Lecture II: The nonlinear analysis of the hydrodynamic model (TT equation)

$$\frac{\partial \vec{v}_\perp}{\partial t} + \lambda_1(\vec{v}_\perp \cdot \nabla_\perp)\vec{v}_\perp = -\nabla_\perp P + D_\perp \nabla_\perp^2 \vec{v}_\perp + D_\parallel \nabla_\parallel^2 \vec{v}_\perp + \vec{\eta}_\perp$$

Is this nonlinear term relevant in the hydrodynamic limit?

Scaling transformation: $\vec{x}_\perp \to s\vec{x}_\perp, x_\parallel \to s^\zeta x_\parallel, \mathrm{t} \to s^z t\,, \vec{v}_\perp \to s^\chi \vec{v}^\perp$

$$\frac{\partial \vec{v}_\perp}{\partial t} + s^{z+\chi-1}\lambda_1(\vec{v}_\perp \cdot \nabla_\perp)\vec{v}_\perp = \cdots + s^{z-2} D^\perp \nabla_\perp^2 \vec{v}_\perp + s^{z-2\zeta} D_\parallel \nabla_\parallel^2 \vec{v}_\perp + s^{z-\chi}\vec{\eta}_\perp(b\vec{x}_\perp, b^\zeta x_\parallel, b^z \mathrm{t})$$

$$< \eta_{\perp,i}(x,t)\, \eta_{\perp,i}(x',t') >= \Delta \delta_{ij} \delta(x-x')\delta(t-t')$$

$$\Delta \longrightarrow s^{2(z-\chi)-(d-1)-\zeta-z}\Delta = s^{z-2\chi-\zeta-d+1}\Delta$$

The scaling exponents for the linear theory

$$\frac{\partial \vec{v}_\perp}{\partial t} + \boldsymbol{s^{z+\chi-1}} \lambda_1 (\vec{v}_\perp \cdot \nabla_\perp) \vec{v}_\perp = \cdots + \boldsymbol{s^{z-2}} \boldsymbol{D}_\perp \nabla_\perp^2 \vec{v}_\perp + \boldsymbol{s^{z-2\zeta}} D_\parallel \nabla_\parallel^2 \vec{v}_\perp + \boldsymbol{s^{z-\chi}} \vec{\eta}_\perp (\boldsymbol{b \vec{x}}_\perp, \boldsymbol{b^\zeta x}_\parallel, \boldsymbol{b^z} \mathrm{t})$$

$$\Delta \longrightarrow s^{2(z-\chi)-(d-1)-\zeta-z} \Delta = \boldsymbol{s^{z-2\chi-\zeta-d+1}} \Delta$$

Linear theory exponents determined by: $z - 2 = z - 2\zeta = z - 2\chi - \zeta - d + 1 = 0$

The linear theory exponents: $\boldsymbol{z = 2}$, $\boldsymbol{\zeta = 1}$, $\boldsymbol{\chi = 1 - d/2}$

diffusive isotropic Loss of LRO in $d \le 2$

Exponent for the nonlinear term: $\boldsymbol{z + \chi - 1 = 2 - d/2}$, which means that the nonlinear convective term is relevant for $d \le 4$

Linearized hydrodynamic theory breaks down for $\boldsymbol{d \le 4}$

The scaling analysis: a simple example

$$\frac{dy}{dx} = x^n \rightarrow y = \int x^n dx = \frac{1}{n+1} x^{n+1}$$

Scaling symmetry (scale invariance): $x \rightarrow \alpha x \qquad y \rightarrow \alpha^{n+1} y$

$$y(x) = \alpha^{n+1} y(\alpha^{-1} x)$$

By setting $\alpha = x$

$$y(x) = y(1) x^{n+1}$$

You get the right answer (up to a constant pre-factor) without doing integration

The scaling hypothesis for the correlation function

There is no typical scale for the correlation function, i.e., it has scale invariance

$$\langle v_{\perp,i}(\vec{\boldsymbol{x}}_\perp, \boldsymbol{x}_\parallel, \mathrm{t}) v_{\perp,j}(\vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}'_\parallel, \mathrm{t}')\rangle = \delta_{ij} |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^{2\chi} f\left(\frac{t-t'}{|\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^{z}}, \frac{\boldsymbol{x}_\parallel - \boldsymbol{x}'_\parallel}{|\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^{\zeta}}\right)$$

The scaling exponents:

$\chi \Rightarrow$ velocity fluctuation exponent (roughness exponent)

