BOULDER SUMMER SCHOOL LECTURE NOTES TOPOLOGICAL SEMIMETALS AND SYMMETRY PROTECTED TOPOLOGICAL PHASES

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1. PART 2 - Interaction Effects and Symmetry Protected Topological Phases

Overview of Part2: We will try to establish the following surprising facts, that arise when interactions are added to the study of topological phases:

- Folklore: *Integer quantum Hall states are classified by an integer $\mathcal{Z}$.* New topological phases from interactions lead to classification by two distinct integers $\mathcal{Z} \times \mathcal{Z}$.
- Folklore: *The 3D topological insulator surface must be gapless or break symmetry.* A fully symmetric but gapped surface state of a TI is possible with strong interactions.
- Folklore: *Theories of strongly interacting topological phases have no experimental consequences.* New insights have led to testable predictions for an experimentally observed state in the half filled Landau level.

1.1. **Quantum Phases of Matter. Short vs. Long Range Entanglement.** How do we distinguish different phases of matter? We will be particularly interested in zero temperature states - i.e. the ground state of an interacting bunch of particles. Typically, the phases of matter are only sharply defined in the limit of an infinite number of particles. Then, two states belong to different phases, if there is necessarily a phase transition separating them, where properties change in a singular fashion. For a while, people thought they had figured out how to diagnose this. The answer they believed had to do with symmetry - at the fundamental level, breaking symmetry in different ways lead to different phases. For example - in the Quantum Ising model, with two level systems arranged on a line:

$$H = -J \sum_i \left(\sigma_i^z \sigma_{i+1}^z + g \sigma_i^x \right) \tag{1}$$

there is a symmetry or flipping the spin $\sigma^z \to -\sigma^z$ (similarly for the y spin direction). This Z_2 symmetry is spontaneously broken if g is small while it is restored if g is sufficiently large. Thus there are two phases which can be distinguished by the order parameter $\langle \sigma_i^z \rangle$. Symmetry is key to having a sharp distinction - if it is broken by hand, eg. by adding a

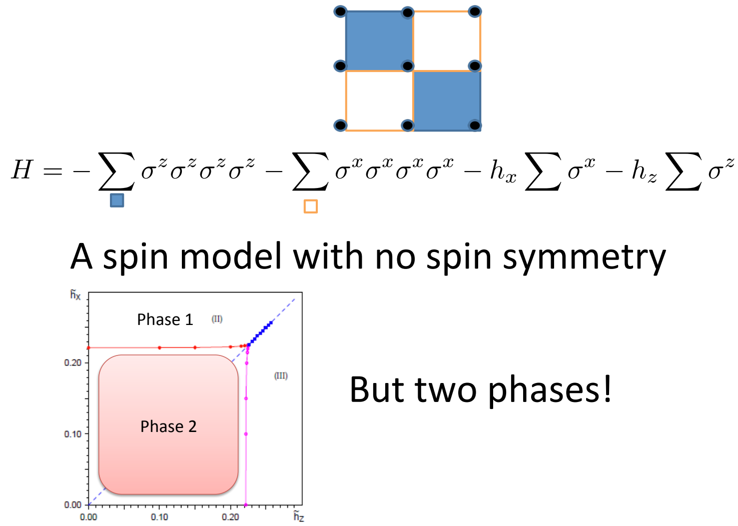


FIGURE 1. The toric code model with generic perturbations, which has two phases although they have the same symmetry. The Phase 2 is gapped but has long range entanglement - as evidenced by having ground state degeneracy with periodic boundary conditions, and anyone excitations with nontrivial mutual statistics.

field along σ^z, then the phase transition can be converted into a crossover. For a while it was thought that all phases of matter (apart from a few well characterized outliers like metals) could be identified by such a procedure.

However, Wegner came up with a model with two phases which shared the same symmetry. Today we understand that they differ in their topology - here is the modern avatar of that model, the Kitaev Toric code, which also has spins on the vertices of a 2D square lattice. The coupling takes the following form - and includes 4 spin couplings around the plaquettes (see figure 1).

$$H = -\sum_{black} \sigma_i^z\sigma_j^z\sigma_k^z\sigma_l^z - \sum_{white} \sigma_i^x\sigma_j^x\sigma_k^x\sigma_l^x - h_z \sum_i \sigma_i^z - h_x \sum_i \sigma_i^x \tag{2}$$

The two phases in this model include a 'trivial' phase which can be thought of as a product state of spins pointing along a certain direction. The other phase does not have any representation as a product state of a site or finite group of sites. It can be thought of as a condensate of closed loops - where the loops are found by linking $\sigma^z = -1$ spins for example. There are two kinds of point excitations in this phase - which violate the individual plaquette terms. One is called a 'charge' and the other the 'vortex'. Despite ultimately being built out of bosons (the spins can equally be thought of as occupying sites with hard core bosons), the excitations have unusual statistics. Taking one around mother leads to a (-1) sign, hence they are mutual semions. This is an example of fractional statistics, in the generalized sense, which includes both exchange of identical particles as well as mutual statistics. This is an indication of long range entanglement (LRE). Another signature is that when the system is defined with periodic boundary conditions (i.e. on a torus), there is a ground state degeneracy. The degenerate states appear identical with respect to any local operator (the degeneracy itself can be understood since the Hamiltonian is a local operator). We define a short range entangled phase as one that does *not* have these properties.

A Short Range Entangled (SRE) state is a gapped phase with a unique ground state on a closed manifold. All excitations (particles with short range interactions) have conventional statistics. For example, if the phase is built of bosons, all excitations are bosonic with trivial mutual statistics.

We also allow for the possibility of a symmetry specified by the group G. We will restrict attention to *internal symmetries* , that is, we do not consider symmetries that change spatial coordinates, like inversion, reflection, translation etc. Common internal symmetries that are encountered in condensed matter physics are charge conservation, various types of spin rotation symmetry, and time reversal symmetry. The advantage of working with internal symmetries is that we can consider disordered systems that respect the symmetry. also the symmetries can be defined at the edge, while for spatial symmetries, one may require a special edge configuration, to preserve symmetry. Some spatial symmetries like inversion are always broken at the edge.

Gapped SRE ground states that preserve their internal symmetries only differ from the trivial phase if they possess edge states. (For 1D systems, the edge states are always gapless excitations, and rigorous statements can be made using matrix product state representation of gapped phases.)

The fact that SRE topological phases only differ at the edge, not in the bulk (unlike LRE state), makes them much easier to study. The set of SRE topological phases in a given dimension with symmetry G actually has more structure than just a set. If we add the trivial phase as an 'identity' element, the set of phases actually form an Abelian group. The operations for the group are shown. The addition operation is obvious: take two states and put them side-by-side. But it is not so obvious that a state has an inverse: how is it possible to cancel out edge states? Two copies of a topological insulator cancel one another, because the Dirac points can be coupled by a scattering term that makes a gap. The inverse of a phase is its mirror image (i.e reverse one of the coordinates). To see this, we must show that the state and its mirror image cancel; the argument is illustrated at the bottom of figure 2. In one dimension, for example, take a closed loop of the state, and flatten it. The ends are really part of the bulk of the loop before it was squashed, so they are gapped. Therefore this state has no edge states. Because topological SRE phases are classified by their edge states, it must be the trivial state. Therefore the Hamiltonian describing the inverse of a particular state $H(x, y, z\dots)$ is, for example $H(-x, y, z\dots)$.

