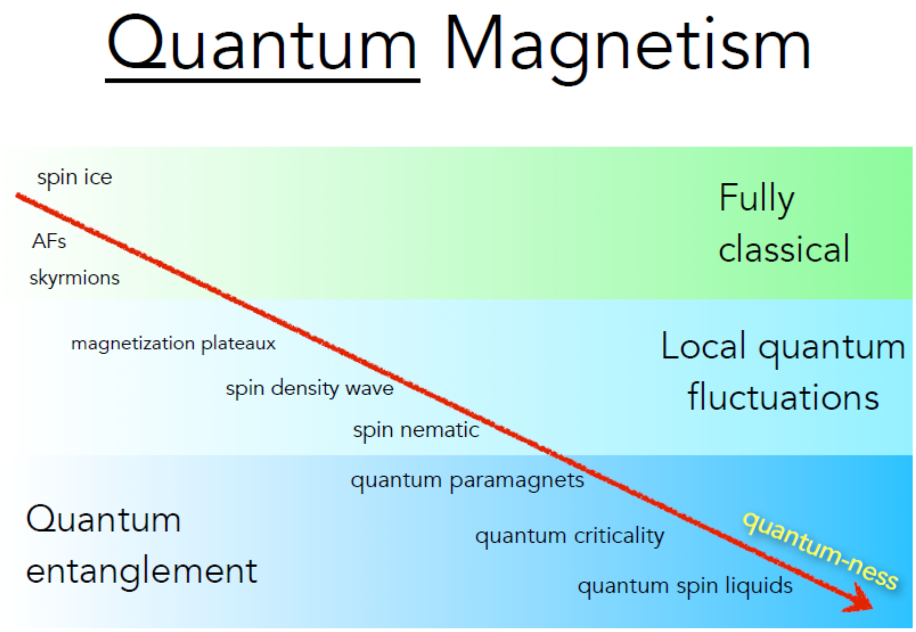
Boulder notes by Victor V. Albert.

I. LEON BALENTS

Notes here are partially taken from notes and slides on https://spinsandelectrons.com/pedagogy/.

I.1. Quantum magnetism and product states



Quantum mechanics is cool because it is non-intuitive. More precisely, quantum systems may contain non-local correlations which are not seen classically (e.g., Schrodinger's cat). However, cat states are exponentially unstable: any local measurement collapses the superposition. They also require an exponentially long time to assemble with local unitary operators. So they, for all practical purposes, do not exist as a phase of matter.

In contrast, equilibrium states of most ordinary matter are local, i.e., can be described by properties that are defined locally (color, elastic moduli, densities, capacities). Global properties can then be obtained from such local measurements.

In virtually all systems studied, the Hamiltonian is **local**, i.e., is a sum of terms involving a small amount of operators $H(x)$ in some region around x:

$$H = \sum_x H(x) \ .$$

Typically, the ground state *approximately* is a product state of blocks (of larger than unit size, i.e., state may not necessarily be separable): $|\psi\rangle = \bigotimes_A |\psi_A\rangle$. Other states which are adiabatically connected to a product state form a **phase**, which has several properties:

- In a generic state of such a phase, entanglement entropy $S_A = -\mathrm{Tr}_A \{\rho_A \ln \rho_A\}$ for a region A (which splits some blocks) follows an area law:

$$S_A \sim \sigma L^{d-1} + O\left(e^{-\alpha L}\right) \qquad \text{(area law)}\,,$$

  where σ, α are numbers. The best example of such a phase is an ordered quantum magnet in the large-S expansion: $H = \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j$ and $|\psi\rangle = \bigotimes_i |\vec{S}_i \cdot \hat{n}_i = S\rangle$.

- Quasiparticle excitations (e.g., excited-state levels of one block) are local. And any plane-wave excitations of the form

$$\sum_{\text{blocks}} e^{ik\cdot r} |\text{one block excited}\rangle$$

form a sharp band. Usually, ground states of such a phase are gapped.

– Ex: given a ground state of spins $|G\rangle = \left|\begin{array}{ccc}\uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow\end{array}\right\rangle$, a spin wave excitation is $|\psi\rangle = \sum_r e^{ikr} \left|\begin{array}{ccc}\uparrow & \uparrow & \uparrow \\ \uparrow & \downarrow_r & \uparrow \\ \uparrow & \uparrow & \uparrow\end{array}\right\rangle$.