$z \Rightarrow$ dynamic exponent

$\zeta \Rightarrow$ anisotropy exponent

$\chi > 0 \Rightarrow$ The system is disordered

$\chi < 0 \Rightarrow$ The ordered state is stable

How do we determine these exponents? -→ The Renormalization Group (RG) Theory

The Renormalization Group Theory: the idea

1) Integrating out the short distance degrees of freedom within $a < |x| < sa$ where $s > 1$ is a scaling factor
equivalently integrating out the large wavelength degrees of freedom within $1/a > |k| > 1/(sa)$

(real-space)

$(k_0 = 1/a)$

(k-space)

2) Scaling transformations: $\vec{\boldsymbol{x}}_\perp \to \boldsymbol{s}\vec{\boldsymbol{x}}_{\boldsymbol{n},\perp}, \boldsymbol{x}_\parallel \to \boldsymbol{s}^{\zeta}\boldsymbol{x}_{\boldsymbol{n},\parallel}, \mathbf{t} \to \boldsymbol{s}^{z}\boldsymbol{t}_{\boldsymbol{n}}, \vec{\boldsymbol{v}}_\perp \to \boldsymbol{s}^{\chi}\vec{\boldsymbol{v}}_{\boldsymbol{n},\perp}$

3) The equations of motion for the the new variables (v') and the new coordinate system (x',t') retain the same form. But they have different coefficients (parameters): $\wp_s = \{D'_\parallel, D'_\perp, \Delta', \lambda'\}$, which depends on the parameters at the original scale: $\wp = \{D_\parallel, D_\perp, \Delta, \lambda\}$

$$\wp_s = R_s(\wp) \qquad\qquad R_{ss'} = R_s R_{s'}$$

Renormalization transformation

Renormalization group: $\{R_s\}$

The Renormalization Group Theory: the scaling

Correlation function at a given scale l with parameters $\wp$:

$$\langle v_{\perp,i}(\vec{\boldsymbol{x}}_\perp, \boldsymbol{x}_\parallel, \mathrm{t}) v_{\perp,i}(\vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}'_\parallel, \mathrm{t}')\rangle = C_{ij}(\vec{\boldsymbol{x}}_\perp \text{-} \vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel, \mathrm{t}-\mathrm{t}' | \wp)$$

The same correlation function can be determined at the new coarse-grained scale sl:

$$C_{ij}(\vec{\boldsymbol{x}}_\perp \text{-} \vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel, \mathrm{t}-\mathrm{t}' | \wp) = s^{2\chi} C_{ij}(\vec{\boldsymbol{x}}_{\boldsymbol{n},\perp} \text{-} \vec{\boldsymbol{x}}'_{\boldsymbol{n},\perp}, \boldsymbol{x}_{\boldsymbol{n},\parallel} \text{-} \boldsymbol{x}'_{\boldsymbol{n},\parallel}, t_n - t'_n | \wp_s) = s^{2\chi} C_{ij}(\frac{\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp}{\boldsymbol{s}}, \frac{\boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel}{\boldsymbol{s}^\zeta}, \frac{t-t'}{s^z} | \wp_s)$$

RG fixed point: $\wp_s \to \wp^*$, i.e., $R_s(\wp_*) = \wp^*$

$$C_{ij}(\vec{\boldsymbol{x}}_\perp \text{-} \vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel, \mathrm{t}-\mathrm{t}' | \wp^*) = s^{2\chi} C_{ij}(\frac{\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp}{\boldsymbol{s}}, \frac{\boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel}{\boldsymbol{s}^\zeta}, \frac{t-t'}{s^z} | \wp^*)$$

$s = |\vec{\boldsymbol{x}}_\perp \text{-} \vec{\boldsymbol{x}}'_\perp|$

The scaling law of the correlation function!

$$C_{ij}(\vec{\boldsymbol{x}}_\perp \text{-} \vec{\boldsymbol{x}}'_\perp, \boldsymbol{x}_\parallel \text{-} \boldsymbol{x}'_\parallel, \mathrm{t}-\mathrm{t}') = \delta_{ij} |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^{2\chi} f(\frac{t-t'}{|\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^z}, \frac{\boldsymbol{x}_\parallel - \boldsymbol{x}'_\parallel}{|\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{x}}'_\perp|^\zeta})$$

The RG flow and the determination of the scaling exponents

The exponents (χ, z, ζ) can be determined by the fixed-point of the RG ”dynamics”

$$R_s(\wp_*) = \wp^*$$

The derivation of the RG “dynamics” (flow equation) is done conveniently in the k-space.

The flow equation is obtained perturbatively by using $\epsilon = d_c - d$ as a small parameter. d_c is the critical dimension, i.e., the linear hydrodynamic theory (or mean field theory) is valid for $d > d_c$. For TT flocking equation, d_c=4.

