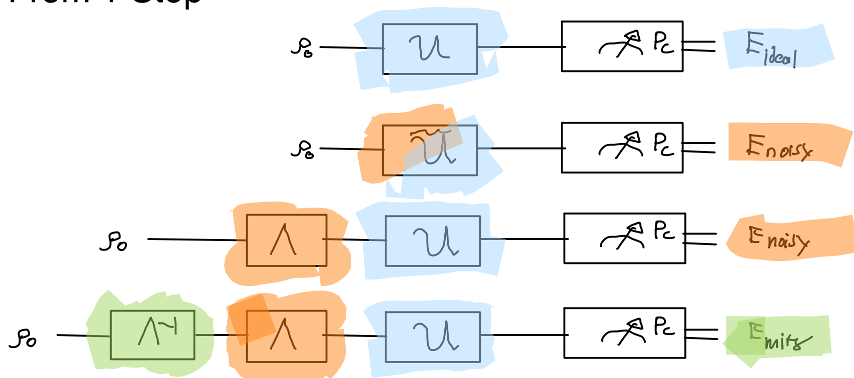
(C65C) PEC Full Derivation

Friday, July 21, 2023   8:09 AM

From 1 Step



Channel definitions

$\mathcal{U} = u \cdot u^{\dagger}$

$\langle \hat{P}_c \rangle(\cdot) = \langle\langle P_c | \cdot$

Need to introduce
- Super-op bra-ket notation ②
- Pauli channel $\mathcal{P}$ & PTM ③
  - super-op notation ④
  - WH ⑤
  - Pauli check representation ①
    - SQ $x \circ \circ y$, $z^{=} z$
    - $\langle a, b \rangle_{sp}$

$$\Lambda = \sum_a f_a |P_a\rangle\rangle\langle\langle P_a| \cdot$$
$$= \sum_b c_b \mathcal{P}_b \cdot$$

$-1 \le f_a \le 1$

$c_b \ge 0 \quad , \sum_b c_b = 1$

$$c_b = \frac{1}{2^n} \sum_a (-1)^{\langle a, b \rangle_{sp}} f_a$$

$\vec{c}_b = W^{-1} \vec{f_a} \qquad \vec{f_a} = W c_b$

$$\Lambda^{-1} = \sum_a f_a^{-1} |P_a\rangle\rangle\langle\langle P_a|$$
$$= \sum_b c_b^{inv} \mathcal{P}_b \cdot$$

$\overrightarrow{c_b^{inv}} = W \overrightarrow{f_a^{-1}}$

$$c_b^{inv} = \frac{1}{2^n} \sum_a (-1)^{\langle a, b \rangle_{sp}} f_a$$

$c_b^{inv} \in \mathbb{R}$

Circuit Expectation Value estimators

$$E_{ideal} := \langle \hat{P}_c \rangle_{\mathcal{U}}$$
$$= \langle\langle P_c | \mathcal{U} | \rho_0 \rangle\rangle$$

ideal exp value with noiseless unitary

$$E_{noisy} := \langle \hat{P}_c \rangle_{\tilde{\mathcal{U}}}$$
$$= \langle\langle P_c | \mathcal{U} \Lambda | \rho_0 \rangle\rangle$$
$$= \langle\langle P_c | \tilde{\mathcal{U}} | \rho_0 \rangle\rangle$$

noisy-gate expectation value

$$E_{mitg} := \langle\langle P_c | \mathcal{U} \Lambda \Lambda^{-1} | \rho_0 \rangle\rangle$$
$$= \langle\langle P_c | \mathcal{U} \Lambda \left(\sum_a f_a^{-1} |P_a\rangle\rangle\langle\langle P_a| \right) |\rho_0\rangle\rangle$$
$$= \langle\langle P_c | \mathcal{U} \Lambda \left(\sum_b c_b^{inv} \mathcal{P}_b \right) | \rho_0 \rangle\rangle$$
$$= \sum_b c_b^{inv} \langle\langle P_c | \mathcal{U} \Lambda \mathcal{P}_b | \rho_0 \rangle\rangle$$

sum of trajectories with weight $c_b^{inv} \in \mathbb{R}$

$$= \underbrace{\sum_b c_b^{inv}}_{\text{classical post process}} \underbrace{\langle P_c \rangle (\tilde{\mathcal{U}}, b)}_{\text{quantum}}$$

Quantum circuit we can execute on HW and find exp. value from.

$\therefore$ To find noise-free val all we have to do is to compute exp-val of all 4^n b-modified circuits! This would give us ideal exp value.

