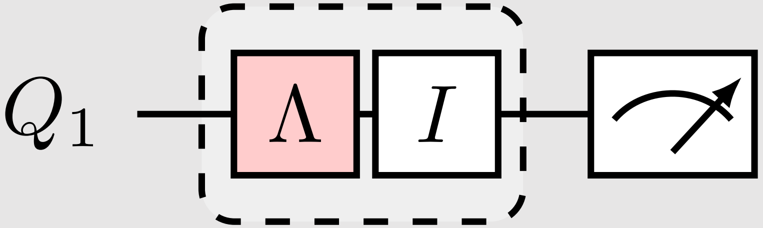
Toy model





Toy model: noise unraveling into quantum trajectories



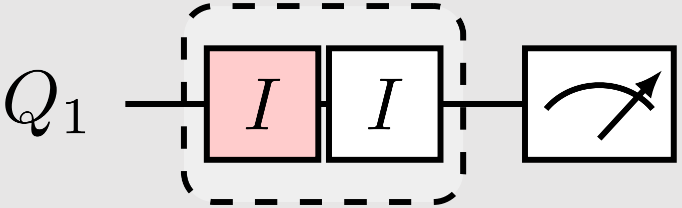


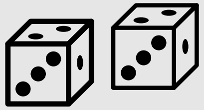
bit-flip noise

=

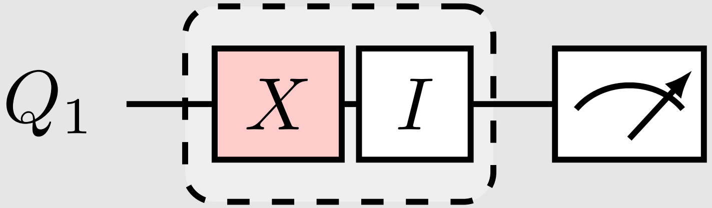
unraveling
(quantum trajectories)

