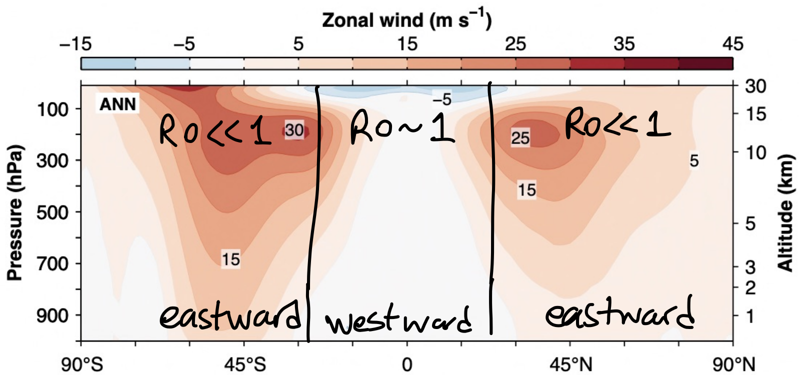
Boulder School 2022 Tiffany Shaw

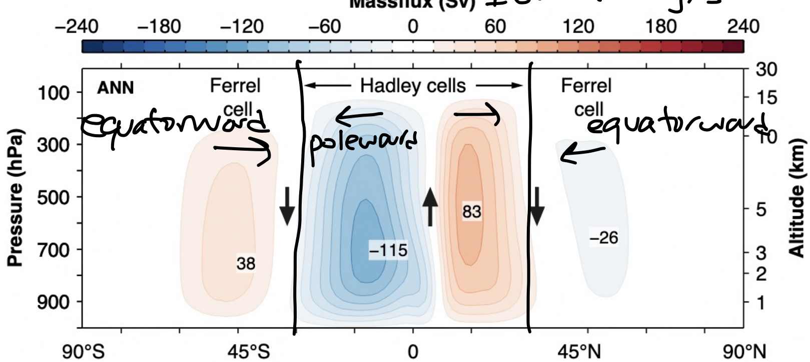
# Lecture 2 - Large scale circulation

Vallis (2006) Chps 11 and 12

## Review of large-scale features




1 Sv = $10^9$ kg/s



# Solutions in different regimes

## Tropical regime : $Ro \sim 1$  $R_1 \leq \varepsilon$

Goals : explain upper level eastward flow
extent of Hadley cell

From Lecture 1 thermal wind balance provides thermal equilibrium solution
$\overline{T} \neq 0$ , $\overline{u} \neq 0$ , $\overline{v} = 0$, $\overline{\omega} = 0$

Model Hadley cell as nearly inviscid axisymmetric circulation $\overline{v} \neq 0$ , $\overline{\omega} \neq 0$
$T \neq 0$, $\overline{u} \neq 0$

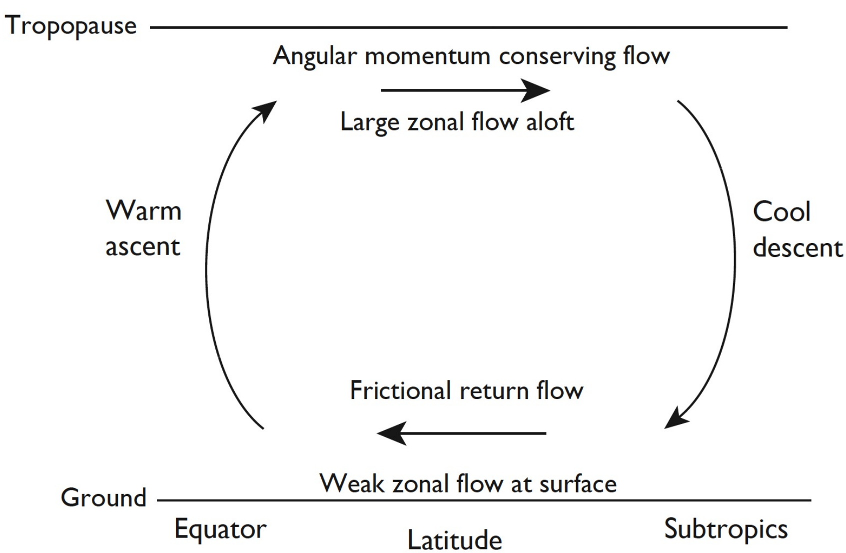



Figure 11.4 in Vallis (2006)