Most quantum matter is not topological and consists of the phases discussed above.

I.2. Non-product states

An example of such an exotic state is the toric code (see Chetan Nayak's notes). For $J_{1,2} \geq 0$,

$$H_{TC} = -J_1 \sum_{\text{vertices } s} \underbrace{\prod_{i \in N(v)} \sigma_i^z}_{S_s} - J_2 \sum_{\text{plaquettes } p} \underbrace{\prod_{i \in p} \sigma_i^x}_{P_p} .$$

All terms of the same type involve σ_z's and terms of opposite types give two minus signs, so everything commutes (frustration-free). In a ground state, $P_p = S_s = +1$, and we now reveal what this state is. Let

$$|\{s_i\}_i\rangle = \bigotimes_i |X_i = s_i\rangle$$

be a basis for the space, and we graphically denote any $s_i = -1$ by coloring the corresponding link. States in which all $s_i = +1$ are all states consisting of closed loops. To find the ground state, we project the state with no loops $|\psi_0\rangle$ onto the $P_p = +1$ subspace:

$$Q_p = \frac{1 + P_p}{2} = \frac{1}{2} \sum_{q_p = 0,1} (P_p)^{q_p} \qquad\qquad |\psi\rangle = \prod_p Q_p |\psi_0\rangle .$$

Writing out the projections yields

$$|0\rangle = \frac{1}{2^{N_p}} \sum_{q_1, \ldots, q_N} \prod_p (P_p)^{q_p} |\psi_0\rangle ,$$

a sum of terms where each term creates a closed loop from the no-loop state $|\psi_0\rangle$. In such a state, all spins are uncertain:

$$\langle 0|X_i|0\rangle = \langle 0|P_p X_i P_p|0\rangle = -\langle 0|X_i|0\rangle = 0 ,$$

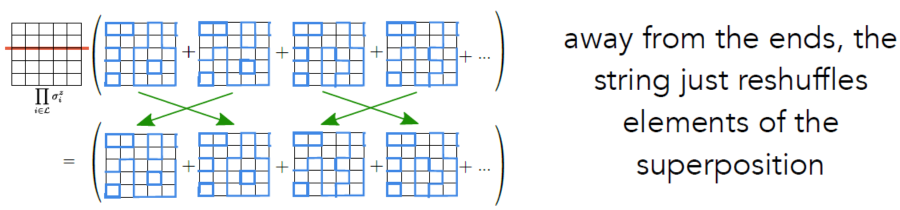
yet there is some structure.

I.2.1. Excitations

These can either $S_s = -1$ (e excitations) or $P_p = -1$ (m excitations). To create one excitation like this, you need an infinite string because **local operators** (e.g., products of Z operators) **create excitations in pairs**. An e charge at position s (given a semi-infinite line ℓ) is

$$|e_s\rangle = \prod_{i \in \ell} Z_i \cdot |0\rangle \qquad\qquad \langle e_s|H|e_s\rangle - \langle 0|H|0\rangle = 2J_2 .$$

This is strange: since Hamiltonian is local, the energy of an infinite excitation like this *should* give an infinite energy (i.e., proportional to the length of the string). However:

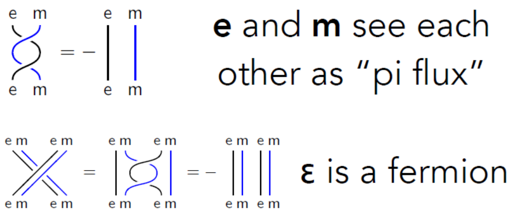


As we saw, local operators do not create single quasiparticles, so quasiparticles are not created by a local operator (although they can have local energy density). This requires entanglement.

There are also m excitations, giving $P_p = -1$,

$$|m_p\rangle = \prod_{i \in \ell} X_i \cdot |0\rangle \,,$$

and ϵ excitations which are just an e and m nearby each other. These are the quasiparticles, and they have non-trivial properties upon braiding/exchange (and ϵ is a fermion). However, braiding two e's or two m's gives no phase, so they are in this sense bosons.



II. LEON BALENTS 2

A **quantum spin liquid** (QSL), loosely speaking, is a phase of matter which has nonlocal excitations. QSLs can be gapped or gapless, and gapped QSLs are usually topological phases. Other QSLs may have excitations that can be observed using local operators far away from them.

