Filament and Membrane

Overview

The key concepts involve the combination of elasticity and statistical mechanics. Why? The answer involves both typicalerry scale and large scale in the problem. The second point really has two issues.

1. We will study rings composed of many atoms—
   a description of the system in terms of atomic bonds
   is not only inconsequential
   but necessary.

   Consider, for example, Filamentous actin. If we want
   to understand this network (Fig 1) or even single
   filament (Fig 2), all atom description (Fig 3) of
   the monomers are not useful, except to provide a
   few parameters for a higher-level, elastic description.

   Atomic interactions $\Rightarrow$ elastic moduli
   and coordinates $(E, \delta)$

   $\{ r_{ij} \mid i=1, \ldots, N > 1 \}$

   Atomic level $| \Rightarrow $ elastic description
Fig 2 - all atomic model of an actin filament (F-actin strand)
Fig 3 - A so-called "ribbon" diagram of an actin monomer w/ bound GDP.
There is a related pt that informs our model of bilamint and membrane elasticity.

2a

\[ L \gg a \]

\[ t \ll \sqrt{A} \]

We can exploit the fact that there are one or two (membrane and bilamnit respectively) small dimensions.

So our goal will be as follows:

\[ \text{(atomic interactions} \Rightarrow \text{elastic constants}) \Rightarrow \text{elasticity + geometry} \]

\[ \neg \text{not cared here.} \]

Effective elastic model of a bilamint or membrane.

Finally, we address easy scalar. For sufficiently small \( a \) or \( t \) (see above) the body experiences only small strain force. This implies what we need...
to account for small fluctuations of these structures in order to treat their mechanics/dynamics on cellular and subcellular scales.

Outline of lecture.

I. Filaments

A. Mechanics of a rod
   1. Torsion
   2. Bend
   3. Elasticity and geometry of a "line" mechanics
   4. Tension mechanics and equations of motion

B. Statistical mechanics of a filament
   1. Persistence length and diffusion on a sphere
   2. Fluctuations and the end-to-end distance
      a. Flexible limit and Kuhn length
      b. Stiff limit of a stick
IV. Semi-flexible Networks.
1. Pulling on a WLC (Worm-Like Chain)
2. Network mechanics

II. Membranes.
A. Mechanics of a thin plate
   1. bending a plate and internal stresses/strain
   2. energy density of a bent plate
      a. the bending modulus
      b. Bulk versus surface terms

B. 3. T=0 mechanics and the equation of equilibrium

B. Another look:
   1. A quick taste of the differential geometry of surfaces
   2. mean and Gaussian curvature
   3. The Helmholtz Hamiltonian
C. Statistical mechanics of surface
   1. Undulation at T=0 and the de Gennes Tapon length
I. Mechanics of Rods.

dNA, F-actin, and so on...

A. Torsion:

\[
\frac{d\varphi}{dz} = \frac{\varepsilon}{\ell} \quad \text{Rotation angle}
\]

\[
Z = 0 \\
Z = \varepsilon
\]

We will assume that $TR \ll 1$ so strains are small.

How do points move in a rod cross section as $z \to z + \delta z$?

\[
\vec{r} \to \vec{r} + \delta \varphi \times \vec{r}
\]

\[
\delta \varphi
\]

\[
\Rightarrow U_x = -T z \delta y \quad \text{and} \quad U_y = -T z \delta x
\]

From

\[
U_\alpha = (\delta \varphi)_{\alpha} = T z (\hat{z} \times \vec{r}) \quad \text{required by condition of twist.}
\]

\[
\alpha = x, y
\]

What about $U_z$?

\[
U_z = T Y(x, y) \quad \text{Torsion function}
\]

\[
\text{must vanish as } T \to 0
\]
From $U_{x\beta} = \frac{1}{2} (\partial_x U_{\beta} + \partial_{\beta} U_x)$
we have $U_{xx} = U_{yy} = U_{zz} = 0$
$U_{xy} = 0$

But $U_{xz} = \frac{1}{2} \frac{1}{T} \left( \frac{\partial^2 y}{\partial x} - y \right)$; $U_{yz} = \frac{1}{2} \frac{1}{T} \left( \frac{\partial^2 x}{\partial y} + x \right)$

Note: $\nabla \cdot \vec{U} = \partial_x U_x = 0$ no volume change.

From $\sigma_{ik} = 2\mu \left[ U_{ik} + \frac{\nu}{1-2\nu} \delta_{ik} \right]$

$\sigma_{xx} = 0$ and $\sigma_{xy} = 0$

Poisson Ratio

But $\sigma_{xz} = \mu T \left( \frac{\partial^2 y}{\partial x} - y \right)$; $\sigma_{yz} = \mu T \left( \frac{\partial^2 x}{\partial y} - x \right)$

Static Equilibrium $\Rightarrow \partial_x \sigma_{x\beta} = 0$ $\Rightarrow$

$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = 0$ $\Rightarrow$ $\nabla^2 y = 0$

where $\nabla^2 = \partial_x^2 + \partial_y^2$

Define $X(x,y)$ such that $\sigma_{xz} = 2\mu T \partial_y X$

$\sigma_{yz} = -2\mu T \partial_x X$

so $\partial_x \sigma_{xz} + \partial_y \sigma_{yz} = 0$ in taken care of automatically.
Now \( \partial_x \gamma = y + 2 \partial_y x \); \( \partial_y \gamma = -x - 2 \partial_x x \)

Using \( \partial_x \partial_y \gamma = \partial_y \partial_x x \)

\[ \Rightarrow 1 + 2 \partial_y^2 x = -1 - 2 \partial_x^2 x \]

\[ \Rightarrow \partial^2 \gamma = -1 \]

What are the boundary conditions on the surface of the rod?

\[ \hat{n} \cdot \mathbf{t} = 0 \]

So \( \partial_{xx} \hat{n}_x + \partial_{yy} \hat{n}_y = 0 \)

(\( i^0 \) i = x, y are simple)

Recall \( \partial_{xx} = \partial_{yy} = \partial_{xy} = 0 \)

\[ \hat{n}_x = 0 \]

Thus:

\( \partial_y x \hat{n}_x - \partial_x x \hat{n}_y = 0 \)

Consider the boundary curve

The local tangent is \( (x'(e), y'(e)) = \hat{t} \)

The local (outward) normal is \( \hat{n} = (+y'(e), -x'(e)) \)

\[ \Rightarrow (\partial_y x) \frac{dx}{de} + (\partial_x x) \frac{dy}{de} = \frac{d}{de} \hat{n}_x = 0 \]

\[ \Rightarrow \gamma = \text{constant on the boundary. As long as there is only one boundary we can take it to be} \]
\( \chi = 0 \) on the boundary.