Recall $Ro \sim 1$ upper troposphere

$Ro \to 1 \Rightarrow (f + \bar{\zeta})\bar{v} \approx 0$

where $\bar{\zeta} = -\dfrac{1}{a\cos\phi}\dfrac{\partial}{\partial\phi}(\cos\phi\,\bar{u})$

N.B. $f + \bar{\zeta} = -\dfrac{1}{a^2\cos\phi}\dfrac{\partial\bar{M}}{\partial\phi}$

where $\bar{M} = (\bar{u} + \Omega a\cos\phi)\,a\cos\phi$

$\Rightarrow -\dfrac{1}{a^2\cos\phi}\dfrac{\partial\bar{M}}{\partial\phi}\bar{v} = 0$

Flow is parallel to angular momentum contours

Angular momentum conserving Hadley cell solution is a consequence of thermal equilibrium solution violating Hide's theorem

Hide's theorem: in steady state there can be no extrema of angular momentum except at lower boundary

Exercise. Prove Hide's theorem. Hint: write equation for angular momentum and prove by contradiction (assume maximum aloft)

Interior $\overline{M}$ cannot exceed $\overline{M}$ at equator at the surface $\overline{M}_{max} = \Omega a^2$

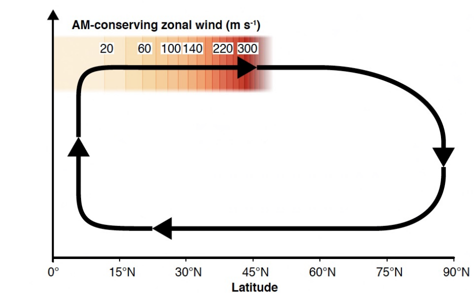
Conservation of $\overline{M}$ at each $\phi$ $\Rightarrow$

$$\overline{M}(\phi) = (\overline{u} + \Omega a \cos\phi) a\cos\phi = \overline{M}_{max} = \Omega a^2$$

$$a\cos\phi\, \overline{u}_{max} + \Omega a^2 \cos^2\phi = \Omega a^2$$

$$a\cos\phi\, \overline{u}_{max} = \Omega a^2 (1-\cos^2\phi)$$

$$\overline{u}_{max} = \frac{\Omega a \sin^2\phi}{\cos\phi}$$




Hide's theorem can be used to quantify extent of the Hadley cell

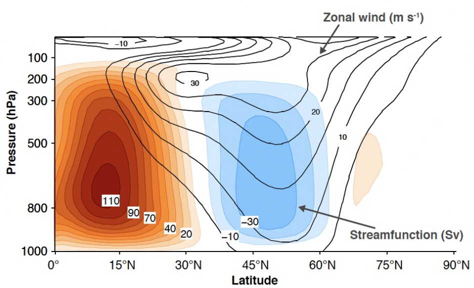



Figure T. Schneider

Relate zonal wind in the Hadley cell to temperature gradient via thermal wind balance

$$2\Omega \sin\phi \frac{\partial u}{\partial p} = \frac{R}{p}\frac{1}{a}\frac{\partial T}{\partial \phi}$$

Vertically integrate $\int_{p_s}^{p_T} \cdot \, dp$

$$2\Omega \sin\phi \left( u_T - \cancel{u_s}^{\text{small}} \right) = \frac{R}{a}\int_{p_s}^{p_T} \frac{\partial T}{\partial \phi} \, d\ln p$$

$$u_T \leq u_{max}$$

$$\frac{R}{a}\int_{p_s}^{p_T} \frac{\partial T}{\partial \phi} \, d\ln p \leq 2\Omega \sin\phi \frac{\Omega a \sin^2\phi}{\cos\phi}$$

$$\boxed{\frac{\partial \langle T \rangle}{\partial \phi} \leq \frac{2\Omega^2 a^2 \sin^3\phi}{R\cos\phi} \approx \frac{2\Omega^2 a^2 \phi^3}{R}}$$