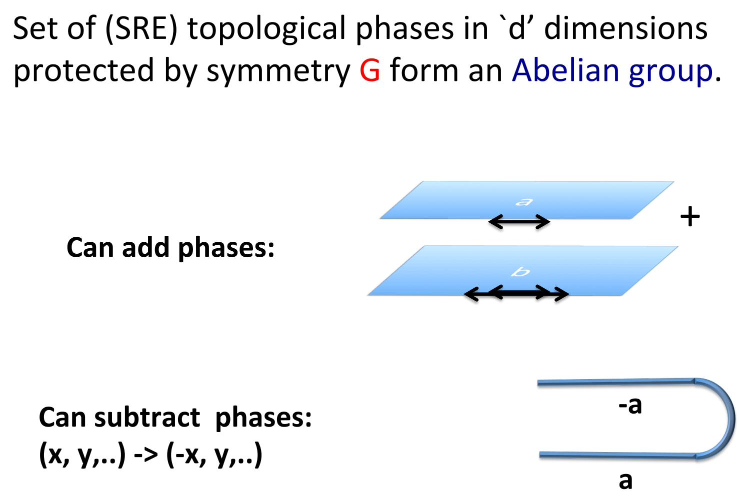


Figure 2. SRE phases that preserve a symmetry must form an Abelian group. This is not true for LRE phases, which typically get more complicated on combining them together.

1.2. **Examples of SRE Topological phases.** Let us give a couple of concrete examples of SRE topological phases of bosons/spins. These are necessarily interacting - unlike free fermions, there is no 'band' picture here.

1.2.1. *Haldane phase of S=1 antiferromagnet in d=1.* The following simple Hamiltonian actually leads to a gapped phase with SRE, but gapless edge states:

$$H = J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

The edge realizes effectively a $S = 1/2$ state, despite the chain being built of $S = 1$ spins. This phenomena has been observed experimentally in some nickel based insulating magnets, e.g. Y_2BaNiO_5. The Ni atoms form S=1 spins, organized into chain like structures that are relatively well isolated from one another . When a non magnetic atom is introduced (like Zn) it disrupts the chains and releases a pair of $S = 1/2$ moments that are nearly decoupled from one another. These contribute to magnetic susceptibility and specific heat in a characteristic way that depends on dilution, and residual interactions between the edge spins leads to a characteristic spectrum that is accessible to spectroscopic experiments [1]

The symmetry that is crucial to protecting this phase is SO(3) spin rotation symmetry. However, it turns putt hat the full rotation symmetry is not required. It is sufficient to just retain the 180 degree rotations about the x, y and z axes. This symmetry group

$\{I, X, Y, Z\}$, contains the identity and the 3 rotation elements. This can be written as $\{I, X\} \times \{1, Y\}$, since $Z = X \times Y$, the combination of two rotations is the third rotation. Mathematically this group is $Z_2 \times Z_2$. We will write down a model with this symmetry (which is not quite reducing the S=1 down to this rotation symmetry), but which has the advantage of being exactly soluble - not just for the ground state but also for all the excited states. This model also has a nice interpretation - of arising from condensing domain walls bound to spin flips.

1.3. **Duality.** It will be useful in many contexts to pass from a description involving local degrees of freedom, to one that focuses on the topological defects. In particular we will see that some SPT phases can be readily accessed in terms of condensing defects to which symmetry quantum numbers are attached. We will focus on this mechanism in the following. The simplest example will be the 1+1D transverse field Ising model, with an additional Z2 quantum number attached.

1.3.1. *An exactly soluble topological phase in d=1.* Consider a spin model with $Z_2 \times Z_2$ symmetry. There is a Z_2 set of topological phases with this symmetry in d=1, and we will explicitly construct the nontrivial topological phase. We will implement this symmetry by a pair of Ising models (labeled σ and τ) that live on a zig-zag lattice as shown in the Figure. Consider beginning in the ordered state of the σ, but where the τ are disordered and point along a transverse field τ^x. Now, we would like to restore the Z_2 symmetry of the σ spins. We do this by condensing the domain walls of the σ spins. If we directly condense domain walls we get the trivial symmetric state. However, we can choose to condense domain walls with a spin flip of τ attached. We will see that this gives the topological phase.

One way to do this is to write down a Hamiltonian that would lead to this binding. Note that the operator $\sigma_i^z \sigma_{i+1}^z$ detects a domain wall. Consider:

$$H = -\sum_i \left(\sigma_{2i}^z \tau_{2i+1}^x \sigma_{2i+2}^z + \tau_{2i-1}^z \sigma_{2i}^x \tau_{2i+1}^z\right) \tag{3}$$

where we have placed the σ (τ) on the even (odd) sites of the lattice. In the absence of a domain wall, we have the usual transverse field term, whose sign changes when a domain wall is encountered. We will show that this is a gapped phase with short range entanglement, but has gapless edge states.

First, consider the system with periodic boundary conditions. We will leave it as an exercise to show that each of the terms in the Hamiltonian 3 commutes with all others. Then, for a system with N sites, we have exactly N terms which can be written as $H = -\sum_i \left(\tilde{\sigma}_{2i}^x + \tilde{\tau}_{2i+1}^x\right)$, where the tilde denote the three spin operators in the Hamiltonian 3. Hence, this simply looks like each site has a modified transverse field, which implies a unique ground state and a gap, in this system with periodic boundary conditions.

Now consider open boundary conditions as shown. Let us focus on the left edge, where the end of the chain implies we lose $\tilde{\sigma}_x$ operator. This will result in a two fold degeneracy as we will show. The first term in the Hamiltonian is now $-\sigma_0^z \tau_1^x \sigma_2^z$. We can easily show that the following two operators commute with the Hamiltonian $\Sigma^z = \sigma_0^z$ and $\Sigma^x = \sigma_0^x \tau_1^z$. However they anti commute with one another. Hence we can show the ground state must

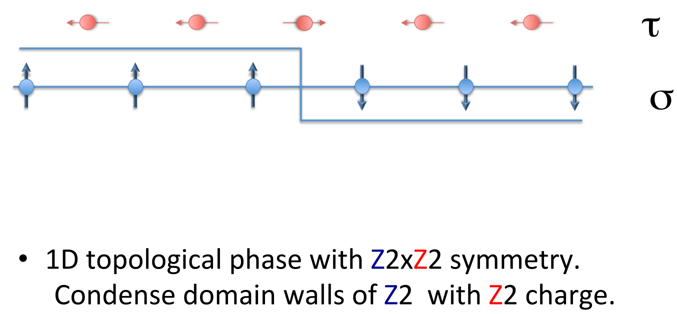


FIGURE 3. An exactly soluble model of a 1D SRE phase with gapless edge states protected by $Z_2 \times Z_2$ symmetry. Terms in this Hamiltonian encourage binding of domain walls to spin flips. The topological phase emerges on condensing these 'decorated' domain walls.

be at least two fold degenerate. Say you had a unique ground state of the Hamiltonian, $|\psi\rangle$. This must be an eigenstate of Σ^z, since it commutes with the Hamiltonian. Let us say $\Sigma^z|\psi\rangle = \lambda|\psi\rangle$, where $\lambda = \pm 1$. However, we can find an independent state $|\psi'\rangle = \Sigma^x|\psi\rangle$. This is a degenerate state since $[\Sigma^x, H] = 0$. It is also a distinct state since it has a different eigenvalue $\Sigma^z|\psi\rangle = -\lambda|\psi\rangle$, due to $\Sigma^z\Sigma^x = -\Sigma^x\Sigma^z$. Hence there are at least two ground states $(|\psi\rangle, |\psi'\rangle)$. They only differ by application of an edge operator, hence this is an edge degeneracy.