probability 1-p



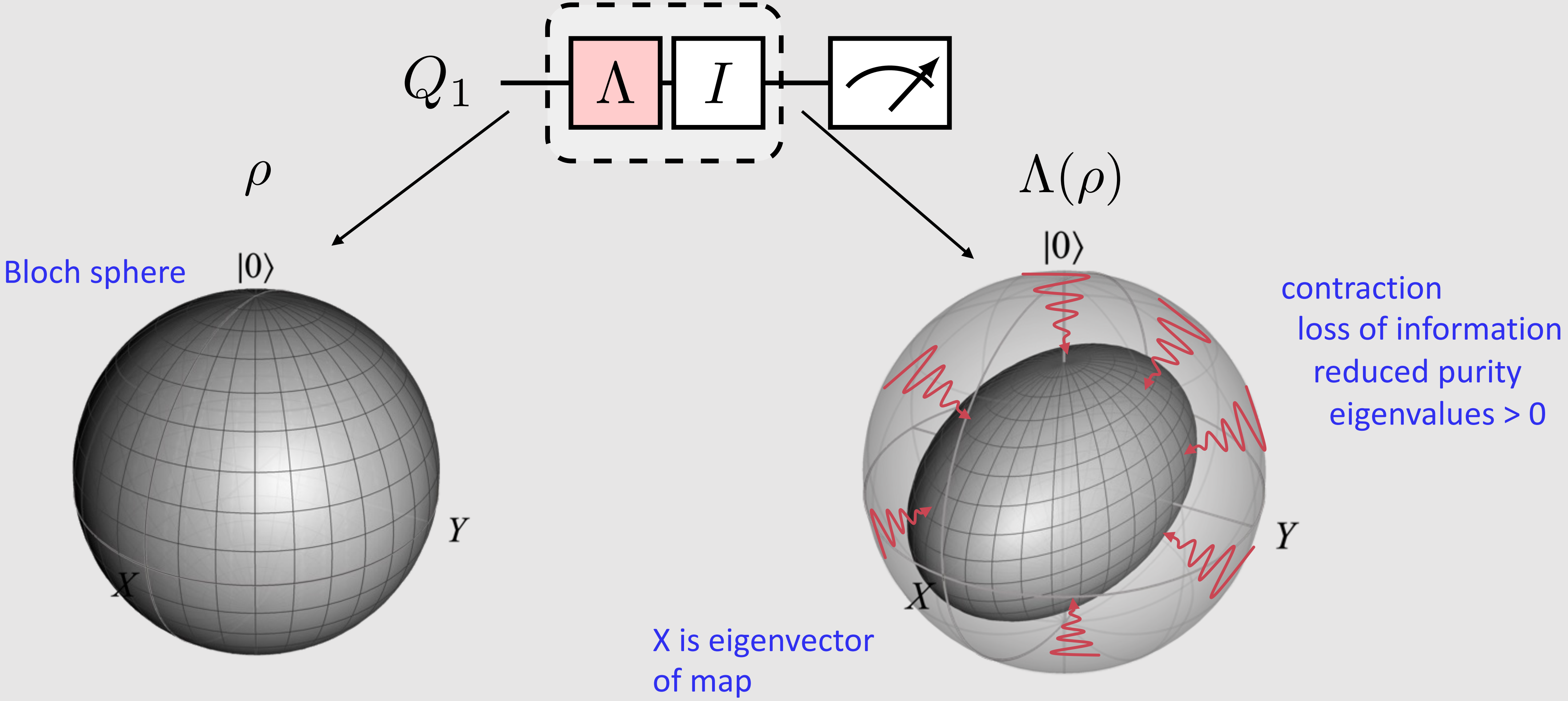


probability p

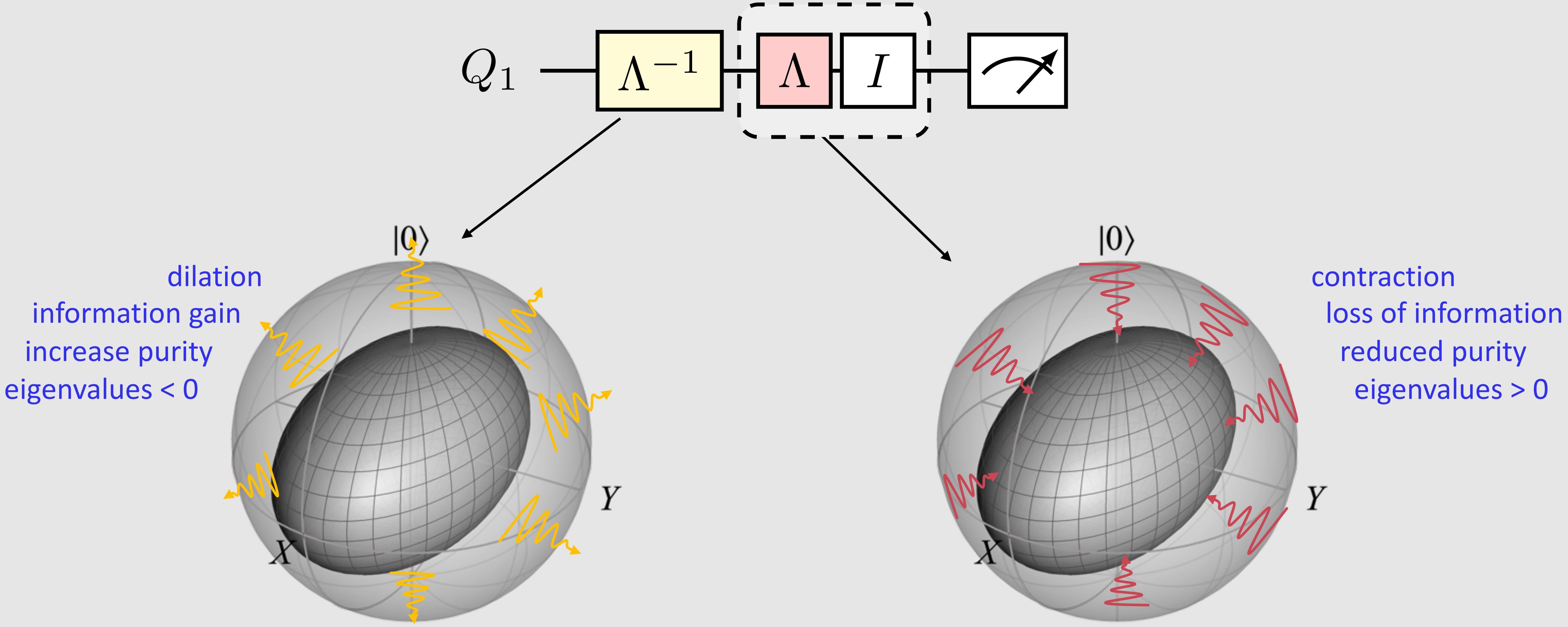


$$\Lambda\left(\rho\right) = (1-p)I\rho I + pX\rho X$$

Toy model: noise unraveling into quantum trajectories



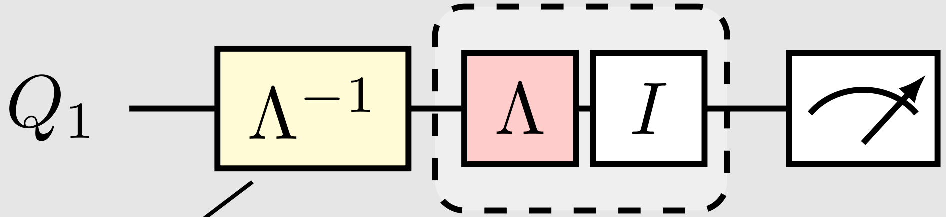
Inverse of noise map is not physical

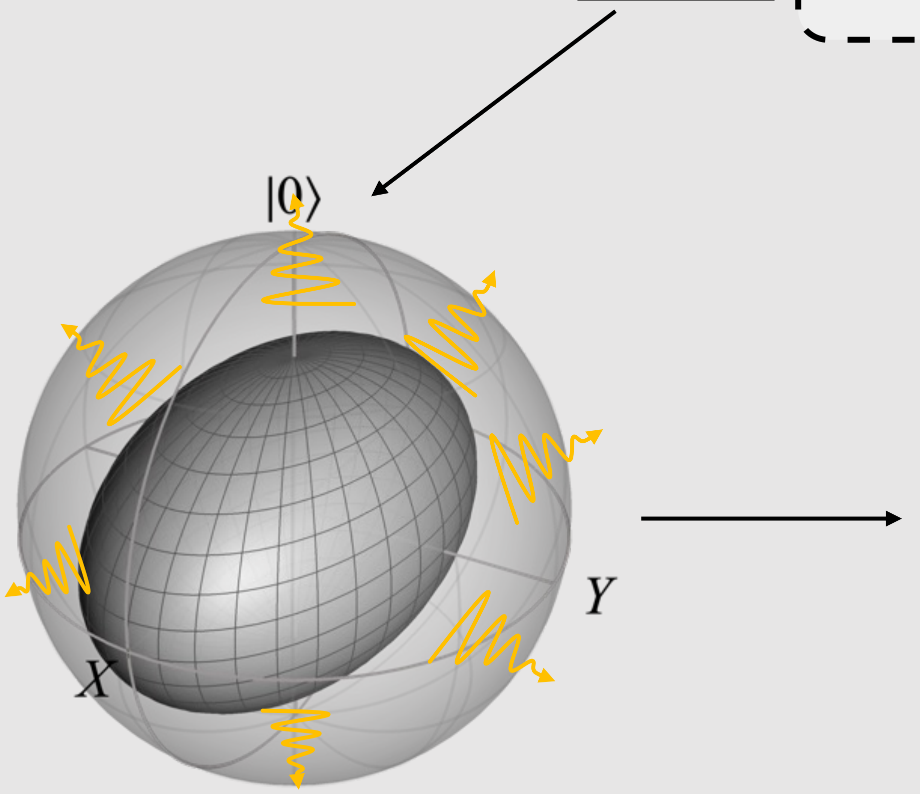


$$\Lambda^{-1}(\rho) = ?$$

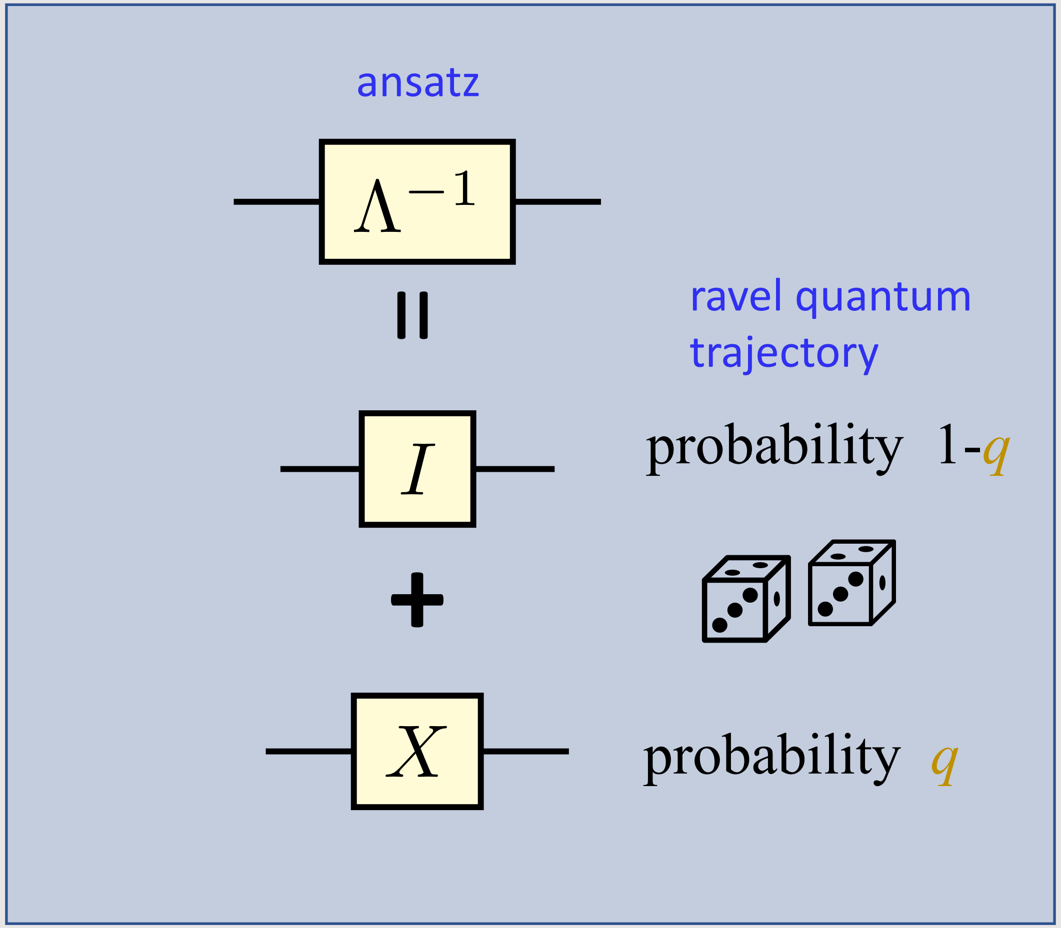
$$\Lambda(\rho)$$

Inverse of noise map is not physical





$$\Lambda^{-1}(\rho) = ?$$



Blackboard derivation

Setup

Details on notation:

Quantum register alphabet $\Sigma = \{0,1\}$

Hilbert space $\mathcal{H} = \mathbb{C}^{\Sigma}$

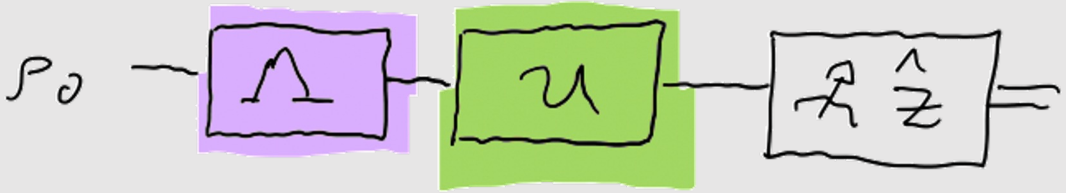
Initial state $\rho_0 \in D(\mathcal{H}) \subset L(\mathcal{H})$