II.1. Quantum spin ice (QSI): XXZ model

This is based on arXiv:cond-mat/0305401. Consider putting the following Hamiltonian on a pyrochlore lattice (a very common lattice in minerals, e.g., rare-earth spins):

$$H = J_{zz} \sum_{\langle ij \rangle} S_i^z S_j^z - J_{\pm} \sum_{\langle ij \rangle} \left(S_i^+ S_j^- + H.c. \right) \,.$$

The lattice makes the model highly frustrated since there are lots of triangles. If $J_{\pm} = 0$, then we obtain the classical limit

$$H = J_{zz} \sum_{\text{tetrahedra } t} \left(\sum_{i \in t} S_i^z \right)^2 - Const.$$

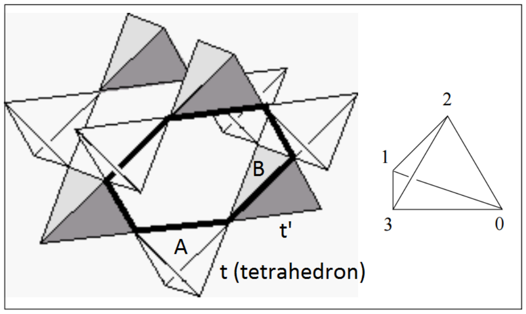
There are many ground states at 0 K, and in each of them $\sum_{i \in t} S_i^z = 0$ ("two in, two out"). We apply perturbation theory to determine effect of $J_{\pm}$, but $P_{GS} H_{\pm} P_{GS} = 0$ (one of the tetrahedra gets its spin risen, leaving the GS subspace). Second order term $-P_{GS} H_{\pm} \frac{1}{E_0 - H_0} H_{\pm} P_{GS}$ is a constant. Third order term is

$$P_{GS} H_{\pm} \frac{1}{E_0 - H_0} H_{\pm} \frac{1}{E_0 - H_0} H_{\pm} P_{GS} \equiv P_{GS} H_{\pm} R H_{\pm} R H_{\pm} P_{GS} \,,$$

where resolvent $R = \frac{1}{E_0 - H_0} (1 - P_{GS})$. Then, up to constants,

$$H_{eff} = -K P_{GS} \sum_{\text{hexagons}} \left(S_i^+ S_j^- S_k^+ S_l^- S_m^+ S_n^- + H.c. \right) P_{GS} \qquad\qquad K = 12 \frac{J_{\pm}^3}{J_{zz}^2} \,,$$

where a hexagon is shown below.



This ring term came from the local constraint $\sum_{i \in t} S_i^z = 0$, so on every tetrahedron we have the conserved quantity

$$Q_t = \sum_{i \in t} S_i^z \qquad\qquad [Q_t, H_{eff}] = 0\,,$$

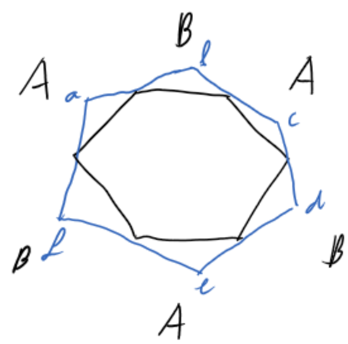
which generates the corresponding symmetry $U_t = e^{i\chi_t Q_t}$ which acts as $S_i^{\pm} \to S_i^{\pm} e^{\pm i\chi_t}$. Such a local symmetry can be called a $U(1)$ gauge symmetry, so this system is a $U(1)$ gauge theory. Each spin lives on two tetrahedra, respectively of types A and B (F1), so we can re-write H_{eff} in terms of a bipartite lattice with spin $i \leftrightarrow \langle t, t' \rangle$ and

$$S_i^z = S_{tt'}^z = S_{t't}^z \qquad\qquad S_i^+ = S_{tt'}^+ = S_{t't}^-\,.$$

Gauge symmetry acts like $S_{tt'}^+ \to S_{tt'}^+ e^{-i(\chi_t - \chi_{t'})}$ and Hamiltonian becomes (see F2)

$$H_{eff} = -KP_{GS} \sum_{\text{hexagons}} \left(S_{ab}^+ S_{bc}^+ S_{cd}^+ S_{de}^+ S_{ef}^+ S_{fg}^+ + H.c. \right) P_{GS} = -KP_{GS} \sum_{\text{hexagons}} \left(\prod_{\text{along hex}} S_{tt'} + H.c. \right) P_{GS}\,.$$