For infinitesimal scaling change, we write $s = e^{dl}$

$$\frac{dD_\perp}{dl} = [z - 2 + G_\perp(g)]D_\perp \qquad \frac{dD_\parallel}{dl} = [z - 2\zeta + G_\parallel(g)]D_\parallel$$

$$\frac{d\lambda}{dl} = [z + \chi - 1 + G_\lambda(g)]\lambda \qquad \frac{d\Delta}{dl} = [z + 1 - d - \zeta - 2\chi + G_\Delta(g)]\Delta$$

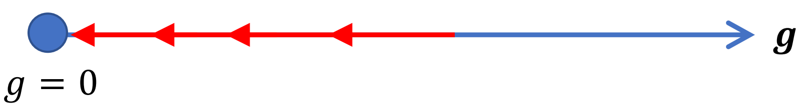
Nonlinear coupling constant $\quad g = \dfrac{\lambda \Delta^{1/2}}{D_\perp^{5/4} D_\parallel^{1/4}}$

G’s can be obtained perturbatively in orders of g by using Feyman diagram.

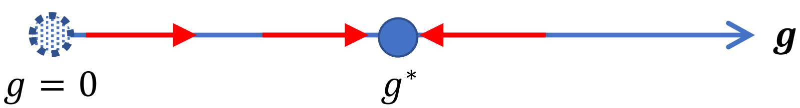
The fixed points of RG flow

$$\frac{dg}{dl} = [\epsilon + G_g(g)]g \qquad G_g(g) = G_\lambda(g) + \frac{1}{2}G_\Delta(g) - \frac{5}{4}G_\Delta(g) - \frac{1}{4}G_\parallel(g)$$

When $\epsilon = 4 - d \leq 0$, the trivial fixed point $g = 0$ is stable , linear hydrohynamics is valid
$\dashrightarrow$ linear exponents are valid



When $\epsilon = 4 - d > 0$, the trivial fixed point $g = 0$ is unstable, there is a nonlinear fixed $g^* > 0$, which is stable.
$\dashrightarrow$ nontrivial exponents can be found



$$z + \chi - 1 + G_\lambda(g^*) = 0 \qquad z + 1 - d - \zeta - 2\chi + G_\Delta(g^*) = 0$$

$$z - 2\zeta + G_\parallel(g^*) = 0 \qquad z - 2 + G_\perp(g^*) = 0$$

The nontrivial exponents in 2D flocking model

$$z + \chi - 1 + G_\lambda(g^*) = 0 \qquad z + 1 - d - \zeta - 2\chi + G_\Delta(g^*) = 0$$

$$z - 2\zeta + G_\parallel(g^*) = 0 \qquad z - 2 + G_\perp(g^*) = 0$$

In 2D, there is only one " $\perp$ " direction, so the nonlinear convective term can be written as a pure derivative term:

$$\lambda v_\perp \partial_\perp v_\perp = \frac{\lambda}{2}\partial_\perp(v_\perp^2)$$

Therefore, in k-space, the corrections due to the nonlinear convective term should all be proportional to $k_\perp^2$.

Since neither the $D_\parallel$ term (diffusion in the parallel direction) and the noise strength Δ contains $k_\perp^2$,
this means that these two terms are not renormalized: $\boldsymbol{G_\Delta(g) = G_\parallel(g) = 0}$

A pseudo-Galilean invariance: $v_\perp \longrightarrow v_\perp + v_0,\ x_\perp \longrightarrow x_\perp - \lambda v_0 t$ for arbitrary constant v_0
→ λ is not renormalized: $\boldsymbol{G_\lambda(g) = 0}$

The nontrivial exponents in 2D flocking model

$$z + \chi - 1 + \cancel{G_\chi(g^*)} = 0 \qquad z + 1 - d - \zeta - 2\chi + \cancel{G_\Lambda(g^*)} = 0$$

$$z - 2\zeta + \cancel{G_\parallel(g^*)} = 0 \qquad z - 2 + G_\perp(g^*) = 0$$

$$z + \chi - 1 = 0$$

$$z + 1 - d - \zeta - 2\chi = 0$$

$$z - 2\zeta = 0$$

$d = 2$ ⇒

$$\chi = -\frac{1}{5}, z = \frac{6}{5}, \zeta = \frac{3}{5}$$

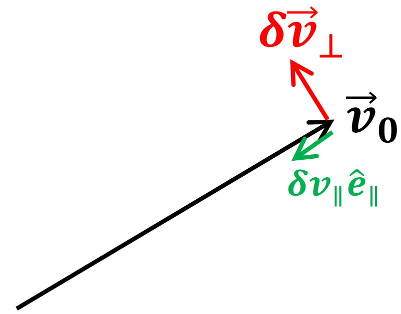
Note that $\chi = -\frac{1}{5} < 0$ $\Rightarrow$ long range order (LRO) is stable in 2D flocking systems!