Ideal unitary $U \in U(\mathcal{H}) \subset L(\mathcal{H})$

Ideal u-channel $\mathcal{U}(\rho) = U \rho U^{\dagger}$

$\mathcal{U} \in C(\mathcal{H}) \subset L(L(\mathcal{H}))$

Noisy gate / circuit $\tilde{\mathcal{U}} \in L(L(\mathcal{H}))$



Decompose noisy gate $\tilde{\mathcal{U}} = \mathcal{U} \Lambda$

Blackboard derivation

Simple Example

Keeping it simple and illustrative, let's do a simple case

Let $U = I$

$\mathcal{U} = I \cdot I$

For the noise, lets play with the simplest bit-flip channel

$$n(\rho) = \underbrace{(1-p)}_{\text{prob of no error}} I \rho I + \underbrace{p}_{\text{prob of error}} X \rho X$$

prob of a bit flip

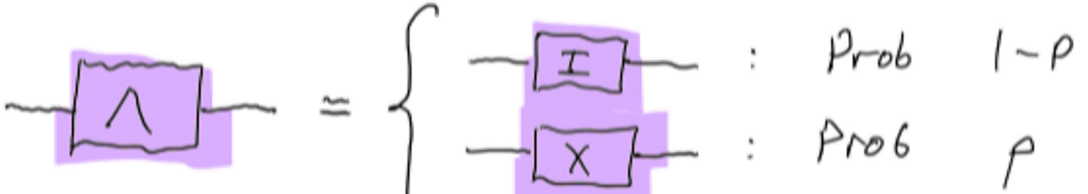
$$\Lambda_\rho = (1-p)\,\mathcal{I}_\rho + p\,\mathcal{X}_\rho$$