However, $|\{b\}| = 4^n$ grows exponentially, hence, infeasible.

but what if we could sample from it to approximate full sum. But... cant sample directly from c_b^{inv} which does not form a valid prob. distribution. Let's solve:

Quasi-Probability Distribution

c_b^{inv} can be outside $[0,1]$

$\sum_b c_b^{inv} =: \gamma \geq 1$ generally for Λ not unitary

e.g. bit flip channel $\Lambda = (1-p)\, I \cdot I + p\, X \cdot X$

$\Lambda^{-1} = \left(1 + \frac{p}{1-2p}\right) I \cdot I + \frac{-p}{1-2p} X \cdot X$

$$\underset{\substack{\uparrow \\ b=(0,0) \\ \text{choose vector}}}{c_I^{inv}} = 1 + \frac{p}{1-2p} \qquad \underset{\substack{\uparrow \\ b=(1,0)}}{c_X^{inv}} = \frac{-p}{1-2p}$$

Turn into probability

$$c_b^{inv} = \underbrace{\mathrm{sgn}(c_b^{inv})}_{\text{sign} \in \{-1,+1\}} \underbrace{\frac{|c_b^{inv}|}{\gamma}}_{\text{prob} \in [0,1]} \underbrace{\gamma}_{\text{scale}}$$

$$\bar{c}_b^{inv} := \frac{|c_b^{inv}|}{\|c_b\|_1} \quad \text{abs val} \qquad \|c_b\|_1 = \sum_b |c_b^{inv}| \leftrightarrow L_1 \text{ norm}$$

$$= \frac{|c_b^{inv}|}{\gamma}$$

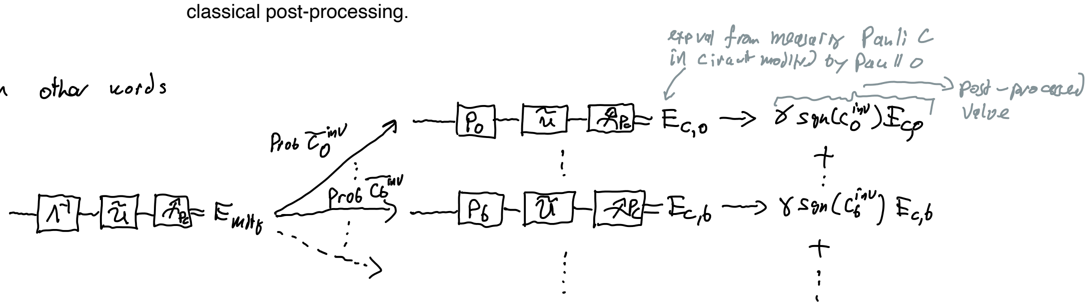
$$E_{mitg} = \sum_b c_b^{inv} \langle \hat{P}_c \rangle (\tilde{U}, b)$$

$$= \sum_b \mathrm{sgn}(c_b^{inv}) \frac{|c_b^{inv}|}{\gamma} \gamma \langle \hat{P}_c \rangle (\tilde{U}, b)$$

$$= \underbrace{\gamma}_{\text{scale}} \sum_b \underbrace{\mathrm{sgn}(c_b^{inv})\, \bar{c}_b^{inv}}_{\text{classical post-processing}} \underbrace{\langle \hat{P}_c \rangle (\tilde{U}, b)}_{\text{valid QC circuit can run \& find value on HW}}$$

In this form, the decomposition of the error error mitigated expectation value is simply a sum over expectation values of Pauli-gate modified circuits,
whose value can be obtained from direct quantum computer execution, weighted by a probability, c bar b inverse, and the sign.
The elements that perform the weighing and rescaling can all be done in classical post-processing.

In other words



$$\underbrace{E_{c,mitg}}_{\downarrow} = \sum_b \gamma\, \mathrm{sgn}(c_b^{inv}) E_{c,b}$$

mitigated value for Pauli C obtained from the quasi-prob distribution

From above, we know this is an unbiased estimator,
but what about the error and sampling

Estimator, Sampling, and Error Bounds

Sample circuits of the form

$$\left\{ \underbrace{\bar{c}_b^{inv}}_{\text{prob}} : -\boxed{P_b}-\boxed{\tilde{U}}-\boxed{A_{P_c}}= \underbrace{Y \in \{-1,+1\}}_{\substack{\text{Pauli observable} \\ \text{single-shot outcome} \\ \text{ie e-vals.}}} \longrightarrow \underbrace{X = \mathrm{sgn}(c_b^{inv})\, Y}_{\substack{\text{post processed value } X \\ X \in \{+1,-1\}}} \right\}$$

Lets say we sample M instances, randomly sampling a single value for σ and obtaining one-shot value on the QC, ie one random instance of $Y=1$ or $Y=-1$, which we then post-process.
The results are thus the classical random variables

$$\{X_1, X_2, \dots, X_M\} \quad \text{or} \quad \{X_m : m=1,\dots,M\}$$

where each $X_m \in \{1,-1\}$ and is distributed to modified Bernoulli distribution with some probability which can be any valid value and can vary from shot-to-shot m.