Tamas Vicsek was very happy when we told him this result!

Now that the ordered state is stable, we can go back and take a look at the modes of fluctuations around the spatially homogeneous flocking state.

Let's do it in 2D

$$\vec{v} = \vec{v}_0 + v_\perp \hat{e}_\perp + \delta v_\parallel \hat{e}_\parallel \qquad |\vec{v}_0| = \sqrt{\frac{\alpha}{\beta}}$$



$$\frac{\partial \vec{v}}{\partial t} + \lambda_1 (\vec{v} \cdot \nabla)\vec{v} + \lambda_2 \ldots = \alpha \vec{v} - \beta |\vec{v}|^2 \vec{v} - \nabla P + D_T \nabla^2 \vec{v} + D_2 \ldots + \vec{\eta}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0$$

$$P(\rho) = \sum_{n=0} \sigma_n (\rho - \rho_0)^n$$

The couple slow dynamics of the density and velocity fluctuations

- The fast mode: $\partial_t \delta v_\parallel = -\sigma_1 \partial_\parallel \delta\rho - 2\alpha\,\delta v_\parallel + \text{irrelevant terms.}$

$$\delta v_\parallel = -D_\rho \partial_\parallel \delta\rho, \qquad D_\rho = \frac{\sigma_1}{2\alpha}$$

- The two coupled slow modes:

$(\gamma = \lambda v_0)$

$$[-i(\omega - \gamma q_\parallel) + \Gamma_v(\vec{q})]v_\perp(\vec{q},\omega) + i\sigma_1 q_\perp \delta\rho(\vec{q},\omega) = \eta_\perp(\vec{q},\omega)$$

$$[-i(\omega - v_0 q_\parallel) + \Gamma_\rho(\vec{q})]\delta\rho(\vec{q},\omega) + i\rho_0 q_\perp v_\perp(\vec{q},\omega) = 0$$

$$\Gamma_v(\vec{q}) = D_\perp^R(\vec{q})q_\perp^2 + D_\parallel q_\parallel^2$$

$$\Gamma_\rho(\vec{q}) = D_\rho q_\parallel^2$$

The renormalized diffusion constant

$$D_\perp^R(\vec{q}_\perp, q_\parallel; \lambda, \rho_0, \sigma_n) = q_\perp^{z-2} f(q_\parallel / q_\perp^\zeta),$$

The density and velocity correlation functions in Fourier space

$(\gamma = \lambda v_0)$

$$[-i(\omega - \gamma q_{\parallel}) + \Gamma_v(\vec{q})]v_{\perp}(\vec{q},\omega) + i\sigma_1 q_{\perp}\delta\rho(\vec{q},\omega) = \eta_{\perp}(\vec{q},\omega)$$

$$\left[-i(\omega - v_0 q_{\parallel}) + \Gamma_\rho(\vec{q})\right]\delta\rho(\vec{q},\omega) + i\rho_0 q_{\perp} v_{\perp}(\vec{q},\omega) = 0$$

$$\langle|\delta\rho(\vec{q},\omega)|^2\rangle = \frac{\Delta q_{\perp}^2 \rho_0^2}{S(\vec{q},\omega)},$$

$$\langle|v_{\perp}(\vec{q},\omega)|^2\rangle = \frac{\Delta[(\omega - v_s q_{||})^2 + D_\rho q_{||}^4]}{S(\vec{q},\omega)},$$

$$S(\vec{q},\omega)=[(\omega - \gamma q_{\parallel})\,(\omega - v_0 q_{\parallel})\text{-}c^2 q_{\perp}^2]^2+[(\omega - \gamma q_{\parallel})\Gamma_\rho(\vec{q}) + (\omega - v_0 q_{\parallel})\Gamma_v(\vec{q})]^2$$

$$(c^2 = \sigma_1\, \rho_0)$$

The mixed velocity-density "sound wave"

Characteristics of the dynamics can be obtained by looking at the poles of the correlation functions