F2



Since we are in a gauge theory, the Q_t's can be thought of as charges,

$$Q_t = \sum_t S_i^z = (-)^{\epsilon_t = A \text{ or } B \text{ sublattice}} \sum_{t'} S_{tt'}^z\,,$$

and the sum is a lattice divergence of a discrete electric field $E_{tt'} \equiv S_{tt'}^z = \pm\frac{1}{2}$. This is slightly inconvenient, and we can embed it in a rotor to work in space of half-integer eigenvalues: $E_{tt'} = \pm\frac{1}{2}, \pm\frac{3}{2}, ... = \mathbb{Z} + \frac{1}{2}$. That way, we can define a conjugate angle $A_{tt'}$ such that it generates translations of E:

$$[A_{tt'}, E_{tt'}] = i \qquad\qquad e^{iA} E e^{-iA} = E \pm 1\,.$$

Then the hoppings on the hexagons are

$$\left(\langle S^z = +\tfrac{1}{2}| \quad \langle S^z = -\tfrac{1}{2}| \right) e^{i\epsilon_t A_{tt'}} \begin{pmatrix} |S^z = +\frac{1}{2}\rangle \\ |S^z = -\frac{1}{2}\rangle \end{pmatrix} = S_{tt'}^+\,.$$

Since we have embedded in a rotor in each site, we need to project back onto the $S^z = \pm\frac{1}{2}$ subspace to rewrite

Hamiltonian. Projection is achieved in the limit $U \to \infty$ of

$$\begin{aligned} H_{eff} &= U \sum_{\langle tt' \rangle} \left(E_{tt'}^2 - \frac{1}{4} \right) - 2K \sum_{\text{hexagons}} \cos\left(A_{ab} + A_{bc} + ...\right) \\ &= U \sum_{\langle tt' \rangle} \left(E_{tt'}^2 - \frac{1}{4} \right) - 2K \sum_{\text{hexagons}} \cos\left(\text{curl } A\right) \\ &= U \sum_{\langle tt' \rangle} \left(E_{tt'}^2 - \frac{1}{4} \right) - 2K \sum_{\text{hexagons}} \cos\left(B\right) \end{aligned}$$

Due to the cosine, this is a compact $U(1)$ lattice gauge theory. Due to projection by P_{GS} (removed above), there is now a constraint

$$(\text{div} E)_t = 0 \qquad\qquad E \in \mathbb{Z} + \frac{1}{2} \,.$$

For all U, K, the system is in the same phase (different when $E \in \mathbb{Z}$). Assuming that large and small U regions are the same, we can then expand the cosine to get the non-compact $U(1)$ gauge theory from E&M:

$$H_{eff} \approx U \sum_{\langle tt' \rangle} \left(E_{tt'}^2 - \frac{1}{4} \right) - 2K \sum_{\text{hexagons}} B^2 + Const.$$

E&M gives a discrete version of a $\frac{1}{r}$ potential between static charges, which are the sums of spins along tetrahedra here:

$$Q_t = \sum_t S_i^z = \begin{cases} 0 & \uparrow\uparrow\downarrow\downarrow \\ +1 & \uparrow\uparrow\uparrow\downarrow \\ -1 & \downarrow\downarrow\downarrow\uparrow \end{cases} .$$

Like in E&M, the charges $Q_t = \pm 1$ will have electric and magnetic field lines emanating from it, so they are non-local excitations. However, these lines can be detected locally using a test charge far away, **different from a topological phase** (in which excitations cannot be locally detected from far away). Flipping a spin creates two charged tetrahedra and creates excitations in pairs, just like in a topological phase. Excitations can be moved due to the (now non-negligible) first-order corrections of H projected onto the excitation subspace. The total spin here is conserved,

$$S_{tot}^z = \sum_i S_i^z = \frac{1}{2} \sum_t \sum_{i \in t} S_i^z \,,$$

where the half is because tetrahedra t share sites. The total spin is always an integer since excitations come in pairs.

In contrast to E&M, in a compact $U(1)$ gauge theory, monopoles m exist as well. Fusing e and m gives dyons ϵ, which are fermions (similar to toric code). Similar to E&M, we also have photons — collective modes living in ground state manifold. Photons make the QSL gapless with dispersion $\omega \approx c\,|k|$. Like the toric code, this model is stable under small perturbations which do not break gauge symmetry. Terms that break gauge symmetry can be expressed as matter fields, which have large gaps when they are small (related: Elitzur's theorem). **While topological phases are stable under small perturbations that don't close the gap, QSLs are protected because all small perturbations have a large gap**.