Now what is the energy density \( \rho \) of the twisted rod, over the cross section?

\[
F = \frac{1}{2} \int \sigma_{ik} U_{ik} = \int \left( \sigma_{xz}^2 + \sigma_{yz}^2 \right) \frac{1}{2\mu} \, d^2 x
\]

In terms of \( \chi \) this is:

\[
F = 2\mu \tau^2 \langle \nabla \chi \rangle^2
\]

\[
\Rightarrow F = \frac{1}{2} \int \tau^2 \, d\zeta \\
\text{when}
\]

\[
c = 4\mu \int d^2 x \langle \nabla \chi \rangle^2
\]

But \( \langle \nabla \chi \rangle^2 = \nabla_{\perp} \cdot (x \nabla_{\perp} \chi) - \chi (\nabla_{\perp}^2 \chi) = \nabla_{\perp} \cdot (x \nabla_{\perp} \chi) + \chi
\]

\[
c = 4\mu \int x \nabla \cdot \nabla_{\perp} \chi \, d\zeta + 4\mu \int d^2 x \chi
\]

\text{boundry term}

With holes we get

\[
c = 4\mu \sum_{k} x_{k} s_{k} + 4\mu \int d^2 x \chi.
\]
So we know the following: (assuming $C$ is independent of $z$)

$$F_{Twist} = \frac{1}{2} C \int c^2 \, dz$$

and for a circular rod cross section:

$$X = (-x^2 - y^2 + R^2) A$$

and $\nabla_x X = -1 \Rightarrow A = \frac{1}{2}$

$$C = 4\mu \int r \, dr \int_0^{2\pi} \, d\phi \frac{(R^2 - r^2)}{4}$$

$$C = \frac{4\mu}{2\pi} \cdot \left[ \frac{R^2 - r^2}{2} - \frac{1}{4} \delta^4 \right]_0^R = 2\pi \mu \left( \frac{R^4}{2} - \frac{R^4}{4} \right)$$

$$C = \frac{1}{4\pi^2} \mu R^4$$

torsional rigidity $\propto R^4$

B. Bending

We work in the same way. To start consider a planar bend...
element length along stretched element

Original undeformed length $dz$

$dz'$ is the arc of a circle

$dz' = \frac{R + x}{R} dz = (1 + \frac{x}{R}) dz$

$\Rightarrow$ recall $de^2 = dl^2 + 2U_{ik} dx_i dx_k$

$\Rightarrow 2U_{zz} = \frac{2x}{R} \Rightarrow U_{zz} = \frac{x}{R}$ for bending

Now $\sigma_{zz} = \frac{Ex}{R}$

$E = \frac{9Ku}{3K+u}$ or $E = \frac{2u}{1+\sigma}$

Young's modulus

Now over the cross-section of the rod

$\int \sigma_{zz} dx = 0$ no tensile force applied.

Pure bending

$\Rightarrow \int x \frac{1}{3} dx = 0.$ implies the neutral surface has coordinate of the center of mass.
Now all \( U_{yy} = 0 \) except

\[
U_{zz} = x/R \quad \text{and} \quad U_{xx} = U_{yy} = -\sigma U_{zz}
\]

What about the displacement field?

\[
U_z = \frac{xz}{R}, \quad U_x = -\frac{\sigma x^2}{2R^2}, \quad U_y = -\frac{xy\sigma}{R} + k(x,z)
\]

and \( U_{xz} = 0 \Rightarrow \partial_x g + \frac{z}{R} + \partial_z h = 0 \) \( I \)

\[
U_{yz} = 0 \Rightarrow \partial_y g + \partial_z k = 0 \quad \text{II}
\]

\[
U_{xy} = 0 \Rightarrow -\frac{\sigma y}{R} + \partial_x k + \partial_y h = 0 \quad \text{III}
\]

\( U_y \) cannot depend on \( z \) \( \rightarrow \) each slice has the same deformation in \( y \) so \( \partial_y g = 0 \) and \( \partial_z k = 0 \) from \( \text{II} \)

From \( \text{I} \)

\[
h = -\frac{z^2}{2R} + \tilde{h}(y) \quad \text{and} \quad \partial_y g = 0
\]

\[
\text{III} \Rightarrow \partial_y \tilde{h} = \frac{\sigma y}{R} \Rightarrow \tilde{h} = \frac{\sigma y^2}{2R} \quad \text{and} \quad \partial_x k = 0
\]

\[
\Rightarrow U_x = -\frac{1}{2R} [z^2 + \sigma (x^2 - y^2)]
\]

\[
U_y = -\frac{xy\sigma}{R}, \quad U_z = xz/R
\]
Notes

1. Each undeformed z-slice will bend under load.

\[ z = z_0 + u_z = z_0 (1 + \gamma R) \]

2. Cross sections typically deform under load. Here is an example.

Consider a slice at \( z = 0.5 \) of an undisturbed circular rod of radius 1. Its boundary at \( (0.5, 0, 0) \) is undeformed (red).

Using \( \sigma = 1/2 \) incompressible.

Now consider the \( z = 0 \) slice. No tilt.

Look at \( h = (x + u_x)^2 + (y + u_y)^2 - a^2 \) on the boundary.
More importantly for our purposes, we get the energy per unit length by integrating over a cross section:

\[ \tilde{F} = \frac{1}{2} \int d^2 \sigma_{ik} U_{ik} = \frac{1}{2} R^{-2} \int x^2 d^2 x \]

Joel the moment of inertia tensor (component) in 2D (without the mass density):

\[ I_{xy} = \int d^2 x \{ \delta_{xy} r^2 - x_x x_y \} = I_{yy} = \int x^2 d^2 x \]

\[ \tilde{F} = \frac{1}{2} EI_{yy} R^{-2} \]

What is the bending moment of the internal stress on a given cross section of the curved rod?

\[ \frac{d}{dx} = \bar{x} \times (\sigma_{zz} dx) \]

Torque. Now integrate over the whole face.

\[ \bar{I}_y = -\frac{E}{R} \int x^2 dx = -\frac{EI_{yy}}{R} \]

\[ \bar{I}_x = 0 \]

\[ \bar{I}_x \rightarrow \text{torque on the element} \]

\[ \bar{I}_x \rightarrow \text{torque on the element} \]

\[ \text{switch sign for moment of force on element} \]
What is $I_{yy} = I_{xx} = I_{zz}$ for a circular cross section? 

$$I_y = \int x^2 \, dx = 4 \int_0^R x^2 \sqrt{R^2 - x^2} \, dx = 4R^4 \int_0^1 x^2 \sqrt{1-x^2} \, dx$$

$$= \frac{\pi}{16} 4R^4 = \frac{\pi R^4}{4}$$

What does this say about biopolymers?

They are all made of essentially the same stuff so

$$\frac{K_1}{K_2} \sim \left( \frac{R_1}{R_2} \right)^4$$

Bending modulus

In terms of persistence length (more on that soon)

$$lp \propto K$$

F-actin $lp = 17 \mu m \quad \text{And} \quad R \sim 7 \mu m$

metabolite $lp \approx 3 \text{mm}$

$$\frac{R_{\text{metabolite}}}{R_{\text{F-actin}}} \sim \frac{(25)}{(7)} = 163$$

$$\frac{lp_{\text{metabolite}}}{lp_{\text{F-actin}}} \sim \frac{3 \times 10^3}{17} = 176$$

Better than we have a right to expect!
Putting it all together: The energy of a deformed rod.

\[ \hat{t} \text{ unit tangent vector.} \]

coordinate system band to the slice.