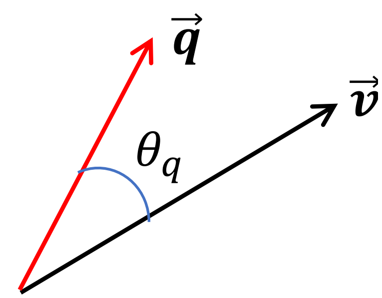
$$S(\vec{q},\omega)=[(\omega-\gamma q_{\parallel})\,(\omega-v_0 q_{\parallel})-c^2 q_{\perp}^2]^2+[(\omega-\gamma q_{\parallel})\Gamma_\rho(\vec{q})+(\omega-v_0 q_{\parallel})\Gamma_v(\vec{q})]^2$$

$S(\vec{q},\omega)=0$

$$\omega \approx C_{\pm}(\theta_q)q \pm i[D_{\perp}^R(\vec{q})q_{\perp}^2 + D_{\parallel}q_{\parallel}^2]$$

Wave propagation

Damping (dissipation)



$$(\omega-\gamma q_{\parallel})\,(\omega-v_0 q_{\parallel})-c^2 q_{\perp}^2=0$$

$$C_{\pm}(\theta_q)=\frac{1}{2}(1+\lambda)v_0\cos(\theta_q)\pm[\frac{1}{4}(1-\lambda)^2 v_0^2\cos^2(\theta_q)+c^2\sin^2(\theta_q)]^{\frac{1}{2}}$$

Characteristics of the correlation function in Fourier space

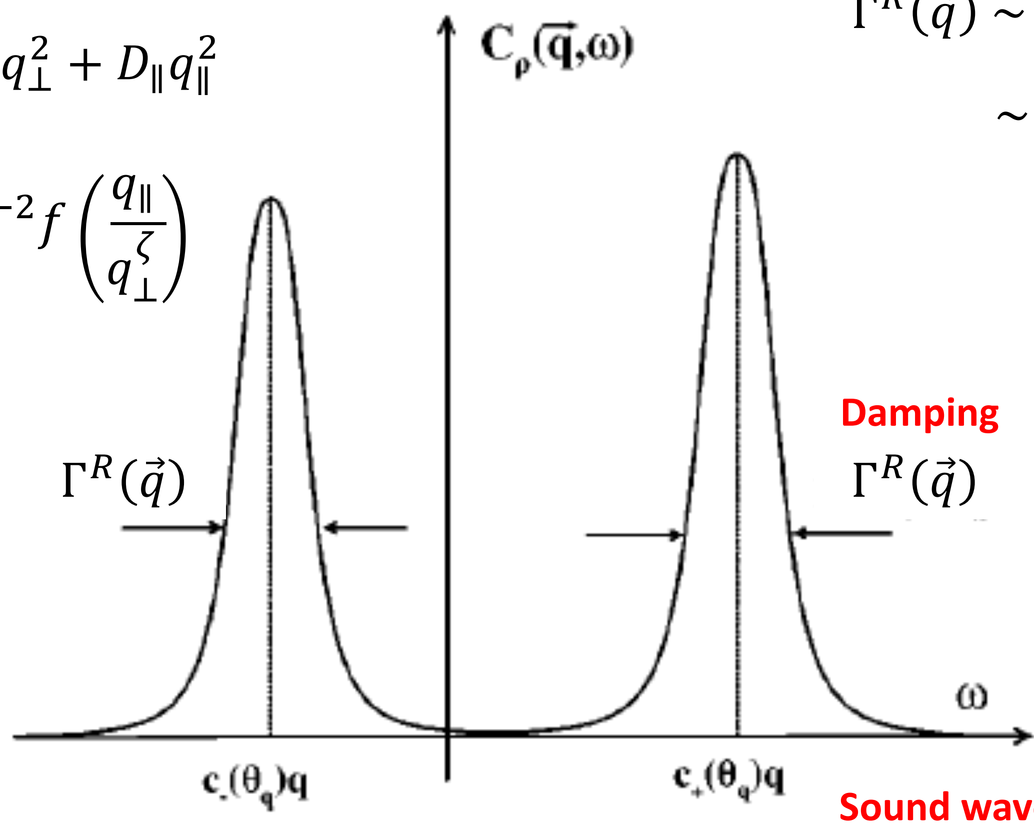
$$\omega \approx C_{\pm}(\theta_q)q \pm i\Gamma^R(\vec{q})$$

$$\Gamma^R(\vec{q}) = D_\perp^R(\vec{q})q_\perp^2 + D_\parallel q_\parallel^2$$

$$D_\perp^R(\vec{q}) = q_\perp^{z-2} f\left(\frac{q_\parallel}{q_\perp^\zeta}\right)$$

$$\Gamma^R(\vec{q}) \sim q_\perp^z\,, \quad when\ q_\parallel \ll q_\perp^\zeta$$