Note, it is important we preserve the symmetry - if we add Σ^a to the Hamiltonian we can gap the edge state, but at the expense of also breaking the $Z_2 \times Z_2$ symmetry. Hence this is called a symmetry protected topological phase (SPT). This model has special properties that make it exactly soluble - but adding general perturbations that are local and preserve the symmetry lead to a more generic state. The presence of an energy gap implies that the state is stable against weak perturbation, which means it will remain in the same phase.

1.4. **A SPT Phase of Bosons in 2+1D.** Here we will show that Bosons with charge q can lead to a SRE topological phase with quantized hall conductance $\sigma_{xy} = 2n\frac{q^2}{h}$. Note, they are quantized to *even* integers [2], which is required by SRE. Therefore unit hall conductance for bosons is considered a fractional quantum Hall state!

We construct this state as follows. We begin with two species of bosons (A,B) and consider a superfluid of one component (A) and an insulator of the other. We then bind a particle of B to a vortex of A)and a hole of B to an antivortex) and condense both these object to obtain an insulator. We will show this is a topological phase protected by $U(1)_A \times U(1)_B$. Finally we break this to a diagonal $U(1)$ to obtain the bosonic integer quantum Hall state.

1.4.1. *Effective Field Theory.* Let us write down an effective theory to describe a fluid built out of boson-vortex composites. The 'A' particles acquire a phase of 2π on circling vortices, hence the effect go vortices can be modeled by a vector potential whose curl is centered at the vortex locations: $\partial_x a_y - \partial_y a_x = 2\pi \sum_j n_j^v \delta(r - r_j^v)$, where (n_j^v, r_j^v) represent the strength and location of the vortices. This vector potential will couple minimally to the current $\mathcal{L} = \vec{j}_A \cdot \vec{a}$, where the vectors are two-vectors. A rewriting o this formalism to include motion of vortices results in the following generalization to the three current $j^\mu = (\rho, j_x, j_y)$, and three gauge potential $a_\mu = (a_0, a_x, a_y)$. Also, since we assume the vortices are bound to the bosons 'B' we can rewrite the equation for a as:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^B = 2\pi j_B^\mu \tag{4}$$

where we have introduced a superscript 'B' for the vector potential. At the same time we can utilize the continuity equation for the current j_A, $\partial_\mu j_A^\mu = 0$ to write:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^A = 2\pi j_A^\mu \tag{5}$$

In order to keep track of the charge density of 'A' and 'B' bosons, it is useful to introduce external vector potentials $A^{(A,\,B)}$ that couple to the currents of the 'A' and 'B' bosons. This leads to our final topological Lagrangian:

$$\mathcal{L}_{topo} = \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\left(a_\mu^A \partial_\nu a_\lambda^B + A_\mu^A \partial_\nu a_\lambda^A + A_\mu^B \partial_\nu a_\lambda^B\right) \tag{6}$$

$$\mathcal{Z}_{topo}[A^A, A^B] = e^{iS_{topo}} = \int \mathcal{D}a^A \mathcal{D}a^B e^{i\int dxdydt\,\mathcal{L}_{topo}} \tag{7}$$

First, we would like to establish that this describes a phase with short range entanglement. Note, the mutual phases involved are all 2π implying the absence of fractional statistics. One can also compute the ground state degeneracy on the torus - this turns out to be directly computable from this theory - if we write

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu^I \partial_\nu a_\lambda^J$$

, the ground state degeneracy is $|detK|$. In this case $K = \sigma^x$, and there is a unique ground state.

Given that it is a SRE phase, we can deduce two important consequences from this theory- the first is regarding edge states, which can be shown to be equivalent to that derived before, and the quantized Hall conductivity. The latter is obtained by integrating out the a fields to obtain an action purely in terms of the external probe fields A. The current is then defined as $j_A = \frac{\delta S}{\delta A^A}$ where $Z = e^{iS}$. A gaussian integration of Eqn. 7 yields:

$$S_{topo} = -\int dxdydt \, \frac{\epsilon^{\mu\nu\lambda}}{2\pi} A^A_\mu \partial_\nu A^B_\lambda \tag{8}$$

thus we have:

$$j^\mu_A = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A^B_\lambda \tag{9}$$

If we consider the spatial components of this equation we find: $j^x_A = \frac{1}{2\pi}E^y_B$, where E_B is the electric field applied on species 'B'. Thus we have a crossed response Hall conductivity $\sigma^{AB}_{xy} = \frac{1}{2\pi}$, which , replacing charge Q_a for the bosons and $\hbar$ gives $\sigma^{AB}_{xy} = \frac{Q_A Q_B}{h}$.

We would like to apply these insights to electronic systems, where one may combine pairs of electrons to form Cooper pairs with charge $Q = 2e$. However, in that case there is a single conservation law. Topological phases with a single $U(1)$ can be described by the field theory above 7 , if we assume that the two species of bosons can tunnel into one another and collapse the combined $U(1)\times U(1)$ symmetry into a single common $U(1)$. This amounts to replacing the pair of external vector potentials by a single one and the resting topological response theory is:

$$S_{topo} = -\int dxdydt \, \frac{\epsilon^{\mu\nu\lambda}}{2\pi} A_\mu \partial_\nu A_\lambda \tag{10}$$

Now, differentiating with respect to the vector potential we get *two* contributions and hence: $j^\mu = \frac{2}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ which implies a Hall conductivity, in units of the boson charge $\sigma_{xy} = 2\frac{Q^2}{h}$. This is the Bosonic Integer Quantum Hall (**BIQH**) phase. Somewhat surprisingly, its Hall conductance is always an *even* integer. Potential realization of this phase in bilayer systems of bosons in the lowest Landau level with net filling $\nu = 2$ have been discussed in recent numerical work [1].

Note we have assumed commensurate filling to admit an insulator. Also, these models are not exactly soluble in the same way that the previous models were - for other approaches to construct models in this phase see [2] .

1.4.2. *Implications for Interacting Quantum Hall State of Electrons:* It is well known that free fermion IQH states have a quantized Hall conductance $\sigma_{xy} = n\frac{e^2}{h}$. At the same time, they have a quantized thermal hall effect $\frac{\kappa_{xy}}{T} = c\frac{\pi^2 k_B^2}{3h}$ where $c = n$. The latter simply counts the difference between the number of right moving and left moving edge states. This equality is an expression of the Wiedemann Franz law that related thermal and electrical conductivity for weakly interacting electrons. This leads to the familiar integer classification of IQH $\mathcal{Z}$. How is this modified in the presence of interactions? We will continue to assume short range entanglement - so that fractional quantum Hall states are excluded from our discussion. It has long been known that n must remain an integer if

[1]See arXiv:1305.0298, arXiv:1304.5716, arXiv:1304.7553

[2]See S. Gerdatis and O. Motrunich, arXiv:1302.1436 (in particular, Appendix C)

charge is to remain unfractionalized. However, the equality $n = c$ can be modified. In fact, if we assume the electrons can combine into Cooper pairs which form the BIQH state, the latter has Hall conductance $\sigma_{xy} = 8\frac{e^2}{h}$ but $\kappa_{xy} = 0$. Thus we can have $n - c = 8m$. Indeed this implies that the classification of interacting quantum Hall states of electrons with SRE is $\mathcal{Z} \times Z$ at least. Note, this also predicts a phase where $n = 0$ but $c = 8$. This can be achieved by combining an $n = 8$ free fermion quantum Hall state with a BIQH state of Cooper pairs to cancel the electrical Hall conductance. The remaining thermal Hall conductance is $c = 8$. It can be shown that a π flux inserted in this state has trivial statistics and can be condensed - which implies that all electrons are confined into bosonic particles without disturbing the topological response of this phase. Alternately, one can show that neutral bosons with short range interaction can lead to a topological phase with chiral edge states, if they appear in multiples of eight[3]. Indeed one can write down a multi component chern simons theory to describe this topological phase of neutral bosons, in terms of a K matrix as described in detail below.