Equivalent superoperator channel representation

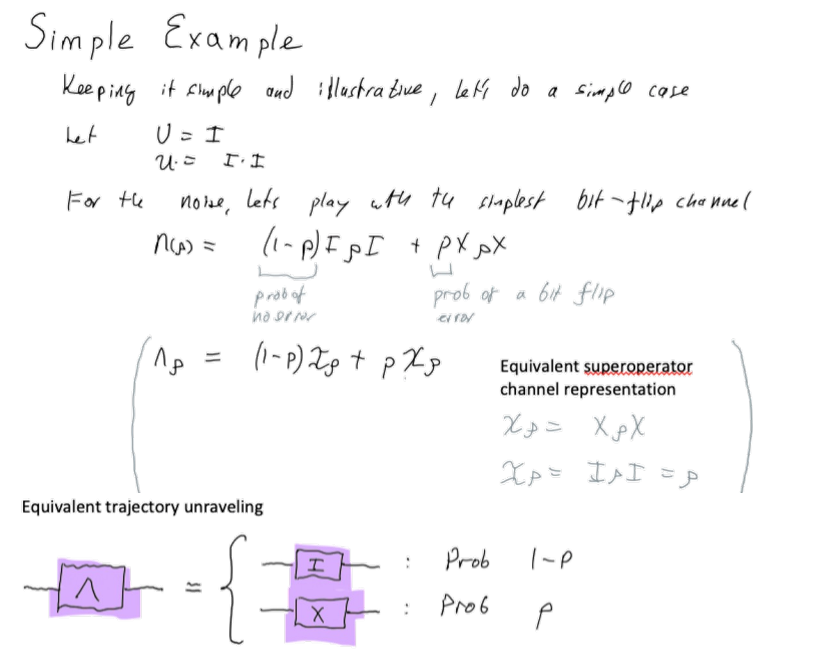
$\mathcal{X}_\rho = X \rho X$

$\mathcal{I}_\rho = I \rho I = \rho$

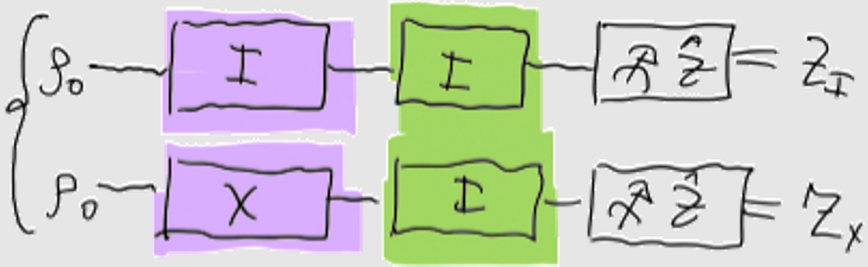
Equivalent trajectory unraveling



Blackboard derivation



Our circuit then is equivalent to either



with prob $1-p$

with prob p

The ideal expectation value is

$$Z_{ideal} = \langle \hat{Z} \rangle = \mathrm{Tr}(Z \mathcal{I} \rho_0) = \mathrm{Tr}(Z \rho_0) = \rho_Z$$

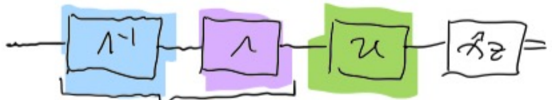
When the channel introduces an error however,

$$\mathbb{E}[Z_X] = \mathrm{Tr}(Z X \rho X) = \mathrm{Tr}(X Z X \rho) = \mathrm{Tr}(-Z\rho) = -\rho_Z$$

Blackboard derivation

Noise Inverse

To undo the noise, we'd like to introduce the inverse noise



cancel

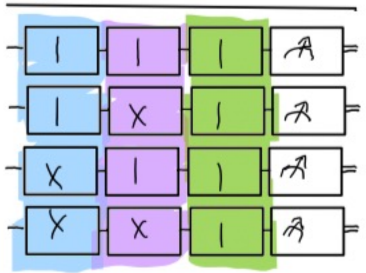
$$\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = I$$

Taking the ansatz $\Lambda^{-1}(p) = (1-r)\, I\cdot I + r(X\cdot X)$

we see 4 cases of unraveling

inverse	noise	no error	prob
I	I	✓	$(1-r)(1-p)$
I	X	X	$(1-r)p$
X	I	X	$r(1-p)$
X	X	✓	rp

Circuit



ideally, we what to interfere trajectories so that the no-error on will coherently add to unity probably, and the ones with an error will cancel.

$\therefore$ (A) $(1-r)(1-p) + r\cdot p = 1$ $\wedge$ (B) $(1-r)p + r(1-p) = 0$

$1 - r - p + 2rp = 1$ $\quad\quad$ $p + r - 2rp = 0$

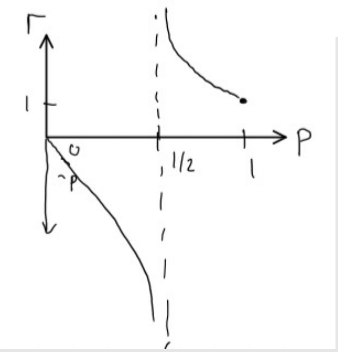
$r + p - 2rp = 0$ $\quad\quad$ same condition

$$\Rightarrow \quad r(1-2p) = -p$$

$$\therefore \quad r = \frac{-p}{1-2p}$$

Recall p is a probability $0 \le p \le 1$,

- $p = 0 \Rightarrow r = 0$ — no noise, no need to do anything
- $p = 1 \Rightarrow r = 1$ — deterministic bit-flip, requires deterministic bit flip as well for Λ^{-1}
- $p = 1/2 \Rightarrow r = \infty$ — singular value, since at $p=1/2$, we fully scramble the state
- $p \ll 1 \Rightarrow r \simeq -p$



Blackboard derivation

Note that we could equivalently have used the algebraic condition and solved for r

$$\Lambda(\Lambda^{-1}(\rho)) = I(\rho) = \rho \qquad \text{solve for } r$$

$$= \Lambda[(1-r)\rho + rX\rho X]$$

$$= \underbrace{(1-p)(1-r)\rho + pr\,\rho}_{\text{no error}} + \underbrace{(1-p)rX\rho X + (1-r)pX\rho X}_{\text{error}}$$

$$= \overbrace{[(1-p)(1-r) + pr]\rho}^{\text{no error}} + \overbrace{[(1-p)r + (1-r)p]X\rho X}^{\text{error}}$$

same conditions as above! solution $r = \frac{-p}{1-2p}$

Blackboard derivation

How to implement? Quasi - Probability

$$\Lambda^{-1} = (1-r) I \rho I + r X \rho X \qquad r = \frac{-p}{1-2p}$$

$$= \left[\frac{|1-r|}{|1-r| + |r|} \operatorname{sgn}(1-r) I \rho I + \frac{|r|}{|1-r|+|r|} \operatorname{sgn}(r) X \rho X \right] (|1-r| + |r|)$$