If $\partial \langle T \rangle / \partial \phi$ is too large the Hadley cell spins up to reduce gradient to satisfy Hide's theorem

Exercise: Estimate minimum extent for Hadley cell by solving for $\phi$ given $\frac{\partial \langle T \rangle}{\partial \phi}$

Including moisture (Emanuel 1995)

$$2\Omega \sin\phi \frac{\partial u}{\partial p} = \frac{1}{a}\frac{\partial \alpha}{\partial \phi}$$ rather than using dry ideal gas law

Let $\alpha = \alpha(s^*, p)$ $\quad s^* = c_p \ln \theta_e^*$ saturation entropy

$$\theta_e^* \simeq c_p T + L q^*$$

$$\Rightarrow \frac{\partial \alpha}{\partial \phi} = \left(\frac{\partial \alpha}{\partial s^*}\right)_p \left(\frac{\partial s^*}{\partial \phi}\right) \stackrel{\text{Maxwell relation}}{=} \underbrace{\boxed{\left(\frac{\partial T}{\partial p}\right)_{s^*}}}_{\text{Moist adiabatic lapse rate}} \left(\frac{\partial s^*}{\partial \phi}\right)$$

$$\Rightarrow \frac{1}{a}\int_{p_s}^{p_T} \left(\frac{\partial T}{\partial p}\right)_{s^*} \frac{\partial s^*}{\partial \phi}\, dp \leq 2\Omega \sin\phi\, \bar{u}_{max}$$

$$\frac{1}{a}\frac{\partial s^*}{\partial \phi}\int_{p_s}^{p_T} \left(\frac{\partial T}{\partial p}\right)_{s^*} dp \leq \frac{2\Omega^2 a \sin^3\phi}{\cos\phi}$$

Moist adiabat $\Rightarrow$ $s^*$ independent $p$

$$\frac{1}{a}\frac{\partial s^*}{\partial \phi}\left(T_T - T_s\right) \leq \frac{2\Omega^2 a \sin^3\phi}{\cos\phi}$$

$$\frac{\partial s^*}{\partial \phi}\left(T_T - T_s\right) \leq \frac{2\Omega^2 a^2 \sin^3\phi}{\cos\phi}$$

Full solution (including energetics) can be found in Vallis (2006) section 11.2

Energy constrains $\bar{\omega}$ by assuming vertical advection balances heat added $\Rightarrow \bar{\psi}$ which gives $\bar{v}$

Model captures:
- subtropical jet $\bar{u} > 0$
- westward surface flow $\bar{u} < 0$
- dry subtropics (sinking)
- bound on Hadley cell edge $\phi$

Discrepancies:
- observed subtropical jet weaker (role of eddies)
- missing observed surface eastward ($\bar{u} > 0$) flow