Deformation \rightarrow \text{ series of rotation } \quad \text{pure twist} \quad \text{pure bending.}

\[ \vec{\Omega} = \frac{d\hat{\Omega}}{dl} \quad \vec{\Omega} \parallel \hat{e} \]

How does \( \hat{t} \) move as we go from \( l \rightarrow l + dl \)?

\[ \frac{dl}{dl} = \vec{\Omega} \times \hat{t} \quad \text{or} \]

\[ \hat{t} \times \frac{dl}{dl} = \hat{t} \times (\vec{\Omega} \times \hat{t}) = \vec{\Omega} - \hat{t} (\hat{t} \cdot \vec{\Omega}) \]

or \[ \vec{\Omega} = \hat{t} \times \frac{dl}{dl} + \hat{t} (\hat{t} \cdot \vec{\Omega}) \]

\[ \theta \text{ is } \Omega \text{ the twist angle.} \]

bending part.
Now \( \frac{dt}{dl} = \frac{1}{R} \hat{t} \) principal normal

radius of curvature

\[ R = \left| \frac{dt}{dl} \right| \]

Fred

We can write the energy of the deformed rod as a quadratic function of \( \hat{t} \)

There can be no terms of the form \( \Omega_2 \Omega_3 \)

Because Fred invariant under \( \hat{t} \rightarrow -\hat{t} \)

⇒ We must have \( \frac{1}{2} E \Omega_3^2 \) for torsion but in the \( \hat{3} \hat{3} \) plane we have a generic bilinear form:

\[
\frac{1}{2} E \left\{ I_{33} \Omega_3^2 + I_{35} \Omega_3^2 + 2 I_{35} \Omega_2 \Omega_3 \right\}
\]

\[
= \frac{1}{2} E \left( \begin{array}{c} \Omega_2 \\ I \end{array} \right) \left( \begin{array}{cc} I_{33} & \Omega_3 \\ \Omega_3 & I_{35} \end{array} \right) \left( \begin{array}{c} \Omega_2 \\ I \end{array} \right)
\]

We can diagonalize by using in the principal axis frame of the \( I \).
\[ F_{\text{rod}} = \int dl \left\{ \frac{1}{2} I_1 \Omega_1^2 + \frac{1}{2} I_2 \Omega_2^2 + \frac{1}{2} C \Omega_3^2 \right\} \]

Energy of a deformed rod.
The equation of equilibrium for rods.

\[ \frac{\partial}{\partial x} \left( \bar{F} + d\bar{F} \right) \]

Body force \( \bar{K} = \text{force/volume on the slab} \)

\[ F_x = \int \bar{K} \, dx \]

\[ \text{Force balance} \Rightarrow \bar{F} + d\bar{F} + \bar{K} \, dx - \bar{F} = 0 \]

\[ \Rightarrow \frac{d\bar{F}}{dx} = -\bar{K} \]

We also have to balance torques.

\[ \bar{M} + d\bar{M} \quad \text{moment of internal stress} \]

Torque balance:

\[ \bar{M} + d\bar{M} + \left[ -\bar{M} + -d\bar{e} \times -\bar{F} \right] = 0 \]
\[
\frac{d\vec{M}}{dt} + \vec{r} \times \vec{F} = 0
\]

\[
\frac{d\vec{M}}{dt} = -\vec{r} \times \vec{F} \Rightarrow \frac{d\vec{M}}{dt} = \vec{F} \times \vec{\hat{r}}
\]

So with no body forces \( \vec{F} = \text{cont} \) and

\[
\frac{d\vec{M}}{dt} = \vec{F} \times \frac{dr}{dt} = \frac{d}{dt} (\vec{F} \times \vec{r}) \Rightarrow
\]

\[
\vec{M} = \vec{F} \times \vec{r}(t) = \text{const.}
\]

A short discussion on boundary conditions:

1. "Clamped"
   \[ \vec{F}(0) \text{ and } \vec{\epsilon}(0) = \left. \frac{d\vec{r}}{dt} \right|_0 \text{ fixed.} \]

2. "Hinged"
   \[ \vec{F}(0) \text{ fixed } \vec{\epsilon}(0) \text{ free } \vec{M} = 0 \]

3. "Supported"
   \[ \vec{F} \perp \vec{\hat{r}} \text{ at } t = 0 \]
   \[ \vec{M} = 0 \text{ at } t = 0 \]

4. "Free"
   \[ \vec{F} = 0 \text{, } \vec{M} = 0 \text{ at the end } (t = 0) \]
Consider a circular cross section so we can eliminate twist and caply.

To see this:
\[ \Omega_z = \hat{e} \cdot \vec{\Gamma} \]

Using \[ \Omega_z = \frac{1}{c} M_z \]

\[ \frac{d}{dl} (\vec{M} \cdot \hat{e}) = c \frac{d}{dl} \Omega_z = \frac{d\vec{M}}{dl} \cdot \hat{e} + \vec{M} \cdot \frac{d\hat{e}}{dl} \]

Using \[ \frac{d\vec{M}}{dl} = \vec{F} \times \hat{e} \] we see that it vanishes.

\[ \Rightarrow c \frac{d}{dl} \Omega_z = \vec{M} \cdot \frac{d\hat{e}}{dl} \text{ and } \frac{d\hat{e}}{dl} = \vec{\Gamma} \times \hat{e} \]

\[ c \frac{d}{dl} \Omega_z = \vec{M} \cdot (\vec{\Gamma} \times \hat{e}) \]

and from \[ M_z = EI_1 \Omega_z, \] \[ M_\gamma = EI_2 \Omega_\gamma \]

we see that if \[ I_1 = I_2 \] then

\[ \vec{M} = EI \vec{\gamma} \]

\[ \text{part } 1 \to \hat{e}. \]
In other words,

\[
\bar{\tau} = EI \hat{t} \times (\hat{\hat{a}} \times \hat{t}) + \hat{t} \cdot \kappa \Omega x
\]

\[
\text{so } \bar{\tau} = EI \hat{t} \times \frac{d\hat{t}}{d\xi} + \hat{t} \cdot \kappa \Omega x
\]

Thus (finally),

\[
C \frac{d\Omega x}{d\xi} = EI \left( \hat{t} \times \frac{d\hat{t}}{d\xi} \right) \frac{d\hat{t}}{d\xi} + 
\]

\[
+ C \Omega x \hat{t} \cdot \frac{d\hat{t}}{d\xi} = 0
\]

Since \( \frac{d(\hat{t} \cdot \hat{t})}{d\xi} = 0 \)

\[
\Rightarrow \text{there is no twist bend coupling.}
\]

We also have a way to look at the shape of a rod.
\[ \dot{\mathbf{M}} = EI \ddot{t} \times \frac{d \mathbf{t}}{d \ell} = EI \frac{d \mathbf{r}}{d \ell} \times \frac{d^3 \mathbf{r}}{d \ell^2} \]

and

\[ \frac{d \dot{\mathbf{M}}}{d \ell} = \mathbf{F} \times \ddot{\mathbf{t}} = \mathbf{F} \times \frac{d \mathbf{r}}{d \ell} \]

so

\[ EI \frac{d \mathbf{r}}{d \ell} \times \frac{d^3 \mathbf{r}}{d \ell^2} = \mathbf{F} \times \frac{d \mathbf{r}}{d \ell} \]