$$\sim q_\parallel^2\,, \quad when\ q_\parallel \gg q_\perp^\zeta$$



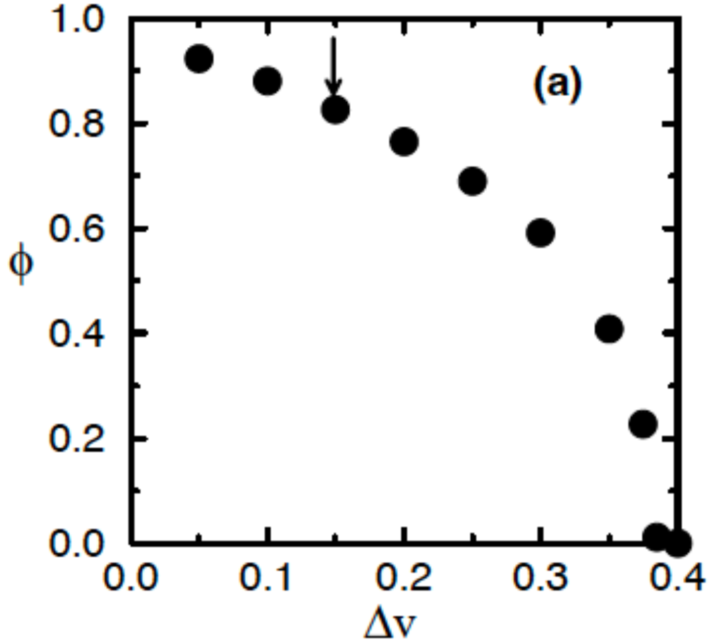
The equal time correlation function

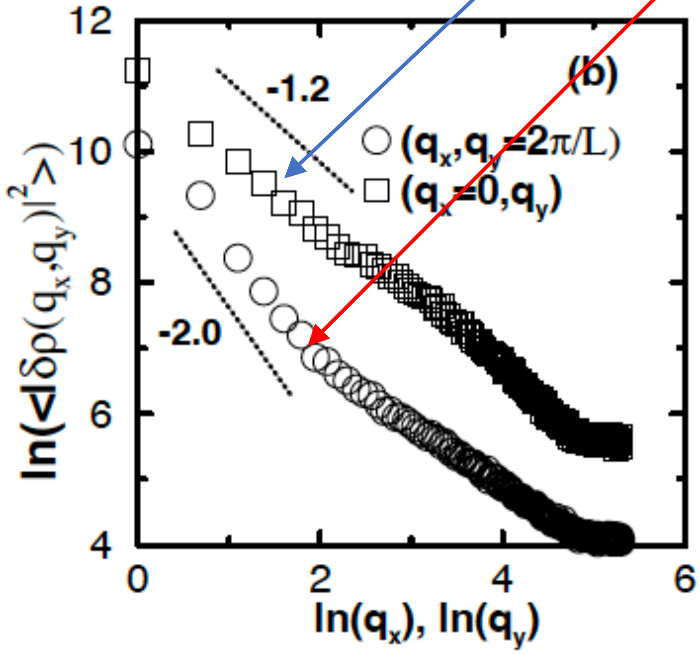
$$C_\rho(\vec{q}) = \int C_\rho(\vec{q},\omega)\,\frac{d\omega}{2\pi} = \frac{2\Delta\rho_0^2}{c^2\Gamma^R(\vec{q})}$$

$$C_\rho(\vec{q}) \sim q_\perp^{-z}, \quad when\ q_\parallel \ll q_\perp^\zeta$$

$$\sim q_\parallel^{-z/\zeta}, \quad when\ q_\parallel \gg q_\perp^\zeta$$

In d=2, $z = \frac{6}{5}, \zeta = \frac{3}{5}$





$(y = \parallel, x = \perp)$

YT, JT, Ulm, PRL, 1998

The giant number fluctuation (GNF)

$$< \delta\rho^2 >= \int C_\rho(\vec{q})d\vec{q} = \int \frac{2\Delta\rho_0^2}{c^2\Gamma^R(\vec{q})} d\vec{q} \sim L^{z-1-\zeta}$$

$$< \delta N^2 >= L^{2d} < \delta\rho^2 > \sim L^{3+z-\zeta}$$ (d=2)

$$< N >= \rho_0 L^2$$

$$< \delta N^2 > \sim L^{3+z-\zeta} \sim N^{(3+z-\zeta)/2} = N^{9/5} \gg N$$

Anomalous diffusion in the perpendicular direction

The dispersion in the perpendicular direction

$$w^2(t) = \langle [y_i(t) - y_i(0)]^2\rangle \sim \int_0^t \int_0^t \langle v_y^i(t') v_y^i(t'')\rangle dt'\, dt''$$

$$\langle v_y^i(0) v_y^i(t)\rangle \sim \langle v_y(\vec{x} + \phi \hat{x} t, t) v_y(\vec{x}, 0)\rangle$$