A phase without topological order is characterized by a symmetric K matrix with $|\det \mathbf{K}| = 1$. A chiral state in 2 + 1-D requires the signature (n_+, n_-) of its K matrix to satisfy that $n_+ \neq n_-$. We therefore seek a K matrix with the following properties (1) $|\det \mathbf{K}| = 1$ (2) the diagonal elements $K_{I,I}$ are all even integers so that all excitations are bosons and (3) a maximally chiral phases, where all the edge states propagate in a single direction. Then, all eigenvalues of K must have the same sign (say positive), so $\mathbf{K}$ is a positive definite symmetric unimodular matrix.

It is helpful to map the problem of finding such a $\mathbf{K}$ to the following crystallographic problem. Diagonalizing $\mathbf{K}$ and multiplying each normalized eigenvector by the square root of its eigenvalue one obtains a set of primitive lattice vectors $\mathbf{e}_I$ such that $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. The inner product of a pair of vectors $l_I\mathbf{e}_I$ and $l'_I\mathbf{e}_I$ are given by $l'_I K_{IJ} l_J$, while the volume of the unit cell is given by $[\text{Det}K]^{1/2}$. The latter can be seen by writing the components of the vectors as a square matrix: $[\mathbf{k}]_{aI} = [e_I]_a$. Then Det$k$ is the volume of the unit cell. However, $K_{IJ} = \sum_a k_{aI}k_{aJ} = (\mathbf{k^T k})_{IJ}$. Thus Det$\mathbf{K} = [\text{Det}\mathbf{k}]^2$.

Thus, for a phase without topological order, we require the volume of the lattice unit cell to be unity $[\text{Det}k] = 1$ (unimodular lattice). Furthermore, for a bosonic state, we need that all lattice vectors have even length $l_I K_{IJ} l_J =$ even integer, since the K matrix has even diagonal entries (even lattice). It is known that the minimum dimension this can occur in is eight. In fact, the root lattice of the exceptional Lie group E_8 is the smallest dimensional unimodular, even lattice[3]. Such lattices only occur in dimensions that are a multiple of 8.

[3]See wikipedia entry for E_8 root lattice (Gosset lattice)

A specific form of the K matrix is:

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{11}$$

This matrix has unit determinant and all eigenvalues are positive. It defines a topological phase of bosons without topological order, with eight chiral bosons at the edge[2]. We will call this the E_8 state since it is related to the E_8 group.

Exercises:

(1) Verify that the terms in Eqn. 3 commute with one another, in a chain with periodic boundary conditions. With open boundaries, explicitly write out a Hamiltonian and check that the edge operators Σ^a commutes with it.

(2) Use the Jordan Wigner procedure to map Eqn. 3 onto fermion operators. Recall, for the 1D quantum Ising model, the transformation is $c_j^\dagger = \sigma_j^+ S_j$ where the string operator is $S_j = \prod_{i>j} \sigma_i^x$. Show that the same mapping leads to a topological phase of these nonocal fermions. Note however there are some important differences with a topological phase of electrons. Argue that in the latter case there must always be a residual symmetry that cannot be broken by any physical operator, unlike in the fermionized version of the problem above.

References

[1] M. Yoshida, K. Shiraki, S. Okubo, H. Ohta, T. Ito, H. Takagi, M. Kaburagi, and Y. Ajiro Phys. Rev. Lett. 95, 117202 (2005).
[2] YM Lu and A Vishwanath, Physical Review B 86 (12), 125119 (2012).
[3] A. Kitaev, Annals of Physics 321, 2 (2006).

Particle vortex duality for Dirac Fermions

(Son; Metltiski &*AV*; Wang-Senthil;Mross-Alicea-Motrunich)

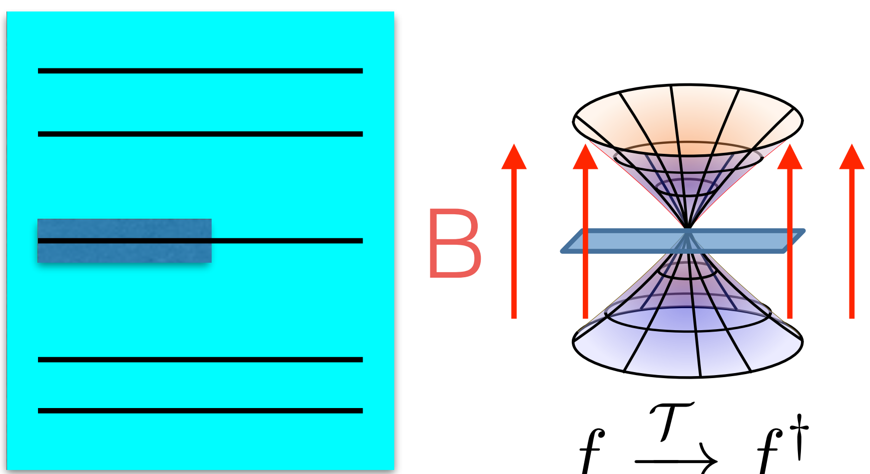
Particle Description	Dual Description
$\mathcal{L} = \bar{\psi}_e (p - eA)_\mu \gamma^\mu \psi_e$	$\mathcal{L} = \bar{\psi}_{CF} (p - a)_\mu \gamma^\mu \psi_{CF} + \frac{\epsilon_{\mu\nu\lambda}}{4\pi} A_\mu \partial_\nu a_\lambda$
Magnetic field B	Number density of composite Fermions: n=B/4π
Electron Density dN	Magnetic field b = Curl a
Particle Hole Symmetry	Time Reversal symmetry

Application I: Surface topological order of Topological Insulator

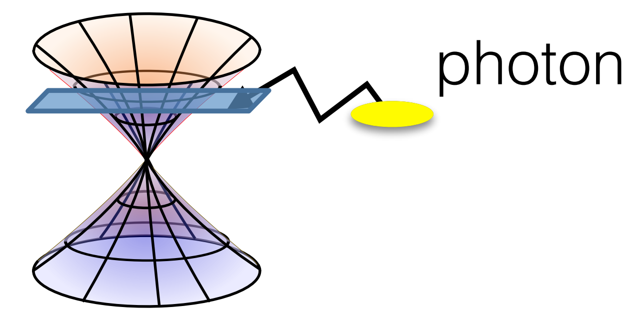
- Surface topological order - strongly correlated state. Hard to access in electronic variables.
- BUT - simple in dual variables!
- Consider superconductivity of composite Dirac fermions - Fu Kane superconductor! This is an INSULATOR of electrons, and it preserves time reversal symmetry. Excitations:
  - “h/2e” vortex (flux = electronic charge. Charge e/4) and Binds a Majorana mode.
  - Topological order T-Pfaffian. (Chen-Fidkowski-AV/Bonderson-Qi-Nayak)
  - Closely related to Read Moore Pfaffian state but compatible with time reversal symmetry. (R-MPfaffian = Ising x $U(1)_8$ while T-Pfaffian = Ising* x $U(1)_8$)