Eqn of equilibrium for a rod.

let's try it out:

\[ \downarrow \mathbf{F} = (0, -F) \]

\[ \mathbf{r} = (x(x), y(x)) \quad \mathbf{t} = \frac{d \mathbf{r}}{d \ell} = (x', y') = (\sin \theta, \cos \theta) \]

\[ \frac{d^3 \mathbf{r}}{d \ell} = \left( \theta'' \cos \theta - \theta'^2 \sin \theta \right) \mathbf{x} + \gamma \left( -\theta'' \sin \theta - \theta'^2 \cos \theta \right) \]

\[ \frac{2}{\ell} \left( \frac{d \mathbf{r}}{d \ell} \times \frac{d^3 \mathbf{r}}{d \ell^2} \right) = -\theta'' \sin^2 \theta - \theta'^2 \cos \theta \sin \theta - (\theta'' \cos^2 \theta - \theta'^2 \cos \theta \sin \theta) \]

\[ = -\theta'' (\sin^2 \theta + \cos^2 \theta) = -\theta'' \]
So the equation of equilibrium becomes:

\[-EI \theta'' = [-\hat{y} F x \left( \sin \theta x + \omega \cos \theta \right)] \cdot \hat{z} = F \sin \theta\]

or

\[- EI \theta'' + F \sin \theta = 0\]

\[\theta'' + \frac{F}{EI} \sin \theta = 0\]

\[\theta' + F \sin \theta = 0 \quad f = \frac{F}{EI}\]

There is a first integral

\[\left[ \frac{1}{2} \theta'^2 + F \cos \theta \right] = 0 \quad \text{as long as } \theta' \neq 0\]

\[\Rightarrow \frac{1}{2} \theta'^2 - F \cos \theta = c\]

torque free

and at \( l = \theta L \) \( \theta' = 0 \), \( \theta = \Theta \) so \( c = -F \cos \Theta \)

\[\frac{1}{2} \theta'^2 - F \cos \theta = -F \cos \Theta \quad \theta' = \sqrt{-F \cos \Theta} \quad \sqrt{2}\]

Choose the + side \( \theta' \),

\[\int_{\theta}^{\theta_{0}} \frac{d \theta}{\sqrt{\cos \theta - \cos \Theta}} = 2\]
so \[ l = \sqrt{\frac{EI}{2F}} \int_{\theta_0}^{\theta_0} \frac{dz}{\sqrt{\cos z - \cos \theta_0}} \]

where we get \( \theta_0 \) from the condition that \( l(\theta_0) = L \leftarrow \text{length of the rod} \).

We recover \( x = \int \sin \theta \, dl \)
\( y = \int \cos \theta \, dl \)

from this solution.

Finally note that for small bending:
\[ L \approx \sqrt{\frac{EI}{2F}} \int_{\theta_0}^{\theta_0} \frac{dz}{\sqrt{-z^2 + \theta_0^2}} \]
\[ L \approx \sqrt{\frac{EI}{2F}} \int_{0}^{1} \frac{\theta_0 \, d\alpha}{\theta_0 \sqrt{\alpha^2 + 1}} = \frac{\pi}{2} \sqrt{\frac{EI}{F}} \]

so if \( F \ll \frac{\pi^2 EI}{4L^2} \) there will be no bent solution \( \Rightarrow \frac{EI}{F} \) is an example of Euler buckling.

Finally \( EI = K \) bending modulus.
Note \[ [K] = \frac{E}{L^4} = E_0 L \]

We typically write it as \( K = k_B T l_p \)

Thermodynamic persistence length.

To see what that means, we should look at the statistical mechanics of a wiggly line or filament.

At temperature \( T \) we might see

What sets the tangent vector correlations?
Persisting length and diffusion on a sphere.

\[ G(\hat{e}_f, \hat{e}_i \mid L) = \text{probability of having a tangent of } \hat{e}_f \text{ gain that the filament has a tangent } \hat{e}_i \text{ at a point } L \text{ to the left.} \]

A couple of general pts:

1. \[ G(\hat{e}_f, \hat{e}_i \mid L) \]
2. \[ G(\hat{e}_f, \hat{e}_i \mid L) = G(\hat{e}_f, \hat{e}_i \mid 1-L) \]
   no special directions.
3. Rotational diffusion of \( \hat{e} \)

\[ \hat{e}_i \quad \hat{e}_f \]

We may not need to know in general what \( G \) looks like, but we do need it for infinitesimal distance (arclength)

\[ G(\hat{e}_b, \hat{e}_a \mid e) \xrightarrow{\text{normalization}} \frac{\varepsilon}{2\pi} \int_0^\varepsilon \sum_{n=1}^\infty \left\{ \frac{\varepsilon n^2}{2\pi} \left( e^{-n^2 \varepsilon^2} \right) \right\} e^{-\frac{K}{2T} \left( \frac{(\hat{e}_b - \hat{e}_a)^2}{\varepsilon^2} \right) \varepsilon} \]

\[ G(\hat{e}_b, \hat{e}_a \mid e) \xrightarrow{\varepsilon \to 0} N \exp \left\{ -\frac{K}{2T} \left( \frac{(\hat{e}_b - \hat{e}_a)^2}{\varepsilon^2} \right) \varepsilon \right\} \]
In the limit $\varepsilon \to 0$ we get

$$S^2(\hat{E}_b - \hat{E}_a) = G(\hat{E}_b, \hat{E}_a; 0)$$

Composition of probability:

$$G(\hat{E}_p, \hat{E}_i | L + \varepsilon) = \int \hat{E}'' G(\hat{E}_p, \hat{E}'' | \varepsilon) G(\hat{E}'', \hat{E}_i | L)$$

Take small $\varepsilon$ (in the $\hat{E}_i$ phase of body at the limit)

$$\hat{E}'' = \hat{E}_p + \vec{\varepsilon}$$

where

$$\vec{\varepsilon} = \vec{\delta} \times \hat{E}_p$$

where $|\vec{\delta}| \to 0$ as $\varepsilon \to 0$

Our plan is to expand both sides order by order in $\varepsilon$ and match terms.