# Extratropical regime : $Ro \ll 1$

Goals : explain equatorward flow aloft
surface eastward flow

Model Ferrel cell as eddy dominated regime

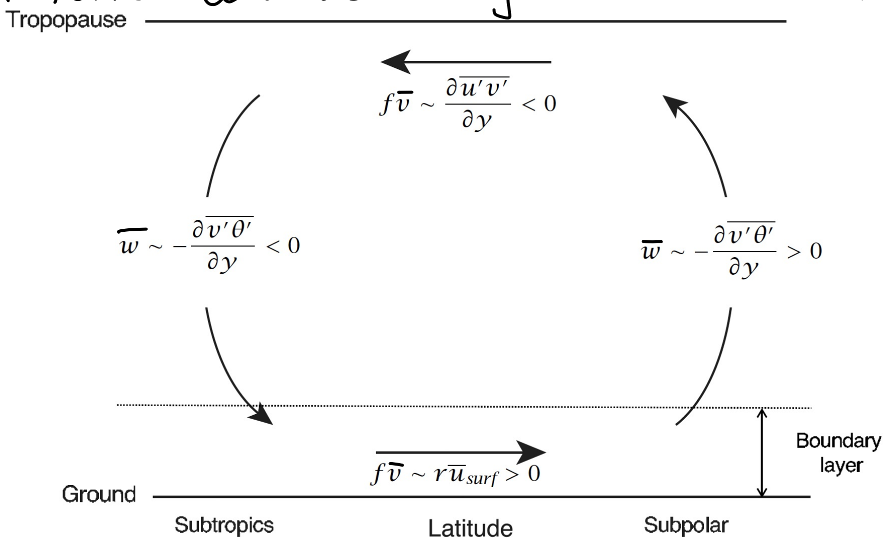



Figure 11.15 Vallis (2006)

Recall $Ro \ll 1$ upper troposphere

$$\Rightarrow \quad f\bar{v} \simeq \frac{1}{a\cos^2\phi}\frac{\partial}{\partial\phi}\left(\cos^2\phi\,\overline{u'v'}\right)$$

What determines $\overline{u'v'}$?

Rossby waves: solutions to the equations of motion in the quasi-geostrophic limit

$Ro \ll 1 \qquad L \sim L_d = \dfrac{NH}{f}$

Asymptotic expansion in Ro:

$Ro^0$ geostrophic balance steady state

$Ro^1$ time dependent quasi-geostrophic equations (combine into single equation)

Assume plane wave solution

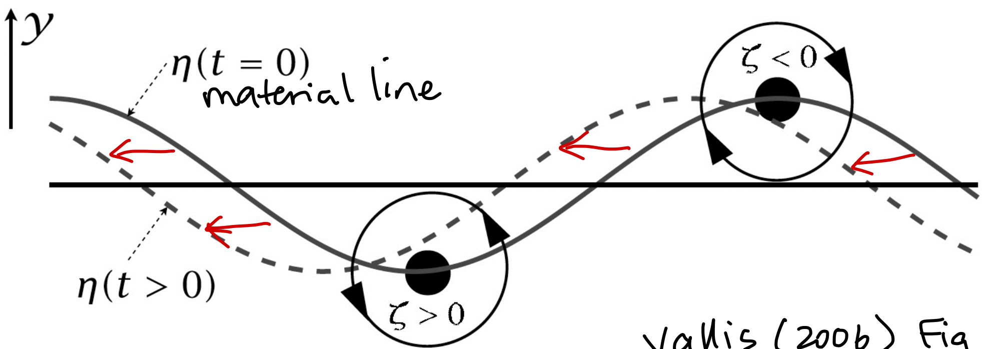
$$\psi = \mathrm{Re}\, A e^{i(kx+ly-\omega t)}$$

Dispersion relation:

$$\omega = ck = \bar{u}k - \frac{\beta k}{k^2+l^2}$$

$f = 2\Omega \sin\phi \approx f_0 + \beta y \qquad \boxed{\beta = \dfrac{2\Omega\cos\phi_0}{a}}$

## Mechanism: $\bar{u} = 0$, $L_d \to \infty$ $\quad \zeta + \beta y = \text{const}$




Vallis (2006) Fig 5.4

## Rossby wave properties:

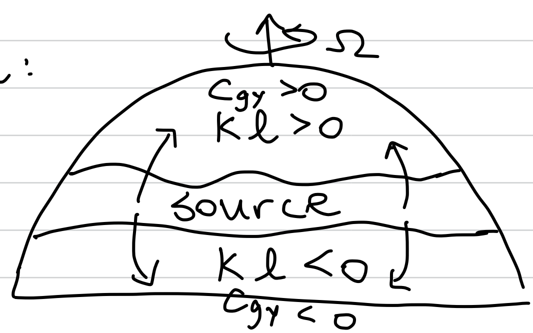
Meridional group velocity

$$C_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{(k^2+l^2)^2}$$

Rossby waves must transport (pseudo) energy away from the source region (radiation condition)

For each $k$ two possible $l$:

$$l = \pm\left(\frac{\beta}{\bar{u}-c} - k^2\right)^{1/2}$$



## (Pseudo) Momentum flux by Rossby waves:

$$\overline{u'v'} = -\frac{1}{2} A^2 k l$$

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Exercise: show $\overline{u'v'} = -\frac{1}{2} A^2 k l$. Hint use streamfunction.