Application II: Half Filled Landau Level

- Magnetic field on a particle hole symmetric Dirac



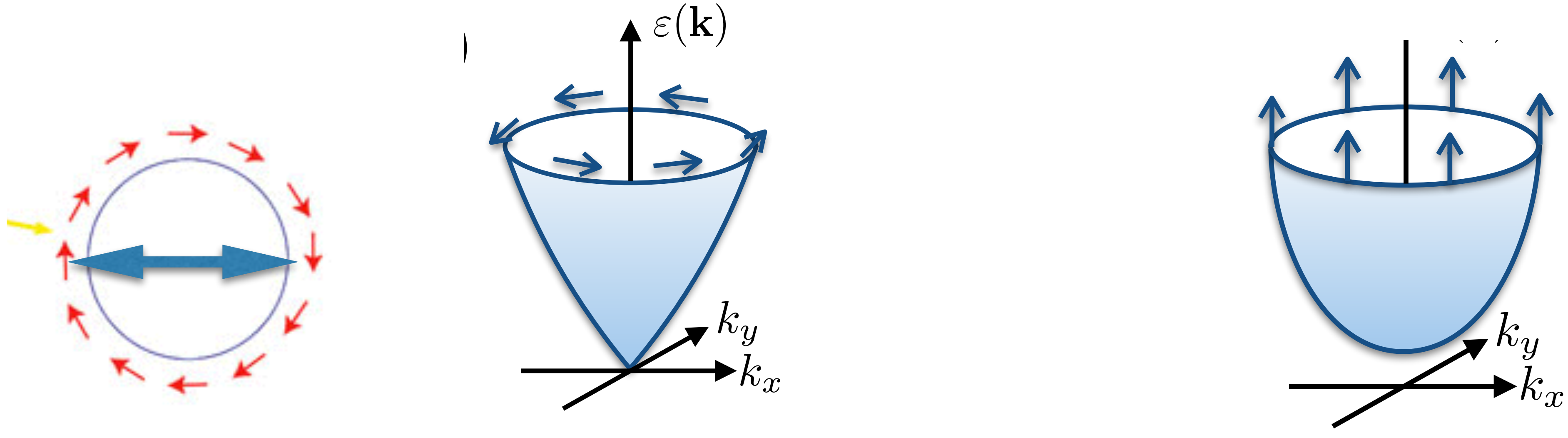
$$f \xrightarrow{\mathcal{T}} f^\dagger$$



$$c_\sigma \xrightarrow{\mathcal{T}} \epsilon_{\sigma\sigma'} c_{\sigma'}$$

Electron	Composite Fermion
$\mathcal{L} = \bar{\psi}_e (p - eA)_\mu \gamma^\mu \psi_e$	$\mathcal{L} = \bar{\psi}_{CF}(p-a)_\mu \gamma^\mu \psi_{CF} + \frac{\epsilon_{\mu\nu\lambda}}{4\pi} A_\mu \partial_\nu a_\lambda$
Time Reversal symmetry	Particle-Hole Symmetry
Magnetic Field B	Composite Fermion Density δN

Experimental Consequences?



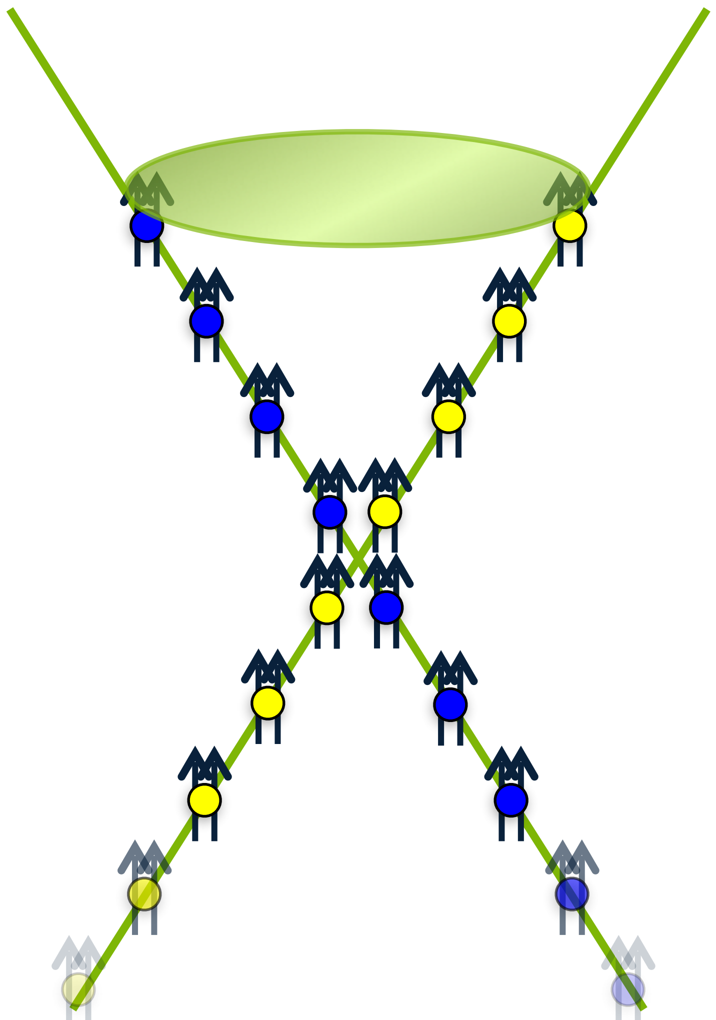
Want to distinguish Dirac from `conventional ‘ Halperin-Lee-Read composite fermions.