RHS is easy...

$$G(\hat{E}_p, \hat{E}_i | L + \varepsilon) = G(\hat{E}_p, \hat{E}_i | L) + \varepsilon \sum_{s=1}^{\infty} G(\hat{E}_p, \hat{E}_i | s) + O(\varepsilon)$$
Now
\[ G(\hat{t}, \hat{t}; 11) = G(\hat{t}, \hat{t}; 11) s^2(\hat{t} - \hat{t}_f) +
(\overrightarrow{\Delta \theta} \times \hat{t}_f) \nabla \nabla G(\hat{t}, \hat{t}; 11) \bigg|_{\hat{t} = \hat{t}_f} + \frac{1}{2} (\overrightarrow{\Delta \theta} \times \hat{t}_f) (\overrightarrow{\Delta \theta} \times \hat{t}_f)^T. \]

\[ \forall \hat{t} \neq \hat{t}_f G(\hat{t}, \hat{t}; 11) \bigg|_{\hat{t} = \hat{t}_f} + O(\overrightarrow{\Delta \theta}^3) \]

and
\[ \exp \left[ -\frac{K}{2 \tau \epsilon} (\hat{t}_f - \hat{t})^2 \right] = \exp \left[ -\frac{K}{2 \tau \epsilon} (\overrightarrow{\Delta \theta} \times \hat{t})^2 \right] \]
\[ = \exp \left[ -\frac{K}{2 \tau \epsilon} \left\{ (\overrightarrow{\Delta \theta})^2 - (\overrightarrow{\Delta \theta} \times \hat{t})^2 \right\} \right] \]

The terms linear in \( \overrightarrow{\Delta \theta} \) must integrate to zero.
There is a \( \overrightarrow{\Delta \theta} \sim -\overrightarrow{\Delta \theta} \) symmetry in the probability distribution

So we are left with:
\[ \mathbb{E}_\epsilon G(\hat{t}_f, \hat{t}_f; 14) = \frac{1}{\epsilon} \int_{-1}^1 d^2 \overrightarrow{\Delta \theta} (\overrightarrow{\Delta \theta} \times \hat{t}_f) (\overrightarrow{\Delta \theta} \times \hat{t}_f). \]

\[ \forall \hat{t}_a \neq \hat{t}_f G(\hat{t}, \hat{t}; 11) N e^{-\frac{K}{2 \tau \epsilon} (\overrightarrow{\Delta \theta})^2} \]

In other words, we need to compute:
\[ I = \left< \epsilon_{ijkl} \epsilon_{k'\ell'} \delta \theta_{j} \delta \theta_{\ell} \right> \overrightarrow{\Delta \theta}_{\ell} \overrightarrow{\Delta \theta}_{k'} \]
\[ I = \langle (S_0^i)^2 \rangle \left( \mathbf{\hat{r}}_i^2 - \mathbf{\hat{r}}_i \cdot \mathbf{\hat{r}}_j \mathbf{\hat{r}}_j \right) \]

Now we can compute the tangent-tangent conductance as follows:

Consider the time-derivative
\[ \partial_t \mathbf{E}(x) = \int d^2 \mathbf{k}_i \mathbf{E}_i \cdot \mathbf{\hat{r}}_i \partial_t G(\mathbf{E}_i, \mathbf{\hat{r}}_i; \mathbf{k}_i) \]

Integrating by parts twice
\[ = D \int d^2 \mathbf{k}_i \mathbf{E}_i \cdot \mathbf{\hat{r}}_i \left( \mathbf{\hat{r}}_i \times \partial_t \mathbf{E}_i \right) G(\mathbf{E}_i, \mathbf{\hat{r}}_i; \mathbf{k}_i) \]

and
\[ (\mathbf{\hat{r}}_i \times \partial_t \mathbf{E}_i) \cdot \mathbf{\hat{r}}_i = -E_{\alpha \beta} \mathbf{\hat{r}}_i \partial_t \mathbf{\hat{r}}_{\alpha \beta} \]

This is related to an old friend "bac-cab" rule...
So
\[ \Delta \left< \hat{t}(s) \cdot \hat{t}(0) \right> = -2D \int d\mathbf{r}_1 d\mathbf{r}_2 \left< \hat{t}_s \cdot \hat{t}_0 \right> G(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) \]
\[ = -2D \left< \hat{t}(s) \cdot \hat{t}(0) \right> \]
\[ \Rightarrow \]
\[ \left< \hat{t}(s) \cdot \hat{t}(0) \right> = \exp \left[ -2D d s l \right] \]

Since \( \left< \hat{t}^2 \right> = 1 \), we know the initial condition

\[ \left< \hat{t}(s) \cdot \hat{t}(0) \right> = e^{-\frac{1}{2} l s l / l_p} \]

when \( l_p = K / T \) is the persistence length.
What do the tangent correlations tell you? (26)

end-to-end length

\[ \langle R^2 \rangle = \langle [\vec{R}(L) - \vec{R}(0)]^2 \rangle \]

Note \( \vec{R}(s) = \int dz \ \hat{e}(z) \) so put the origin at \( \vec{R}(0) \)

\[ \langle R^2 \rangle = \langle \int_0^L ds \ \hat{e}(s) \cdot \int_0^L ds' \ \hat{e}(s') \rangle \]

\[ = \int_0^L ds \int_0^L ds' \langle \hat{e}(s) \cdot \hat{e}(s') \rangle \]

\[ = 2 \int_0^L ds \int_0^s ds' \ e^{-|s-s'|/l_p} = 2 \int_0^L ds \ e^{-s/l_p} \ e^{-|s'/l_p} \]

\[ \langle R^2 \rangle = 2 l_p \int_0^L ds \ [1 - e^{-s/l_p}] = 2 l_p \left[ L + l_p e^{-L/l_p} \right] \]
\(< R^2 \) = 2l_p L \left[ 1 + \frac{\rho_p}{L} \left( e^{-\frac{L}{l_p}} - 1 \right) \right]

Does this make sense? Check the limits.

\( L \ll l_p \) \quad \text{rod-like filament}

\(< R^2 \) \approx 2l_p L \left[ 1 + \frac{\rho_p}{L} \left( 1 - \frac{1}{2} \frac{L^2}{l_p^2} \right) \right]

\approx 2l_p L \left[ 1 - 1 + \frac{1}{2} \frac{L^2}{l_p^2} \right] = l_p^2 + O(4l_p)

\text{essentially straight filament}

\( l_p \gg \) \quad \text{flexible filament}

\(< R^2 \) \approx 2l_p L \left[ 1 + \frac{l_p}{L} \right] \approx 2l_p L

\(< R^2 \) \approx (2l_p) L \ll L^2 \quad \text{coiled filament}

we can write this as

\(< R^2 \) = b^2 N \quad \text{where} \quad N = \frac{L}{2l_p} \ll \# \text{segments}

b = 2l_p \quad \text{Kuhn length}

A short side box on random coils. This is an easy way to see this result.
A lattice model with no correlations:

\[ R^2 = \sum_{n,m=1}^{N} (\hat{t}_n \cdot b) \cdot (\hat{t}_m \cdot b) \]

The legs of step = b Kuhn length

\[ \langle R^2 \rangle = b^2 \sum_{n,m=1}^{N} \langle \hat{t}_n \cdot \hat{t}_m \rangle = b^2 \sum_{n=1}^{N} \langle \hat{t}_n^2 \rangle = Nb^2 \]