Our mitigation estimator is then for M shots:

$$E_M := \gamma \frac{1}{M} \sum_{m=1}^{M} X_m = \frac{1}{M} \sum_{m=1}^{M} \gamma \, \mathrm{sgn}(\underbrace{c^{inv}_{\sigma_m}}_{\text{Pauli chosen for } m\text{-th shot}}) \overset{\text{rand outcome of } m\text{-th shot for } \sigma\text{th circ.}}{Y_{m_\sigma}} (\overset{\text{noisy circuit}}{\tilde{U}}, \sigma_m)$$

There are now 2 random processes:

σ_m : which pauli σ we pick for shot m

Y_{σ_m} : which outcome ± 1 we get for σ_m circuit of shot m

Unbiased Estimator of the Ideal, noise-free circuit expectation

$$\mathbb{E}[E_M] = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}[\gamma X_m] \qquad \text{iid rand vars}$$

$$= \mathbb{E}[\gamma X_m] \qquad \text{no } X_m \text{ is different}$$

$$= \mathbb{E}\left[\gamma \, \mathrm{sgn}(c^{inv}_{\sigma_m}) \, Y_{\sigma_m}(\tilde{U}, \sigma_m)\right] \qquad \text{where rand var is } \sigma_M \text{ now, not just } m, \text{ so}$$

$$= \mathbb{E}_{\sigma, Y}\left[\gamma \, \mathrm{sgn}(c^{inv}_\sigma) \, Y_\sigma \underbrace{(\tilde{u}, \sigma)}_{\text{drop notation}}\right] \qquad \mathrm{Prob}[\sigma] = \bar{c}^{inv}_\sigma$$

$$= \sum_\sigma \mathbb{E}_Y[\gamma \, \mathrm{sgn}(c^{inv}_\sigma) Y_\sigma] \, \mathrm{Prob}[\sigma]$$

$$= \sum_\sigma \underbrace{\gamma \, \mathrm{sgn}(c^{inv}_\sigma)}_{\text{post-process of outcome}} \underbrace{\bar{c}^{inv}_\sigma}_{\text{sample prob}} \mathbb{E}_Y[\underbrace{Y_\sigma}_{\text{rand outcome}}] \longrightarrow$$

> note: $\langle \hat{P}_c \rangle(\tilde{u}, \sigma) = \langle\langle P_c | \mathcal{U} \Lambda \mathcal{P}_\sigma | \rho_0 \rangle\rangle$
> $= \mathbb{E}[Y_\sigma]$
> Rand variable ± 1 for output of the σth pauli circuit.
>
> $\therefore$ for some classical func of σ $f(\sigma)$ which does not depend on the value Y_σ but only on the label σ:
>
> $$\mathbb{E}[f(\sigma) Y_\sigma] = f(\sigma) \langle \hat{P}_c \rangle (\tilde{u}, \sigma)$$

$$= \sum_\sigma \gamma \, \mathrm{sgn}(c^{inv}_\sigma) \frac{c^{inv}_\sigma}{\gamma} \langle\langle P_c | \mathcal{U} \Lambda \mathcal{P}_\sigma | \rho_0 \rangle\rangle$$

$$= \langle\langle P_c | \mathcal{U} \Lambda \Big(\sum_\sigma c^{inv}_\sigma \mathcal{P}_\sigma \Big) | \rho_0 \rangle\rangle$$

$$= \langle\langle P_c | \mathcal{U} \Lambda \Lambda^{-1} | \rho_0 \rangle\rangle$$

$$= \langle\langle P_c | \mathcal{U} | \rho_0 \rangle\rangle$$

$$= \langle \hat{P}_c \rangle (\mathcal{U}_{ideal}) \qquad \text{without noise!}$$

unbiased estimator of the true noise-free, ideal value of the circuit

(Optional step) Variance

Variance of E_M

$$\mathbb{V}_{\underbrace{\{\sigma_m, X_m\}}_{\text{with respect to}}}[E_M] = \frac{\gamma^2}{M^2} \sum_{m=1}^{M} \mathbb{V}[X_m] \qquad \begin{array}{l} X_m \text{ iid} \\ \text{can drop subscript } m \\ \text{and emphasize } \sigma \text{ and value } X \end{array}$$

$$= \frac{\gamma^2}{M} \mathbb{V}[X(\tilde{u}, \sigma)] \qquad \begin{array}{l} \text{note the same variance is just} \\ \text{rescaled by } \gamma^2 \text{ due to } \gamma \end{array}$$