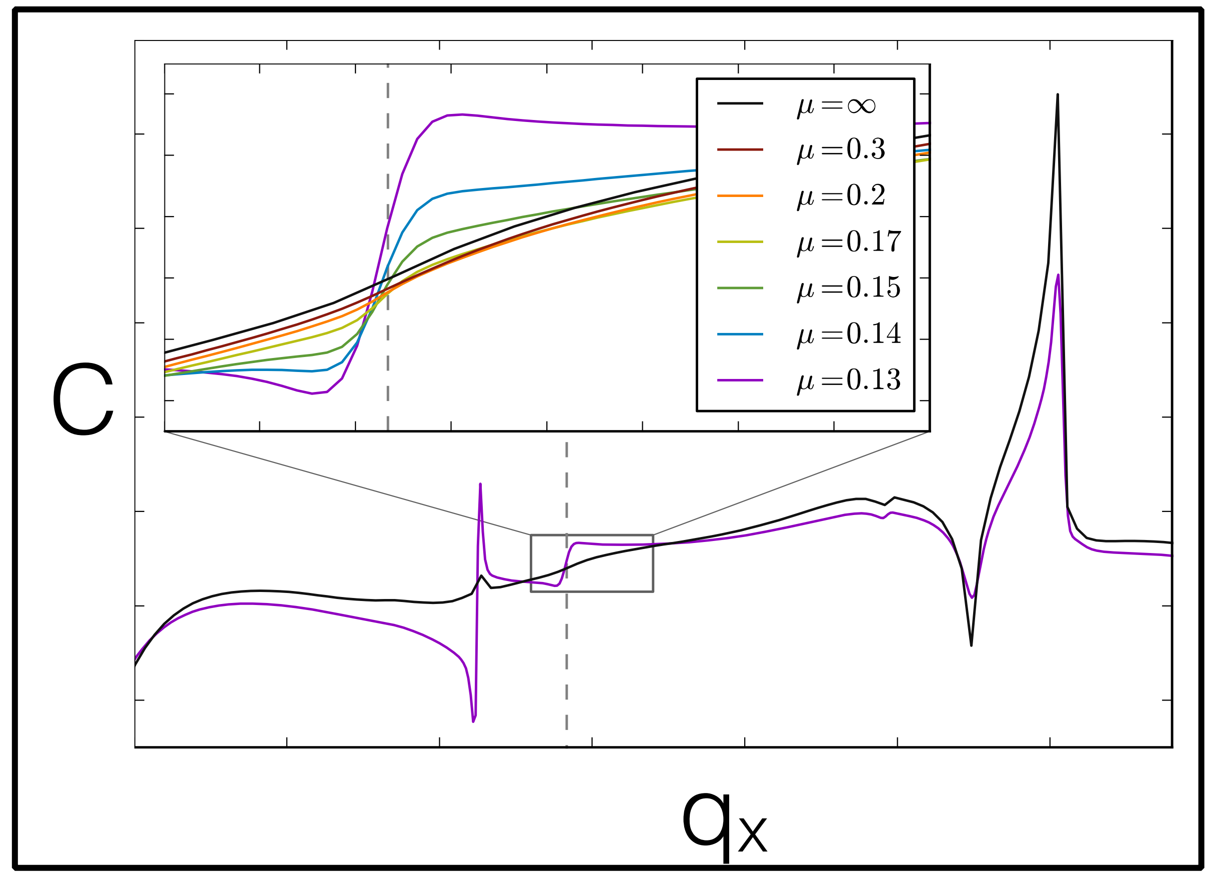
Measure the Berry phase at Fermi surface?

- Physical consequence - suppressed $2k_F$ backscattering by particle-hole symmetric scattering.

- Broken PH - Berry phase can interpolate between pi and 0.

Dual Description of Dirac Fermions



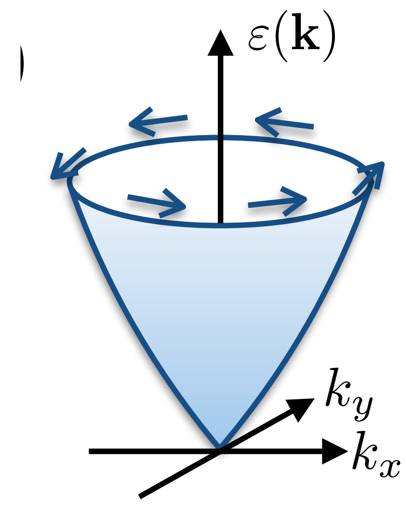


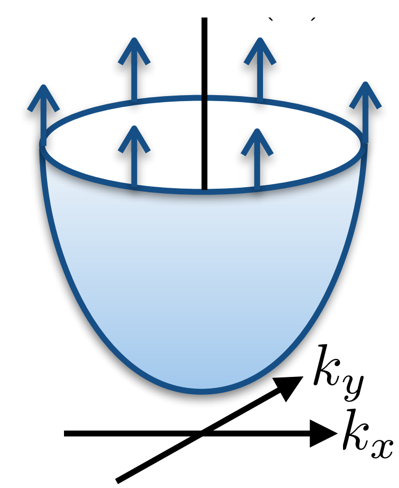
$2k_F$ signal reappears upon breaking PH symmetry (purple)

Suppression of $2k_F$ backscattering with particle hole symmetry. Seen in numerical (DMRG) simulations of 1/2 filled Landau level with Coulomb Interactions.

Gerdaets, Zaletel, Metltiski, AV, Motrunich.

Experimental Consequences?





Distinguish Dirac from `conventional‘ Halperin-Lee-Read composite fermions. Measure the Berry phase.

Our proposal - thermoelectric transport (in particular Nernst effect). (Potter, Serbyn, AV - arxiv:1512.06852)

$$E = S\nabla T$$

S_{xx} - Thermopower

S_{xy} - Nernst effect

$$\frac{S_{xy}}{S_{xx}\sigma_{xx}} = 2\frac{h}{e^2} \quad \text{(Dirac)}$$

$$= 0 \quad \text{(HLR)}$$

Duality of $d = 2$ XY Quantum Model

Focus on topological defects (vortices).

Superfluidity => no mobile vortices

Vortex :- $\psi = |\psi| e^{i\varphi}$

$\varphi(r)$ winds by 2π

eg

unit vortex :-
$\varphi = \theta + \text{const}$

Far away -> ground state. Hence defect has low energy. Map circle at large radius to ground state manifold. Both are circles (S^1), one parameterized by angle θ, other by phase φ.

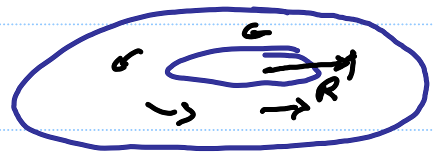
Map $S^1 \to S^1$

characterized by integer winding: $\varphi = n\theta + \text{const}$

Strength 'n' vorticity.

Connection to Superfluidity

Consider flow in an annulus

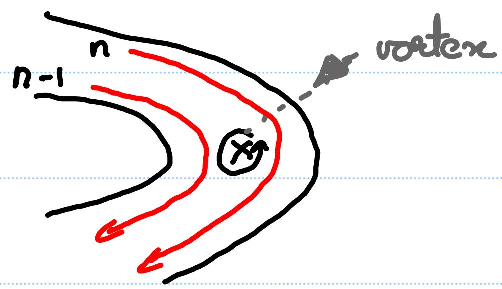


$$\phi = n\Theta$$

$$J = \rho_s \nabla\phi$$

$$J \simeq \rho_s \frac{n}{2\pi R} \hat{\Theta}$$

Current only decays if vortex number 'n' changes. This requires tunneling of a vortex from inside to outside



Therefore, if vortices are gapped in the bulk, there is an infinite tunneling barrier for a thick annulus.

	Boson	Vortex
Super-fluid	condensed	gapped
Mott Insulator	gapped	condensed

Dual Variables

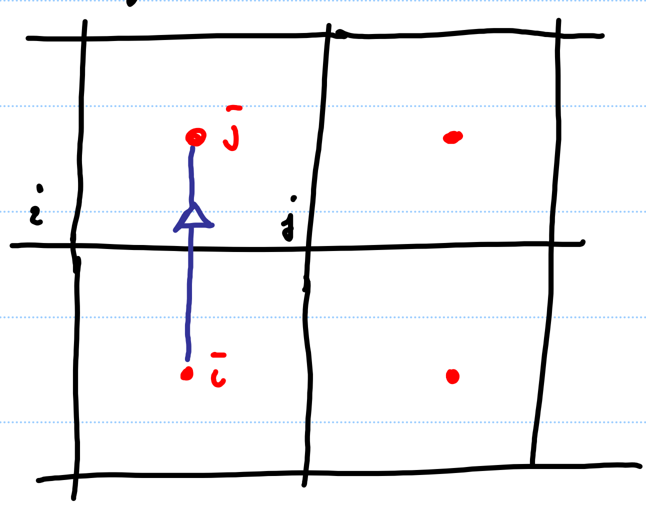
Lets rewrite

$$H = \frac{U}{2} \sum{}' (n_i - \bar{n})^2 - J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)$$

in vortex variables. We assume the offset $\bar{n}$ = integer, which is equivalent to $\bar{n} = 0$.