Looking at the small bending limit allows you to make reasonable simplifying approximations.
Small bendy limit:
\[
\frac{dt}{d\varepsilon} = \frac{d^2r}{d\varepsilon^2} = \frac{d^2r}{dz^2} 
\]
so \[\Omega_y = -\frac{d^2u_y}{dz^2}, \quad \Omega_z = \frac{du_x}{dz^2}\]

\[
\Rightarrow F_{nd} = \frac{1}{2} E \int_0^L \left\{ I_1 \left( \frac{d^2u_x}{dz^2} \right)^2 + I_2 \left( \frac{d^2u_y}{dz^2} \right)^2 \right\} \, dz
\]

To get the equations of static equilibrium:
\[
\frac{\delta F}{\delta u_x} = 0 \Rightarrow \frac{1}{2} \int_0^L K \left[ \varphi_z^2 (u_x + su_z) \right] \, dz
\]

Integrate by parts.

\[
= \frac{1}{2} \int_0^L K \varphi_z^2 u_x \varphi_z^2 u_x \, dz
\]

\[
= K \int_0^L \left[ \varphi_z^2 u_x \varphi_z^2 u_x \right] - \varphi_z^2 u_x (\varphi_z^2 u_x) \, dz
\]

\[
= K \varphi_z^2 u_x^3 \varphi_z^3 u_x \bigg|_0^L - K \int_0^L \varphi_z^2 u_x \varphi_z^2 (su_z) \, dz
\]

\[
= K \varphi_z^2 u_x^3 \varphi_z^3 u_x \bigg|_0^L - K \left( \varphi_z^2 u_x \right) \, su_z \bigg|_0^L + K \int_0^L \varphi_z^2 u_x \varphi_z^2 su_z \, dz
\]
We could include localized forces acting on the light ray by incorporating a term like:

\[ \int_{0}^{L} f_x(z) u_x(z) \, dz \]

We get

\[ \frac{S F_{rod}}{\delta u_x} = 0 \Rightarrow 0 = K \frac{\partial^2 u_x}{\partial z^2} \left| u_x \right|_0^L - K (\partial^3 u_x) \left( \frac{1}{\partial z^4} \right) u_x \left|_0^L \right. \]

\[ + K \partial^4 u_x - f_x(z) \]

It is easy to identify the body terms:

1. \[ K \frac{\partial^2 u_x}{\partial z^2} \left| u_x \right|_0^L = \text{energy proportional to local angle} \]

\[ u_x \uparrow \hspace{1cm} \alpha = \tan \theta \approx \alpha \approx \frac{u_x}{z} \]

\[ z = 0 \]

\[ \Rightarrow K \frac{\partial^2 u_x}{\partial z^2} = \text{torque at the body} \]

2. \[ -K \left. \frac{\partial^3 u_x}{\partial z^3} \right| u_x \left|_0^L \right. \text{energy proportional to load displacement} \]

\[ \Rightarrow -K \frac{\partial^3 u_x}{\partial z^3} = \text{torque at the body} \]
example: clamped end at 0 free end at L.

\[ \begin{align*}
8U_x|_0 &= 0 \quad 8U_x'|_0 = 0 \quad \text{and} \quad \partial^2_x U_x|_L = 0 \\
\partial^3 U_x|_L &= 0
\end{align*} \]

Subject: \[ U_x'' = 0 \Rightarrow \]

\[ U_x(x) = A + Bz + Cz^2 + Dz^3 \]

b.c. at \( z = 0 \): \( A = 0 \), \( B = 0 \)

\[ U_x(x) = C + 6DL = 0 \]

\[ U_x'''(L) = D = 0 \Rightarrow U_x'(L) = 0 \quad \text{of con.} \]

Now put a body moment at the end.

\[ U_x'' = T \neq 0 \Rightarrow 2C = T \]

and \( D = 0 \)

\[ U_x(z) = \frac{T}{2} z^2 \]

Now try:

\[ U_x'(0) = \alpha \quad U_x'(L) = -\alpha \]

\[ U_x(0) = U_x(L) = 0 \]

\[ A = 0 \quad \text{and} \quad B + CL^2 + DL^3 = 0 \]

\[ B = \alpha \quad \text{and} \quad B + 2CL + 3DL^2 = -\alpha \]
\[ C = -\alpha / L \quad \text{and} \quad D = 0 \]

\[ U_x(z) = \alpha z - \frac{\alpha z^2}{L} = \alpha z \left[ 1 - \frac{z}{L} \right] \]

and finally \( U_x'(0) = \alpha \quad U_x'(4) = 0 \)
\( U_x(0) = U_x(L) = 0 \)

\[ C = -2\alpha / L \quad D = \alpha / L^2 \]

\[ U_x(z) = \alpha z - \frac{2\alpha z^2}{L} + \frac{\alpha z^3}{L^2} = \alpha z \left[ 1 - 2 \frac{z}{L} + \left( \frac{z}{L} \right)^2 \right] \]

Try it w/ a force applied to the end.

Now consider a thermally filament.
\[ F_{\text{tot}} = \frac{1}{2} K \int_{0}^{L} U_{x}^{n}(z)^2 \, dz \quad \text{expand in sine series} \]

\[ U_{x}(z) = \sum_{n=1}^{\infty} A_{n} \sin \left( \frac{n \pi}{L} z \right) \]

\[ \bar{z} = \frac{z}{L} \]

\[ F_{\text{tot}} = \frac{1}{2} K \sum_{m=0}^{\infty} \int_{0}^{L} A_{n} A_{m} \sin \left( \frac{n \pi}{L} z \right) \sin \left( \frac{m \pi}{L} z \right) \, dz \left( \frac{n \pi}{L} \right)^2 \left( \frac{m \pi}{L} \right)^2 \]

\[ F_{\text{tot}} = \frac{1}{2} K \sum_{n,m=1}^{\infty} \left( \frac{n \pi}{L} \right)^2 \left( \frac{m \pi}{L} \right)^2 A_{n} A_{m} \int_{0}^{1} d\bar{z} \sin \left( n \pi \bar{z} \right) \sin \left( m \pi \bar{z} \right) \]

\[ F_{\text{tot}} = \frac{1}{2} K \sum_{n,m=1}^{\infty} \left( \frac{n \pi}{L} \right)^2 \left( \frac{m \pi}{L} \right)^2 A_{n} A_{m} L \frac{1}{2} \delta_{nm} \]

\[ F_{\text{tot}} = \frac{1}{2} KL \sum_{n=1}^{\infty} \frac{(n \pi)^4}{L^4} A_{n}^2 \]

The partition sum \( Z = \frac{1}{\beta} \int dA_{n} e^{-\frac{KL}{4} \sum_{n=1}^{\infty} \frac{n^4 \pi^4}{L^4} A_{n}^2} \)

We see that (through the magic of Gaussian integrals)

\[ \langle A_{n} \rangle = 0 \quad \langle A_{n} A_{m} \rangle = \delta_{nm} \frac{1}{2} \frac{KL}{4T} \frac{n^4 \pi^4}{L^4} \]

\[ \langle A_{n}^2 \rangle = \frac{L^2}{n^2 \pi^2} \quad \frac{2T}{K'} = \frac{2L^3}{\xi_{p} n^4 \pi^4} \]
So what is \( \langle U^2(z) \rangle \)?

\[
\langle U^2(z) \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \langle A_n \sin \left( \frac{n\pi z}{L} \right) A_m \sin \left( \frac{m\pi z}{L} \right) \rangle \\
= \sum_{n=1}^{\infty} \frac{2L^3}{n\pi^4} \frac{1}{n^4} \sin^2 \left( \frac{n\pi z}{L} \right)
\]

dominated by the \( n=1 \) mode.

\[ \langle U^2(z) \rangle \]

\[ \begin{align*}
\text{from } n=1 \\
\text{all other modes.}
\end{align*} \]

What about pulley on a filament.