Similar to toric code, the Hamiltonian delocalizes the system amongst loops along the hexagons (vacuum fluctuations). The ground state is then a sum of all the loops, just like in the toric code. However, on surfaces with non-contractible loops, such loops of will have an energy above that of the ground state, so there is no BC dependent ground-state degeneracy.

In general, taking a classical system with a constraint and adding "quantum" perturbations leads to gauge theories, but sometimes those theories may contain matter terms (i.e., not be pure). Here, we see emergence of a gauge theory, which in essence describes why E&M occurs in minerals. Generalizations of this model can then be used to explain emergence of other fundamental forces in nature.

III. LEON BALENTS 3

III.1. Partons

We want to construct good ground states for the spin/fermionic Hamiltonian

$$H = \sum_{ij,\alpha\beta} J_{ij}^{\alpha\beta} S_i^\alpha S_j^\beta \, .$$

We want to use a highly entangled variational wavefunction, so we introduce free fermions c_k such that the spins are then $\vec{S} = c^\dagger \frac{\vec{S}}{2} c$. The free-fermion wavefunctions, with and without spin, are then

$$|\psi_{FF}\rangle = \prod_{k\in FS} c_k^\dagger |0\rangle \qquad\qquad |\psi_{FFw/S}\rangle = \prod_{k\in FS} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger |0\rangle \, .$$

These wavefunctions are associated with some **free fermion Hamiltonian** H_{FF}. We project to guess the ground state to be

$$|\psi_{FFS}\rangle = P_G |\psi_{FF}\rangle \, ,$$

where the **Gutzwiller projection** $P_G = \prod_i \hat{n}_i \left(2 - \hat{n}_i\right)$ projects out all unoccupied and doubly occupied sites and leaves us with one occupation number (i.e., one spin) per site. This $\hat{n}_i = 1$ is roughly associated with a gauge constraint.[1] There is a belief that each such projected wavefunction represents a true QSL phase, in which the partons become the non-local quasiparticles - "spinons" - and they are coupled to an effective gauge field. A guess for an excited state could be $P_G \left[c_{k\uparrow}^\dagger |\psi_{FF}\rangle \right]$.

We have three choices for the corresponding Hamiltonians H_{FF}:

1. H_{FF} with $U(1)$ symmetry. Fermi surface becomes projected and there is an emergent $U(1)$ gauge symmetry.
2. For cases such as graphene, we can put the Fermi level at a Dirac node. Projecting creates a gauge-symmetric $U(1)$ Dirac state.
3. No $U(1)$ symmetry, e.g., a BCS Hamiltonian. Projecting yields a $\mathbb{Z}_2$ gauge-symmetric toric-code like wavefunction.

In the ground state subspace, the Hamiltonian becomes

$$H_{eff} = P_G H P_G = \sum_{\langle ij,\alpha\beta\rangle} t_{ij,\alpha\beta} c_{i\alpha}^\dagger c_{j\beta} \, .$$

We now gauge H_{eff} by adding conjugate variables $[E_{ij}, A_{ij}] = i$ to obtain

$$H_{eff} = \sum_{\langle ij,\alpha\beta\rangle} t_{ij,\alpha\beta} c_{i\alpha}^\dagger c_{j\beta} e^{-iA_j} + \frac{1}{2} U E_{ij}^2 - K \cos\left(\nabla \times A\right) \, .$$

In the large U limit, we hope to reproduce the original spin model H.

As another example, consider the BSC Hamiltonian

$$H_{FF} = t c_i^\dagger c_i + \Delta_{ij} c_i^\dagger c_j^\dagger + H.c. \, ,$$

which gauges to

$$H_{eff} = t_{ij} \sigma_{ij}^z c_i^\dagger c_j + \Delta_{ij} \sigma_{ij}^z c_i^\dagger c_j^\dagger + H.c. \, .$$

[1] While in the previous lecture, a gauge theory was more or less derived by projecting, below the terms will just be added. However, such a theory can be derived more rigorously via a path integral representation of H with the constraint that $\hat{n}_i = 1$ or other means. In general, the connection between projecting and gauging is **not** rigorous.