Introduce bond variables $e_{\bar{i}\bar{j}}$ on dual lattice. Note $e_{\bar{i}\bar{j}} = -e_{\bar{j}\bar{i}}$.

Use right hand rule to relate: $e_{\bar{i}\bar{j}} = \frac{\phi_i - \phi_j}{2\pi}$



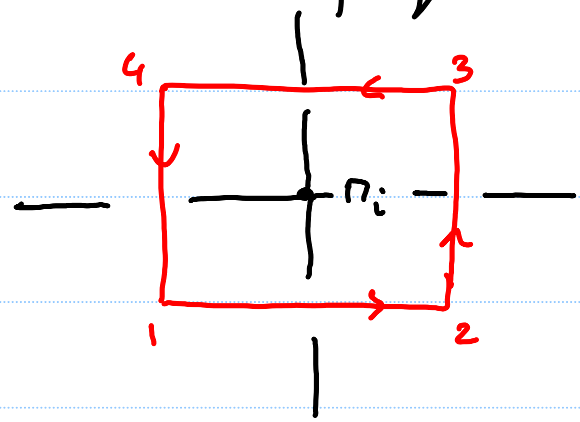
which is a lattice version of:

$$\vec{e} = \frac{1}{2\pi} \hat{z} \times \vec{\nabla}\phi$$

1

So
$$e_x = -\frac{1}{2\pi} \Delta_y \phi$$
$$e_y = \frac{1}{2\pi} \Delta_x \phi$$

The conjugate variable of e_{ij} on a bond is a_{ij}. Since the operator n_i translates ϕ_i, which changes e_{ij} around a plaquette



$$2\pi n_i = a_{12} + a_{23} + a_{34} + a_{41}$$

This is the lattice version of $n \sim \frac{1}{2\pi}(\nabla \times a)$

Thus the Hamiltonian is exactly :-

$$H = \frac{U}{2} \sum \left[\frac{\Delta \times a}{2\pi}\right]^2 - J \sum_{\langle ij \rangle} \cos(2\pi e_{ij})$$

However there is a constraint on electric fields :

Naively, since $e \sim \frac{1}{2\pi} \hat{z} \times \nabla\phi$

$\nabla \cdot e = 0$

However, since ϕ is compact $(\phi = \phi + 2\pi)$

$\oint \nabla\phi \cdot dl = 2\pi n_v$

Thus $\int \nabla \cdot e \, dA = \frac{1}{2\pi} \oint \nabla\phi \cdot dl = n_v$

So $\nabla \cdot e = n_v \delta^2 (r - r_0)$

These are the vortices, and they are quantized to have <u>integer</u> strength.

Note :- Since 'e' is not an integer, the electrodynamics we obtain is non-compact. Unlike compact electrodynamics, we have 2 phases in $d = 2+1$.

Analyzing the Dual Theory

$$H = \frac{u}{2}\sum\left(\frac{\Delta\times a}{2\pi}\right)^2 - g\sum\cos 2\pi e_{ij}$$

$$\Delta\cdot e = 2\pi n_v \quad \{\text{definition of } n_v\}$$

To analyze this we expand the cosine, but implement the fact that there are only $2\pi\times$ integer displacements of the 'a' field, by introducing the potential "$-t\cos a$".
This however violates the constraint, unless we also displace n_v. Therefore introduce the conjugate phase χ : $[n_v,\chi]=-i$ on every dual site. Then we have:-

$$H \cong \sum \frac{u}{2}\left(\frac{\Delta\times a}{2\pi}\right)^2 + \frac{g}{2}(2\pi e)^2 - t\cos(\Delta\chi - a)$$

and $\Delta\cdot e = n_v$.

This model has two pheses.

(i) When J is large, there are strong fluctuations of the cosine term, since the conjugate electric field gets well defines. Ignoring the cosine, we get Maxwell Hamiltonian

$$H = e^2 + b^2$$

which has a gapless 'photon' mode. This is the Goldstone mode in the dual language, and this is the superfluid phase.

(ii) When J is small the cosine term is relevant and condenses vortices. This leads to an 'a^2' term on expanding the cosine, which gaps out the photon. This is the Mott phase.

Lecture 2 - Sep 3, 2013. Spontaneous Symmetry Breaking in the Quantum ising Model in D=1+1.

Phy 250 - Ashvin Vishwanath

PACS numbers:

I. OVERVIEW OF SYMMETRY

We will be studying the Hamiltonian:

$$H = -J\left[\sum_i \sigma_i^z \sigma_{i+1}^z + g\sum_i \sigma_i^x\right] \tag{1}$$

We note that ther eis a competition between the two terms, leading to interesting physics. The first ferromagnetic term wants adjacent pairs of spins to be both up or both down, while the second term wants to polarize the spins in the horizaontal direction. However there is more to that - consider for example the Hamiltonian: $H = -J\left[\sum_i \sigma_i^z + g\sum_i \sigma^x\right]$. Here too there is a competition of the two terms, but the evolution as a fiunction of g is smooth.

The key difference is one of symmetry - the first Hamiltonian posesses a Z_2 symmetry that is spontaneously broken at small values of g , giving rise to two phases. Intuitively, the symmetry is the fact that the Hamitlonain does not distinguish between spins being up or down.

To formalize this - the mathematical structure that describes symmetry is group theory. Symmetry is an invariance (you make a transformation to the system and ask if that brings it to a distinct but equivalent state). Clearly, a combination of two such invariance operations is also an invariance. This is the property of closure that group elements satisfy, where the group product is performing one action after another, and the group elements are the transformations that leave the system in an equivalanet state. Similarly the other properties of groups - associativity, and existence of an identity and inverse, can be redily checked.

What is the group here? Our symmetry takes up to down spins - this is implemented as $(\sigma_x,\, \sigma_y,\, \sigma_z) \to (\sigma_x,\, -\sigma_y,\, -\sigma_z)$. Obviously doing this twice is doeing nothing. So the group is Z_2, the elements are (1,U), where U^2=1. The operator that implements this symmetry is $U = \prod_i \sigma_i^x$. Clearly $H = U^\dagger H U$ hence this is a symmetry.