[From Köis Mackintosh Janney 1995]

\[ T_{\text{small}} \]

Pulley at the filament w/ tension \( T \).
How much light is "used up" by transverse thermal vibrations?

For a configuration $u(x)$ we have analytically

$$L_\infty = \int_0^L \sqrt{1 + u'^2} \, dz \approx L + \frac{1}{2} \int_0^L u'^2 \, dz$$

$$SL = L_\infty - L = \frac{1}{2} \int_0^L u'^2 \, dz$$

We add a term to the energy of the form: $\tau SL$

$$F_{\text{tot}} = \frac{1}{2} \int_0^L \left\{ K u''^2 + \tau u'^2 \right\} \, dz$$

Using the Fano sine series again,

$$F_{\text{tot}} = \frac{1}{2} \sum_{n=1}^{\infty} \left\{ \frac{K n^2 \pi^4}{L^4} + \tau n^2 \pi^2 \right\} (A_n)^2$$
Compute \( \langle S_L \rangle_T \) — ensemble over time \( T \).

\[
\langle S_L \rangle_T = \sum_{n=1}^{\infty} \frac{1}{2^4} \left\langle A_n \frac{n^2 \pi^2}{L^2} \right\rangle = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{L^2} \langle A_n^2 \rangle
\]

and \( \langle A_n^2 \rangle = \frac{1}{2^4} \frac{T}{\pi} \frac{1}{2} \left[ \frac{n^2 \pi^2 k + 2 n^2 \pi^2}{L^2} \right] \)

\[
\Rightarrow \langle S_L \rangle_T = \sum_{n=1}^{\infty} \frac{T}{2^4} \frac{n^2 \pi^2}{L^2} \left[ \frac{n^2 \pi^2 k + 2 n^2 \pi^2}{L^2} \right] \times \frac{1}{2^4} \frac{T}{\pi} \frac{1}{2} \left[ \frac{n^2 \pi^2 k + 2 n^2 \pi^2}{L^2} \right]
\]

\[
\langle S_L \rangle_T \approx T \sum_{n=1}^{\infty} \frac{1}{2^4} \frac{T}{\pi} \frac{1}{2} \left[ \frac{n^2 \pi^2 k + 2 n^2 \pi^2}{L^2} \right] \langle A_n^2 \rangle
\]

\[
\langle L_\infty - L \rangle \sim \sum_{n=1}^{\infty} \frac{T}{T + kn^2 \pi^2 L^2}
\]

as \( T \) gets bigger, more and more terms are of the form \( \frac{1}{T} \) instead of \( \frac{1}{n^2} \) so the sum decreases w/ increasing \( T \). — As it should!
Now for small $T$ we can write:

\[ \sum_{n=1}^{\infty} \frac{1}{T + \frac{K n^2 \pi^2}{L^2}} = \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right) \frac{L^2}{K \pi^2} - T \sum_{n=1}^{\infty} \frac{1}{(K n^2 \pi^2)^2} + T^2 \sum_{n=1}^{\infty} \frac{1}{(K n^2 \pi^2)^3} = \frac{\pi^2 L^2}{6 K \pi^2} - \frac{T L^4}{K^2 \pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} + \frac{T^2 L^6}{K^3 \pi^6} \sum_{n=1}^{\infty} \frac{1}{n^6} \]

or

\[ \langle L \rangle \approx L_0 - \frac{T L^2}{6 K} + \frac{T L^4}{K^2 \pi^4} + \frac{T^2 L^6}{K^3 \pi^6} \]

\[ \langle L \rangle \approx L_0 - \frac{L^2}{6 \ell_p} + \frac{T L^4}{K \ell_p \pi^4} + \frac{T^2 L^6}{K^2 \ell_p \pi^6} \]

equilibrium short linear first nonlinear correction at $T=0$ response

In particular $T \sim SL \frac{K^2}{K^2 \ell_p 90}$ or $T \sim 90 \frac{K^2}{TL^4}$

third spin-cont

of the filament
What is the modulus of a filament network?

1. Each filament gets stretched.

\[ SL \sim \theta L e \]

Length between entanglement or crosslink.

2. There are \( \frac{1}{\xi^2} \) filaments per unit area.

\[ \sigma = \frac{\text{Force}}{\text{area}} \times \frac{1}{\xi^2} \times T \sim \frac{K^2 \theta L e}{T L e} \frac{1}{\xi^2} \]
G modules (chain) is given by

$$G \sim \frac{K^2}{T \xi^3}$$  \hspace{1cm} \text{flexible chain} \hspace{1cm} \frac{T}{\xi^3}

if the network is densely crosslinked \( \xi \ll \xi_c \)

$$G \sim \frac{K^2 \xi^{-5}}{T} \sim \frac{K^2}{T} \xi^{-\frac{1}{2}}$$  \hspace{1cm} \text{for densely crosslinked gel}

$$\xi \sim \frac{1}{(ac)^{1/2}}$$  \hspace{1cm} \text{number density of spheres}
Bending a plate: Mechanics [LL Chapter II]  

1. On the neutral surface \( U_x = U_y = 0 \) \( \rightarrow 2^{nd} \) order in \( \xi(x,y) \)

\[ U_z = \xi(x,y) \]

2. Take \( \hat{n} \), small bend.

\[ \nabla_k \hat{n}_k = 0 \quad \text{on surface} \Rightarrow \overline{\sigma}_{xz} = \overline{\sigma}_{yz} = \overline{\sigma}_{zz} = 0 \]

3. For linear elasticity

\[ \overline{\sigma}_{xz} = \frac{E}{1+\sigma} U_{zz} ; \quad \overline{\sigma}_{zz} = \frac{E}{(1+\sigma)(1-2\sigma)} \left\{(1-\sigma)U_{zz} + \sigma(U_{xx} + U_{yy})\right\} \]

These must vanish.