$$= \gamma \left[s_I p_I I \rho I + s_X p_X X \rho X \right]$$

with

$$\gamma = |1-r| + |r|$$

$$p_I = \frac{|1-r|}{\gamma} \qquad s_I = \operatorname{sgn}(1-r)$$

$$p_X = \frac{|r|}{\gamma} \qquad s_X = \operatorname{sgn}(r)$$

valid prob distribution

$$0 \le p_I, p_X \le 1 \quad \text{and} \quad |p_I| + |p_X| = 1$$

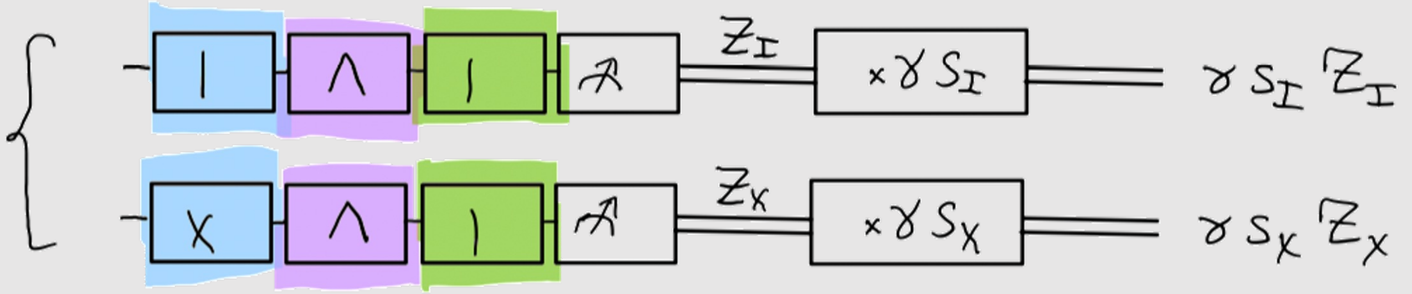
Blackboard

How to sample?

$$\langle Z \rangle = \mathrm{Tr}(Z \mathcal{I} \Lambda \Lambda^{-1} \rho_0)$$
$$= \mathrm{Tr}(Z \tilde{\mathcal{I}} [\gamma s_I p_I \rho_0 + \gamma s_X p_X X \rho_0 X])$$
$$= \gamma s_I p_I \mathrm{Tr}(Z \tilde{\mathcal{I}} \rho_0) + \gamma s_X p_X \mathrm{Tr}(Z \tilde{\mathcal{I}} X \rho_0 X)$$
$$= \gamma [s_I p_I \langle Z \rangle_I + s_X p_X \langle Z \rangle_X]$$

quantum circuit exp. val we can find

Equivalent interpretation: sample prob



: p_I

: p_X

Blackboard

Estimator $\quad E_{mitig} = \gamma s_I Z_I + \gamma s_X Z_X$

$$\mathbb{E}[E_{mitig}] = \langle \hat{Z} \rangle_{ideal}$$

$$\mathbb{V}[E_{mitig}] = \mathbb{V}[\gamma s_I Z_I] + \mathbb{V}[\gamma s_X Z_X]$$

$$= \gamma^2 \mathbb{V}[Z_I] + \gamma^2 \mathbb{V}[Z_X]$$

$$= \gamma^2 (2\sigma_{ideal}^2)$$

$$\sigma_{ideal}^2 = \mathbb{V}[Z_I] = 4q(1-q)$$

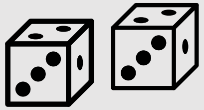
$$q = 1 - 2p_Z = \langle \frac{1-\hat{Z}}{2} \rangle$$

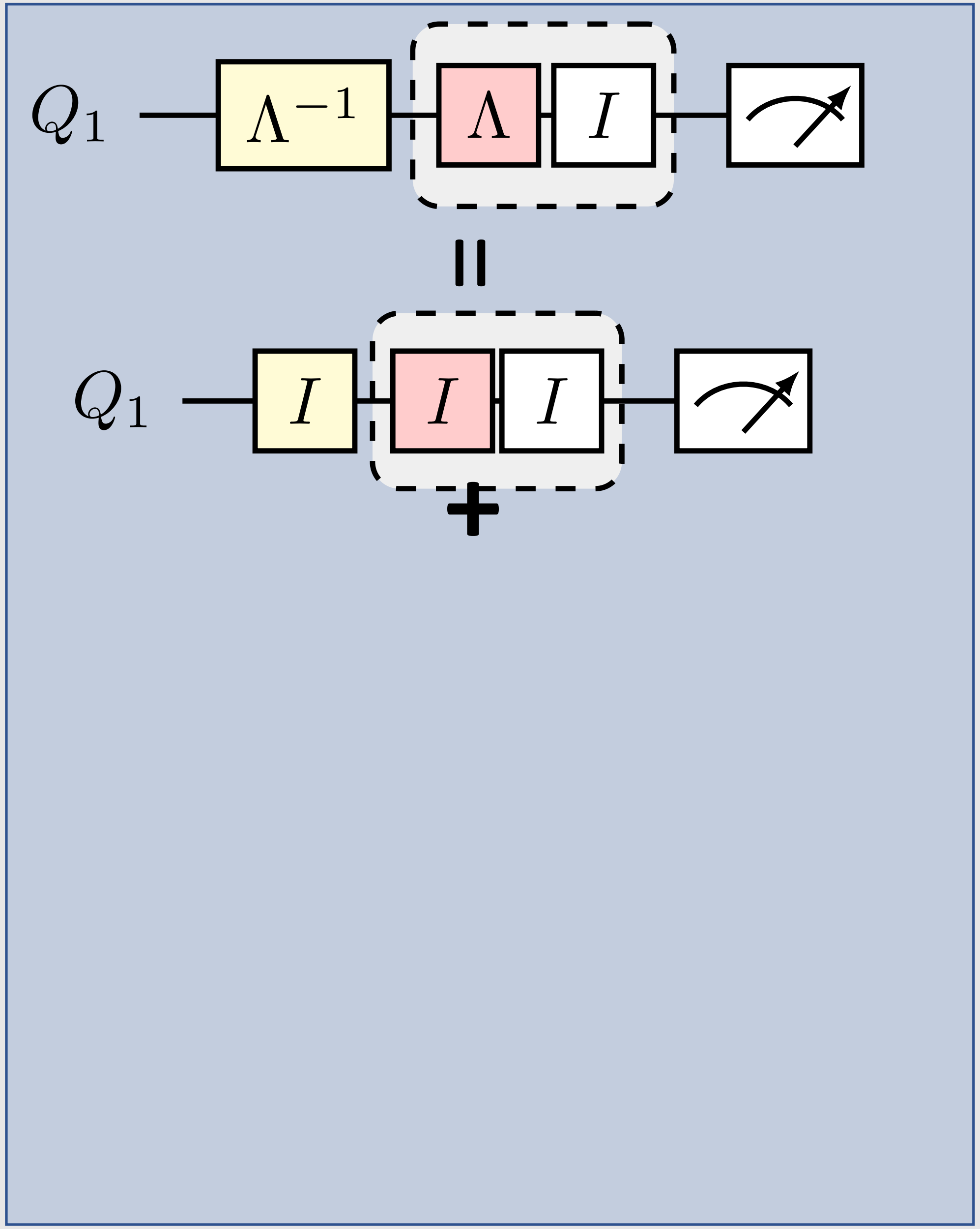
Since the X just flip $Z \to -Z$ of ρ, it follows that the variance is the same, since symmetric