II. QUANTUM DISORDERED STATE (SYMMETRY PRESERVED)

Analyze the limit $g \gg 1$. First we ignore the ferromagnetic term and just satisfy the second term. This gives a unique ground state with all spins along the 'x' direction which we represent as $|0\rangle = |\rightarrow\rightarrow\rightarrow\cdots\rightarrow\rangle$. Clearly this ground state respects the symmetry as can be seen by applying $U:\; U|0\rangle = |0\rangle$. The first excited states are gotten by reversing the spin at some point 'i': $|1\rangle = |\leftarrow\rightarrow\rightarrow\cdots\rightarrow\rangle$, $|i\rangle = |\rightarrow\rightarrow\rightarrow\cdots\leftarrow\cdots\rightarrow\rangle$. These excited states have energy $\Delta E = 2gJ$ above the ground state, whose energy we label as E_0. However there are 'N' of them for a length 'N' chain and are simply localized in this limit. However adding the ferromagnetic term as a perturbation gives them a dispersion. Restricting to the low energy space of a single spin flip, the ferromagnetic term induces a transition from$|i\rangle \to |i+1\rangle,\, |i-1\rangle$. So we can write the approximate Eigenvalue equation:

$$H|i\rangle = -J\left[|i+1\rangle + |i-1\rangle\right] + (E_0 + 2gJ)|i\rangle$$

It is simplest to consider the system on a circle, with 'N' sites, so the displaced indices $i \pm 1$ are interpreted with periodic boundary conditions at the edges. Then, this is readily solved by a Fourier transform. By writing in Fourier space: $|i\rangle = \frac{1}{\sqrt{N}}\sum_j e^{ikr_j}|k\rangle$, where the locations of the spins are at $r_j = ja$, we see that the momentum labels 'k' must satisfy $k = 2\pi\frac{m}{Na}$, in order to satisy periodic boundary conditions with integer 'm'. Furthemore, only $m \in 1,\, 2,\, \ldots,\, N$ are distinct. In the limit of $N \to \infty$, we get all values in the line $k \in [-\pi/a,\, \pi/a]$

In these variables we find:

$$[H - E_0]\,|k\rangle = [-2J\cos k + 2gJ]\,|k\rangle$$

thus, for a given momentum thereis a specific energy $\epsilon(k) = 2J[g - \cos k]$. This is a *particle* excitation - for a given momentum there is a fixed energy. Hence the lowest excitations above the symmetric state are gapped particles, with energy gap $\Delta = 2J(g-1)$ and a dispersion $\epsilon(k) \sim \Delta + Jk^2$, for small momenta. The operator that measures the number of particles is σ_i^x while the one that creates them is σ_i^z. Since these particles have a Z_2 character, the creation and annihilation operators are identical.

Naively, if we simply extend the calculation well beyond its regime of validity at $g \gg 1$, we anticipate a transition at $g = 1$, where the gap to these excitations close, and they 'condense'. Serendipiously this turns out to be the exact value.

III. BROKEN SYMMETRY STATE

Consider now the weak coupling limit - $g \ll 1$. Actually, let us set $g = 0$, and then carefully discuss what happens at finite but small g. Now, since the first term is the only one it is minimized by states $|+\rangle = |\uparrow\uparrow\uparrow \ldots \uparrow\rangle$ and by $|-\rangle = |\downarrow\downarrow\downarrow \ldots \downarrow\rangle$ and by any combination of them. Clearly these ground states break symmetry so that $U|+\rangle = |-\rangle$. However, the linear combinations $(|+\rangle \pm |-\rangle)$ do respect the symmetry. If we are at any nonzero g, there is a finite matrix element for $|+\rangle$ to mix with $|-\rangle$. Thus they are not eigenstates. However, the matrix element is vanishingly small in the themrodynamic limit of $N \to \infty$. We will argue this has to do with the gapped excitations in these states. However, if we assume that for a moment, then we can see how spontaneous symmetry breaking appears. While at any finite system size the ground states actually respect the symmetry, if one prepares the system in say all spin up, it takes an extremely long time to evolve out of it and demonstrate that it is not an eigenstate. So for all practical purposes we can consider it to be an eigenstate, and one can make the approximation better and better by going to larger systems, keeping all parameters fixed. We will see later that

Let us begin with the state$|+\rangle = |\uparrow\uparrow\uparrow \ldots \uparrow\rangle$. What is the lowest energy excitation here? For simplicity consider an open chain. One may naively say this is again a spin flip - now the flip occurs from up to down, and costs $\Delta E = 4J$. However there is actually a lower energy excitation - a domain wall with energy 2J. On the periodic boundary conditions system we need to make a pair of domain walls, however these are independent excitations - i.e. the single spin flip is not a 'particle' it does not have a fixed energy momentum relation in general. A Given energy may be divided into various momenta. So we get a '2-particle continuum'. However there are particle like excitations - domain walls, which however are non local objects. You need to change the state of a macroscopic number of spins to create them.

As before, we see that the fist excited states are at energy 2J, and represent single domain walls, that are localized in the limit of g=0. Consider now the action of the perturbation $g\sum_i \sigma_i^x$. This will cause the Domain walls to move. The domain walls are most naturally represented as living on the bonds i.e. $|\bar{i} = i + 1/2\rangle$. Flipping a spin will move it either to the left or right. Hence, we have a very similar situation as above, which can be written as :

$$(H - E_0)|\bar{i}\rangle = -gJ\left[|\bar{i}+1\rangle + |\bar{i}-1\rangle\right] + 2J|\bar{i}\rangle$$

Again, by Fourier transforming we get the spectrum $\epsilon(k) = 2J\left[1 - g\cos k\right]$- for the domain walls. (We have glossed over the fact that we now have periodic boundary conditions and hence have an even number of domain walls, but this can be justified). Again one may expect a transition when the domain walls condense at g=1.

Now, one can see why the broken symmetry state is stable - one needs to create a pair of domain walls and make them move around the system and annihilate. However these are gapped excitations - the energy cost to do this is 2J, while the perturbation gJ is the one that moves it through the system. Thus, the action cost is $S \sim e^{-N\log\frac{1}{g}}$ which vanishes in the thermodynamic limit.

IV. DUALITY

There is a suggestive symmetry between dispersion of single domain wall and single spin flip - seem to require g\rightarrow 1/g. Indeed this duality between strong and weak coupling (and spin flips and defects) can be made completely rigorous in the quantum Ising model as we explain below.

First assume open boundaries. Then, the state of the spin system can be completely specified by a knowledge of the location of domain walls and one spin (say the first one). So we can rewrite the problem in terms of domain wall variables. We would like to know the operators that create a domain wall and measure it, like for the case of spin flips. By analogy we will call them $\tau_{\bar{i}}^z$ and $\tau_{\bar{i}}^x$. Clearly:

$$\tau^x_{\bar{i}} = \sigma^z_{\bar{i}+\frac{1}{2}}\sigma^z_{\bar{i}-\frac{1}{2}} \tag{2}$$

while the operator to insert a single domain wall is: $\tau^z_{\bar{i}} = \prod_{j>\bar{i}} \sigma^x_j$. Furthermore this can be inverted to give:

$$\sigma^x_i = \tau^z_{\bar{i}-\frac{1}{2}}\tau^z_{\bar{i}+\frac{1}{2}} \tag{3}$$

Clearly these may be represented as Pauli spin operators. Now, we may represent the Hamiltonian as:

$$H = -J\left[g\sum_{\bar{i}} \tau^z_{\bar{i}}\tau^z_{\bar{i}+1} + \sum_{i} \tau^x_{\bar{i}}\right] \tag{4}$$

Thus the Hamiltonian is self dual, which exchanges weak and strong coupling $g \to 1/g$. Thus if there is a single transiton it must occur at $g = 1$.

Note, there is not a perfect symmetry between the two sides of the phase diagram - we have glossed over a few details while doing this duality and we will come back and fix it in the problem sets.

For example, if a field along 'z' is appled, the domain walls are confined. These are analogs of quarks bound into a meson and show resonances. Later we will discuss an experimental observation of these resonances.

A. References:

Subir Sachdev. 'Quantum Phase Transitions' 1st edition. Page 39-49