4. From

\[ \partial_z U_x = -\partial_x U_z \]
\[ \partial_z U_y = -\partial_y U_z \]

and

\[ U_{zz} = -\frac{\sigma}{(1-\sigma)} \left[ U_{xx} + U_{yy} \right] \]
Replace $U_z$ by $\frac{x}{2}(x, y)$

\[
\begin{align*}
\partial_z U_x &= -2x \frac{\partial}{\partial x} U_x \\
\partial_z U_y &= -2y \frac{\partial}{\partial y} U_y
\end{align*}
\]

\[
\Rightarrow U_x = -z \frac{\partial}{\partial x} U_x ; U_y = -z \frac{\partial}{\partial y} U_y
\]

set $U_x = U_y = 0$ at the neutral axis. Then $U_x = -2x$

\[
\int \frac{1}{2} x^2
\]

Then $U_x = -z x$

Now we can directly compute $U_{xx}, U_{yy}$:

\[
U_{xx} = -z \frac{\partial^2}{\partial x^2} U_x, \quad U_{yy} = -z \frac{\partial^2}{\partial y^2} U_y, \quad U_{xy} = -z \frac{\partial}{\partial x} \frac{\partial}{\partial y} U_y
\]

$U_{xz} = U_{yz} = 0$ and $U_{zz} = \frac{5}{1-\sigma} \frac{z}{2} \left( \nabla^2 \frac{1}{2} \right)

\[
= \frac{c_x^2 + c_y^2}{2}
\]

Now we can write the energy of deformation.
\[ F = \frac{E}{2(1-\sigma)} \left[ \varepsilon_{yy}^2 + \frac{\sigma}{1-2\sigma} \varepsilon_{xx}^2 \right] \]

\[ F = \frac{E}{1+\sigma} \left[ \frac{1}{2(1-\sigma)} \left( \nabla^2 \varepsilon \right)^2 + \left[ \partial_y^2 \varepsilon - \partial_x^2 \varepsilon \right]^2 \right] \]

all \( z \) dependence is here. We can get an effective energy for the plate by integrating on the thickness:

\[ \int_{-h/2}^{h/2} z^2 \, dz = \frac{1}{3} h^3 \]

\[ \frac{1}{24} h^3 \]

Plate elastic energy \( \Rightarrow \) for structures like rod body:

\[ F_{pl} = \frac{E h^3}{24(1-\sigma^2)} \int dxdy \left[ \left( \nabla^2 \varepsilon \right)^2 + 2(1-\sigma) \left( \partial_y^2 \varepsilon - \partial_x^2 \varepsilon \right)^2 \right] \]

area integral

The first variation and the equation of equilibrium for plates:

We need to minimize \( F \) with respect to \( \varepsilon(x,y) \)

\[ \frac{S F_{pl}}{S \varepsilon(x,y)} = 0 \Rightarrow \text{A P.D.E.} \]
The surface term will tell us about the boundary condition on the field \( \psi \).

The Berry phase is \( K_{2D} = \frac{Eh^3}{24(1-\sigma^2)} \) and

- has dimensions of energy,
- has max for incompressible material \( \sigma = 1/2 \).

1st variation: First term \((\nabla^2 \psi)^2\)

\[
\frac{1}{2} \int \left( \nabla^2 \psi \right)^2 \, dx = \frac{1}{2} \int \nabla^2 (\nabla^2 \psi + \delta \nabla^2 \psi) \, dx - \left( \nabla^2 \psi \right)^2
\]

so we need to look at

\[
I = \int \left( \nabla^2 \psi \right) \nabla^2 (\delta \psi) \, dx = \int \left[ \nabla \cdot \left( \nabla \psi \nabla \delta \psi \right) \right] \, dx
\]

where \( \nabla^2 = \nabla \cdot \nabla \)

\[
I = \int \left( \hat{n} \cdot \nabla \psi \right) \nabla^2 \psi \, ds - \int \nabla \cdot \left( \nabla \psi \nabla \psi \right) \, dx
\]

\( \text{boundary of area } A \)
We have converted a surface integral into another one plus a boundary line integral.

\[ A = \boxed{A} + \]

\begin{align*}
- \int \tilde{\nabla}_T (\nabla^2 \phi) \cdot \nabla_T S^k \, d^3 \mathbf{x} &= - \int \nabla \cdot \left[ \tilde{\nabla}_T (\nabla^2 \phi) S^k \right] \, d^3 \mathbf{x} + \\
+ \int (\nabla^2 \phi) S^k \, d^3 \mathbf{x} &= - \int [\hat{\mathbf{n}} \cdot \tilde{\nabla}_T (\nabla^2 \phi)] S^k \, d^3 \mathbf{x} \\
\int (\nabla^2 \phi) S^k \, d^3 \mathbf{x} &\quad \text{so collecting all the terms we get}
\end{align*}

Now do it again!
\[
\frac{1}{2} \int_A \left( \nabla^2 \varphi \right)^2 \, d^2x = \int_A \left( \nabla^2 \varphi \right) \, d^2x - \int_{\partial A} \varphi \left( \nabla \varphi \right) \cdot (\nabla \varphi) \, d\ell +
\]

It we had added a normal force to the plate,

\[ SF_{\text{plat}} \rightarrow SF_{\text{plat}} - \int_A P(xy) \, \varphi \, d^2x = \]

equation of equilibrium:

\[ \kappa_{2D} \nabla^2 \varphi - P = 0, \]

what about the other term?

It turns out that it is all surface terms in the\n
words it is a total derivative of

\[ \frac{\partial}{\partial x} \left( \begin{array}{c} \psi_x \\
\psi_y \end{array} \right)^2 - (\psi_x^2)(\psi_y^2) \right) \, d^2x \]

\[
= \int A 2 \left( \psi_x^2 \psi_y^2 - \psi_x^2 \psi_y^2 \right) - \psi_x \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) - \psi_y \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int A \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]

\[
= \int \left( \psi_x^2 \psi_y^2 \right) - \psi_x \psi_y \frac{\partial}{\partial x} \left( \psi_y^2 \psi_x^2 \right) + \psi_y \psi_x \frac{\partial}{\partial y} \left( \psi_x^2 \psi_y^2 \right) \, d^2x \]
\[ S \int_A \left[ \left( \frac{\partial^2}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x \partial y} \right) \right] \, d^2 \mathbf{x} \]

\[ = \int_A \mathbf{\nabla}_T \cdot \mathbf{V} \, d^2 \mathbf{x} = \oint_{\partial A} \mathbf{n} \cdot \mathbf{V} \, ds \]

Of course, \( \mathbf{V} \) is a tensor. But it is proportional to \( \mathbf{\hat{n}} \cdot \mathbf{V}_{\text{rel}} \) and \( \mathbf{\hat{n}} \cdot \mathbf{V}_{\text{rel}} \) is evaluated on the boundary.

If we take \( z = 0 \), \( \mathbf{n} \cdot \mathbf{V}_z = 0 \) on the boundary, then all the surface terms vanish.

\[ \text{--- clamped b.c.} \]

Also, the terms \( \alpha \, \mathbf{\hat{n}} \) give us the force excited on the boundary.

And the terms \( \alpha \mathbf{n} \cdot \mathbf{V}_{\text{rel}} \) give us the torque excited by the boundary.
We get \( F = -D \left[ \frac{\partial^2 E}{\partial n^2} \right] \) \( \text{Force} \)

\[ \hat{n} \cdot \vec{F} = \vec{M} \]

\( M = D \frac{\partial^2 E}{\partial n^2} \) \( \text{Torque} \)

where

\[ \int ds \Theta \]

so \( E_x = \cos \Theta \) \( E_y = \sin \Theta \)

So, why did the result come out \( E_y \)?

why these two terms and why does one contribute only a surface term?