Raveling quantum trajectories to undo noise

probability

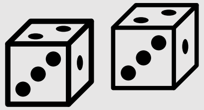
No error (1-q) (1-p)

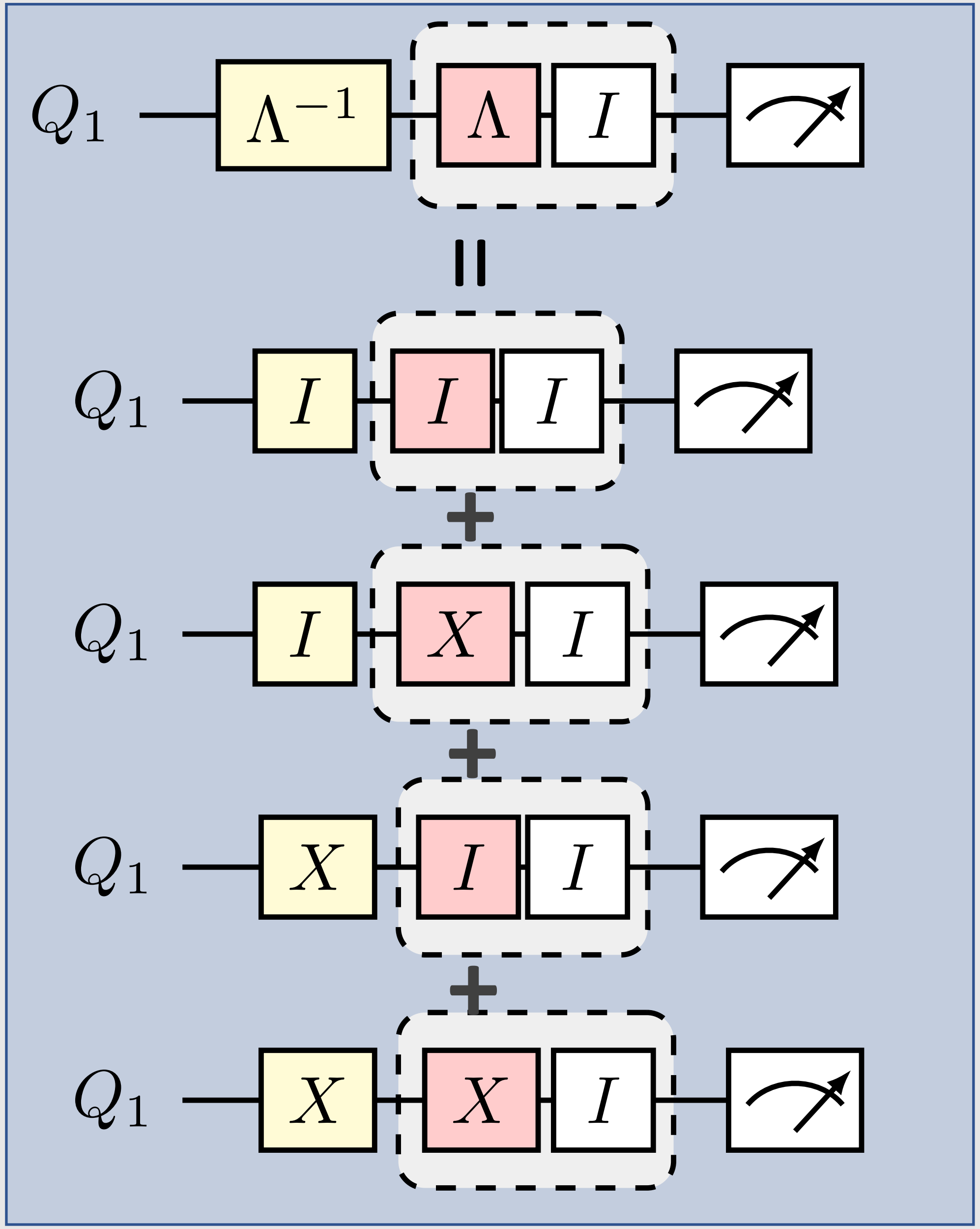




Raveling quantum trajectories to undo noise

	probability
No error	(1-q) (1-p)
ERROR!	(1-q) p
ERROR!	q (1-p)
Error CANCELED!	q p

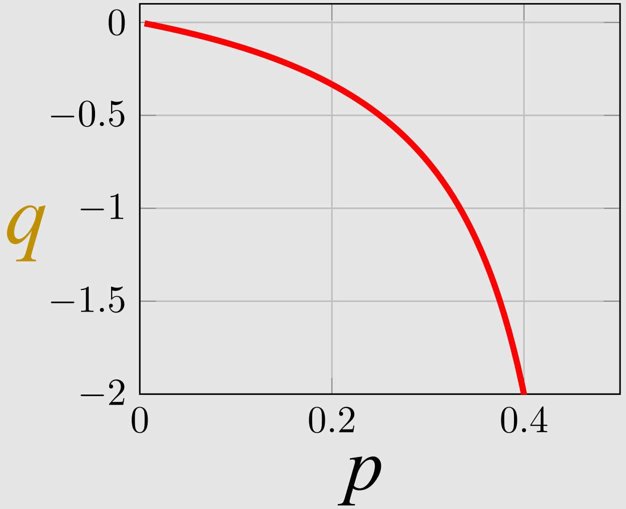




Solution to noise free!