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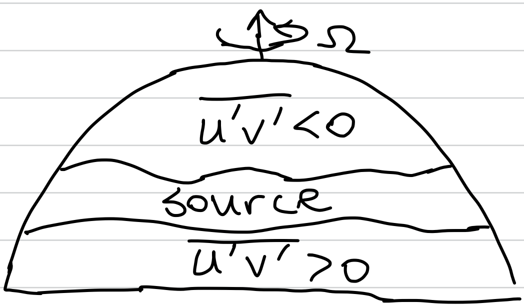


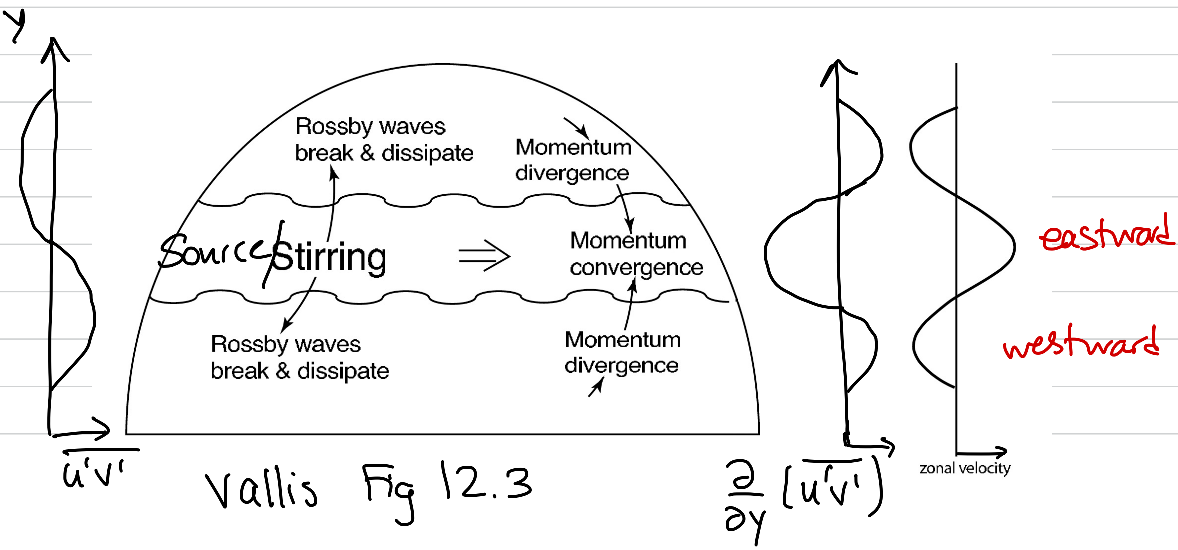


Vallis Fig 12.3

How do we get southward flow aloft?

$$f\bar{v} \approx \frac{\partial}{\partial y}(\overline{u'v'}) < 0$$

How do we get surface eastward flow?

Aloft $\quad f\bar{v} \approx \frac{\partial}{\partial y}(\overline{u'v'}) < 0$

Surface $\quad f\bar{v}_s = r\,\bar{u}_{surf}$

Take vertical integral, mass conservation $\Rightarrow \bar{v}_s = -\bar{v}$

$$f\bar{v} = -f\bar{v}_s = -r\,\bar{u}_{surf} = \frac{\partial}{\partial y}(\overline{u'v'})$$

$$\Rightarrow \quad \bar{u}_{surf} = -\frac{1}{r}\frac{\partial}{\partial y}(\overline{u'v'}) > 0$$

What happens at intermediate values $Ro$, $R_1$? Everything matters

Focus of current research

How do the regimes change under climate change?

Focus of Lecture 3!