$$q = \frac{-p}{1-2p}$$

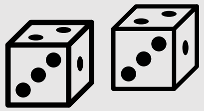
Sign & scale:
quasi-probability



How to implement?

probability

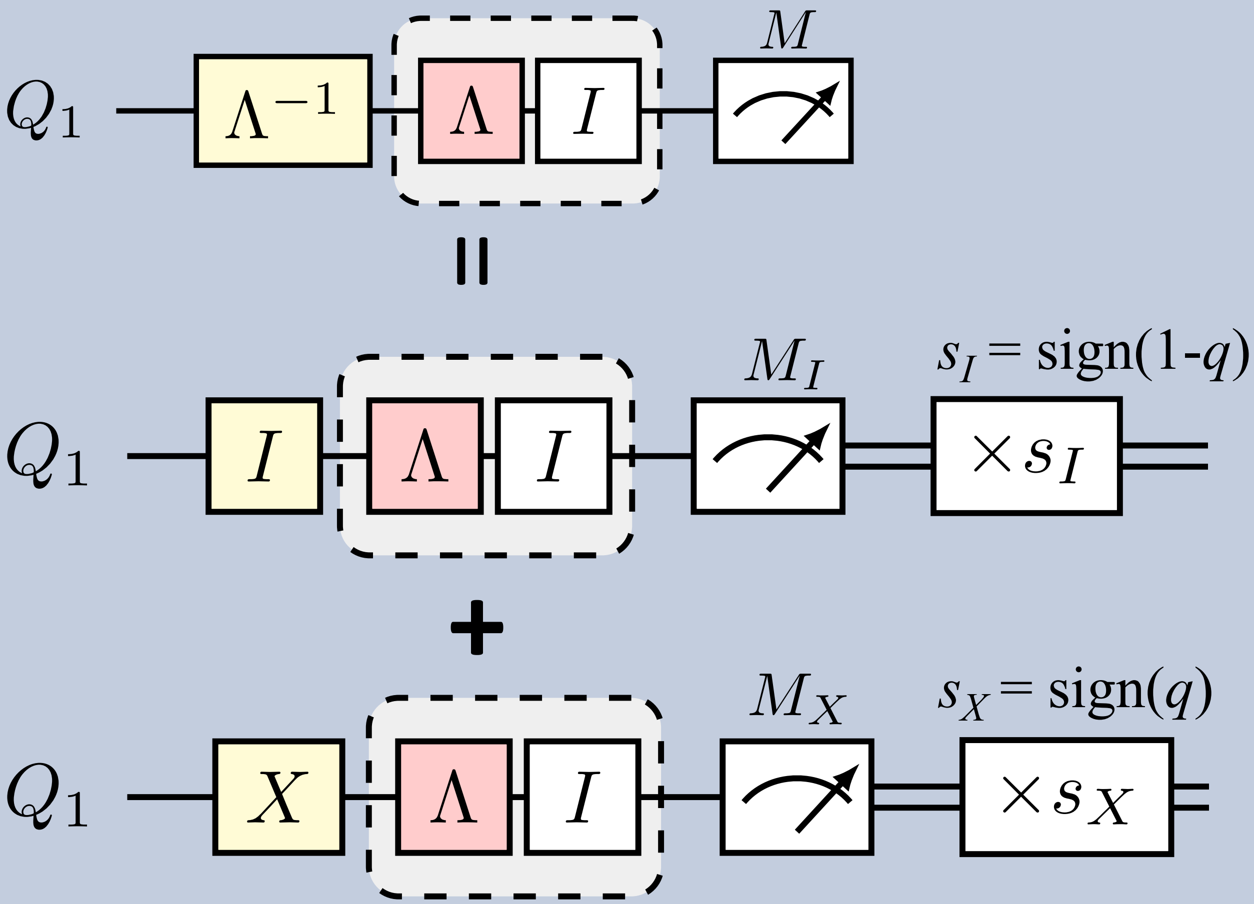
$P_I = |1\text{-}q|/\gamma$



$P_X = |q|/\gamma$

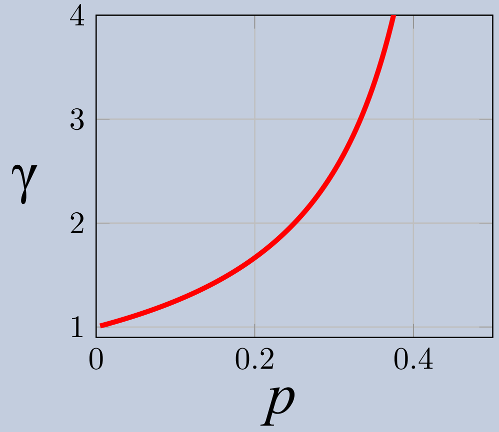
$P_I + P_X = 1$

random circuits
(quantum trajectories)



sampling overhead

$\gamma = |1\text{-}q| + |q|$



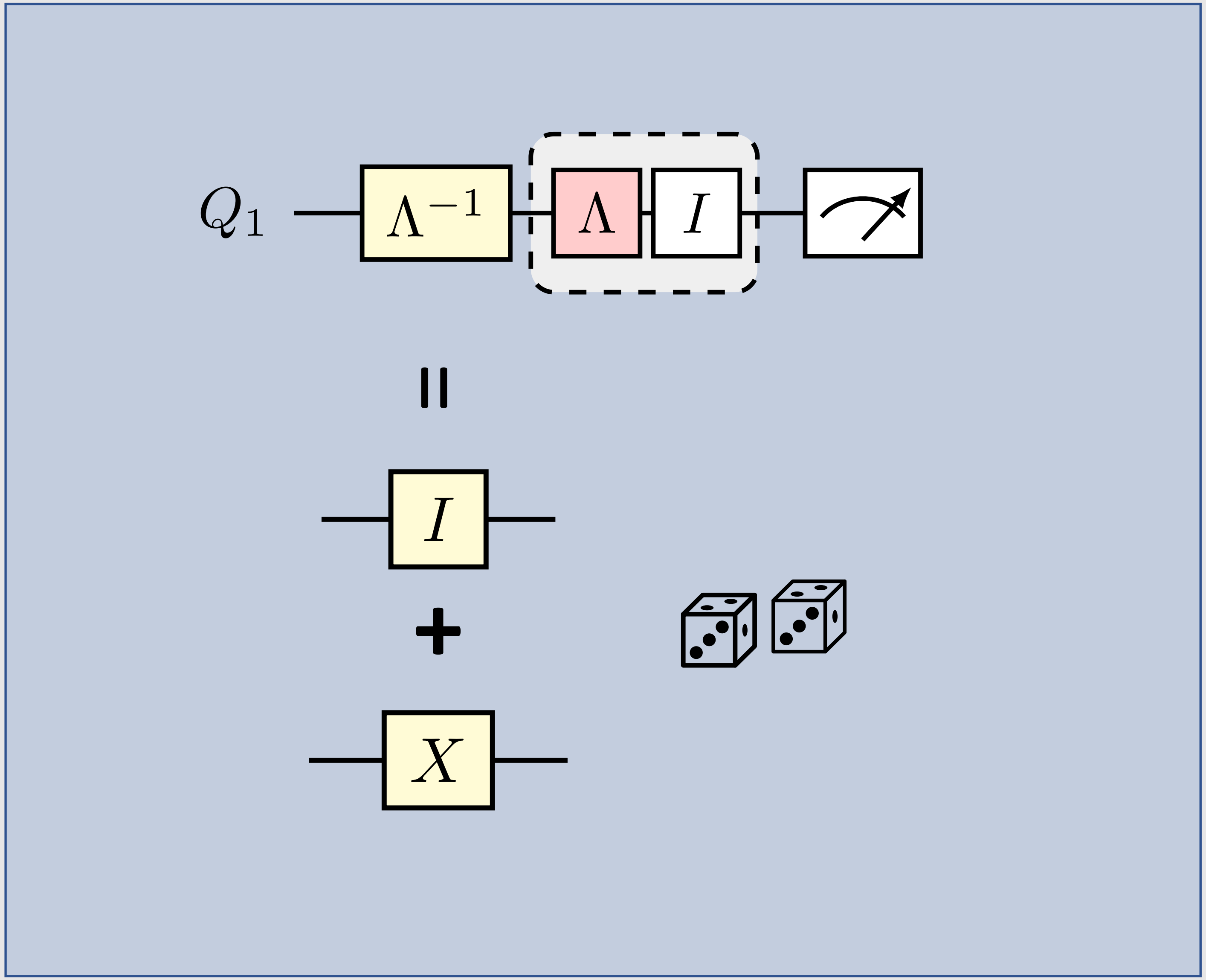
mitigated expectation

$$\langle M \rangle = \gamma\left(s_I P_I M_I + s_X P_X M_X\right)$$

Gain: Bias-free estimate!

Cost: Variance

Canceling noise with noise



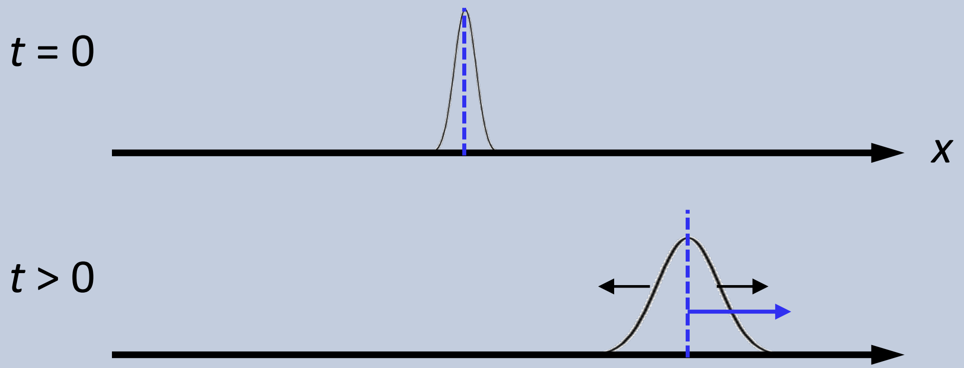
Canceling noise with noise: Drunkard's classical random walk analogy

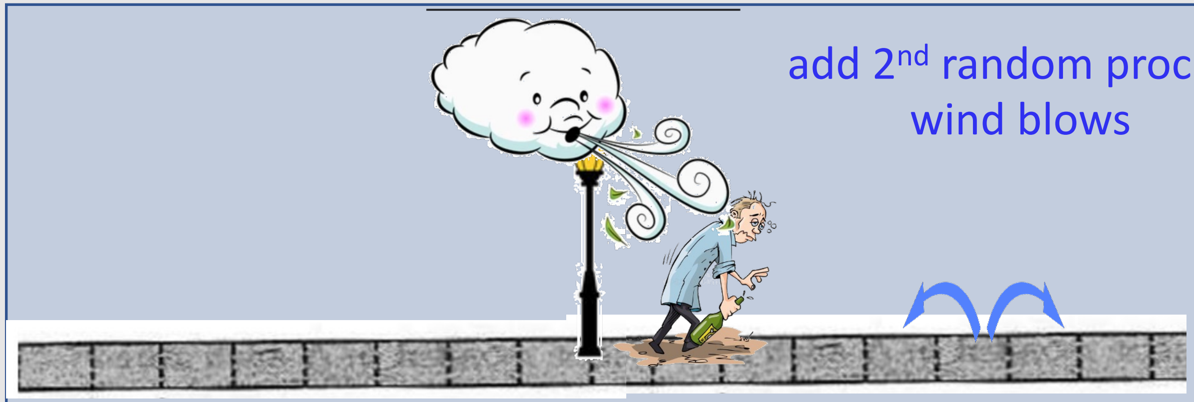
Random step

P(1 step left) = ½-p

P(1 step right) = ½+p

Distribution of random walk





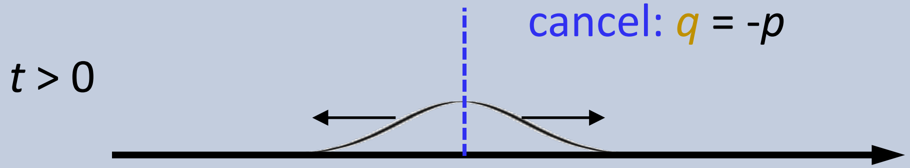
add 2nd random process

wind blows

P(1 step left) = ½+q

P(1 step right) = ½-q

Distribution of random walk with wind



Gain: Bias-free estimate!

Cost: Variance

Image: A. Brand