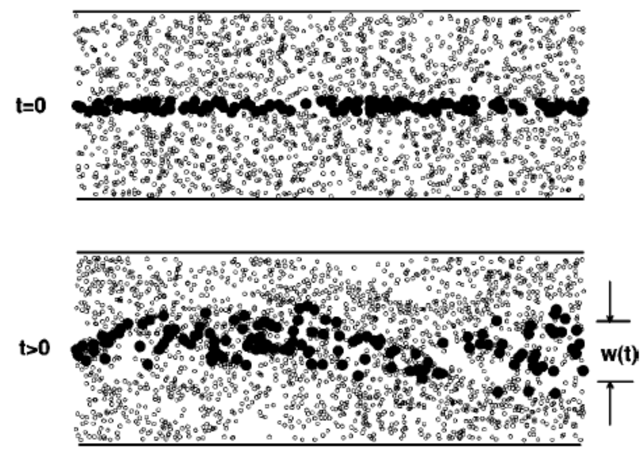
$$= \int \frac{\exp[i(\omega - \phi q_{||})t]\Delta(\omega - v_s q_{||})^2 d^2q\, d\omega}{S(\vec{q}, \omega)} \sim t^{1-1/\zeta}$$

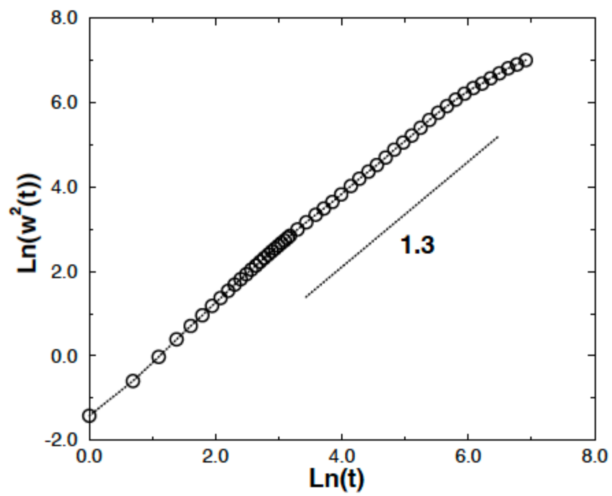
$$w^2(t) \sim t^{3-1/\zeta} = t^{4/3}$$

$(\zeta = \frac{3}{5}$ in 2D$)$

Super-diffusion

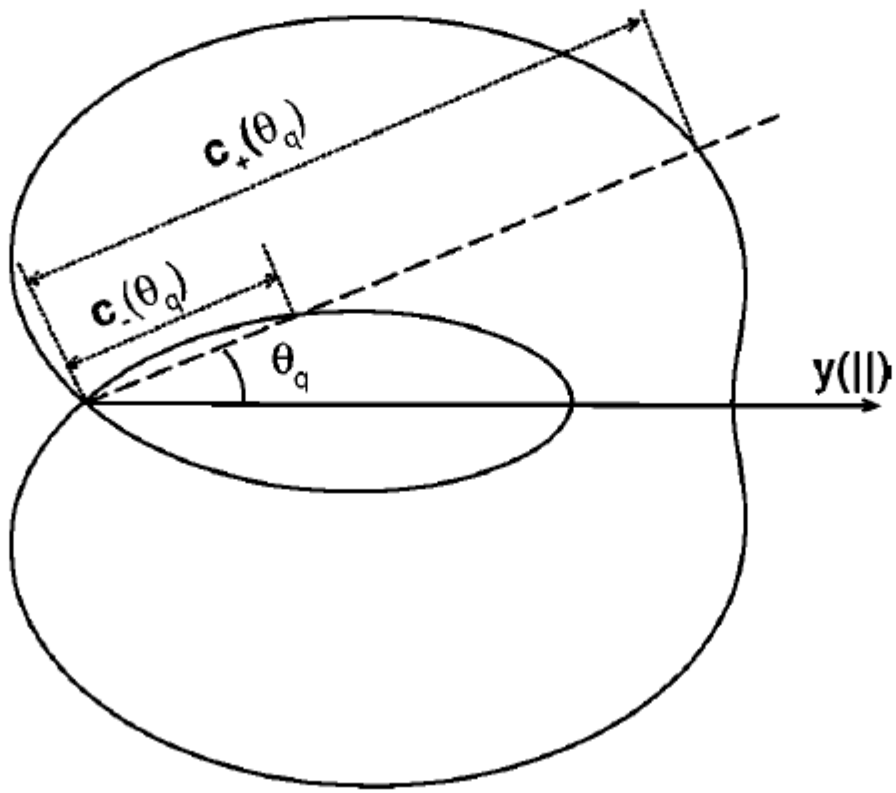
YT, JT, Ulm, PRL, 1998





The anisotropic sound speed

$$C_{\pm}(\theta_q) = \frac{1}{2}(1+\lambda)v_0\cos(\theta_q) \pm [\frac{1}{4}(1-\lambda)^2 v_0^2\cos^2(\theta_q) + c^2\sin^2(\theta_q)]^{\frac{1}{2}}$$



YT, TT, Ulm, PRL, 1998
JT & YT, PRE, 1998

In the flocking direction (θ_q=0)

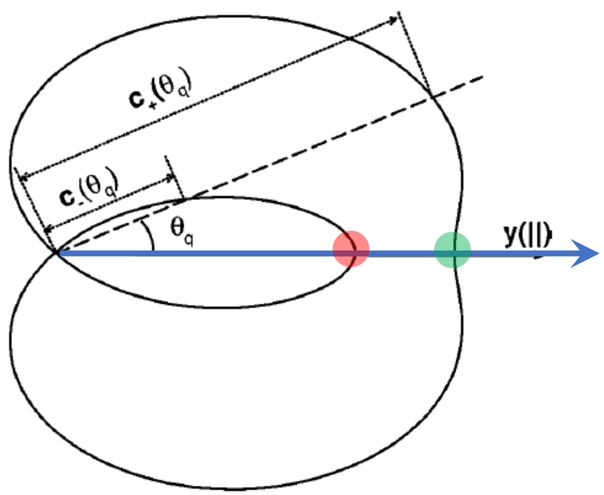
$$C_{\pm}(0) = \frac{1}{2}(1+\lambda)v_0 \pm \frac{1}{2}(1-\lambda)v_0$$

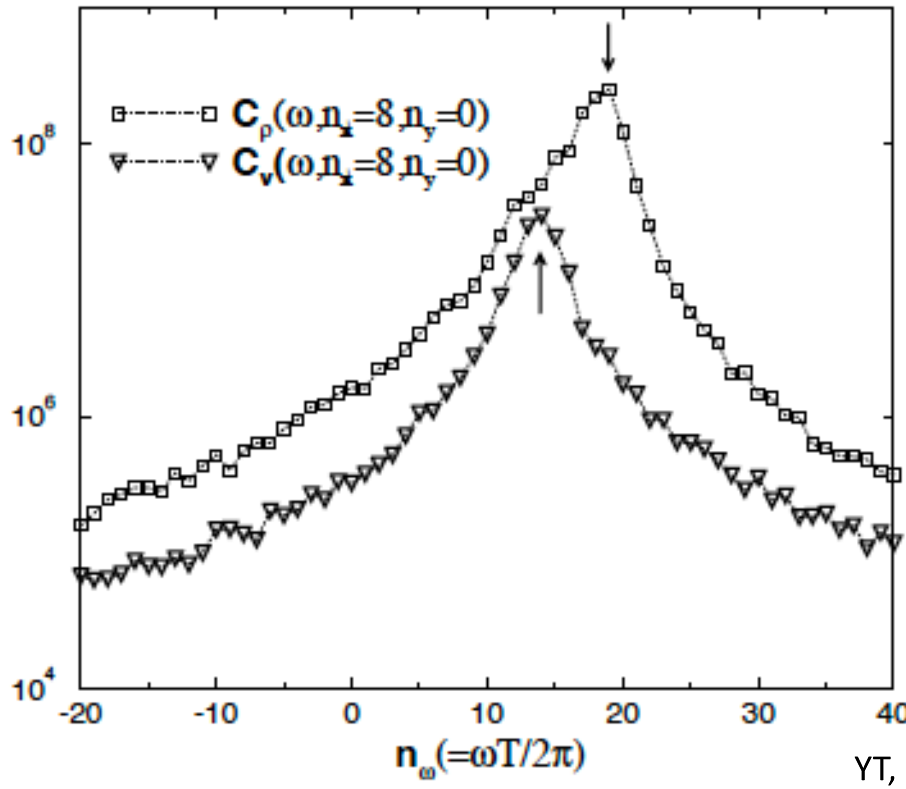
Velocity and density fluctuations decoupled!

They travel at different (advection) speeds:

v_0 -- density wave (green dot)

λv_0 -- velocity wave (red dot)



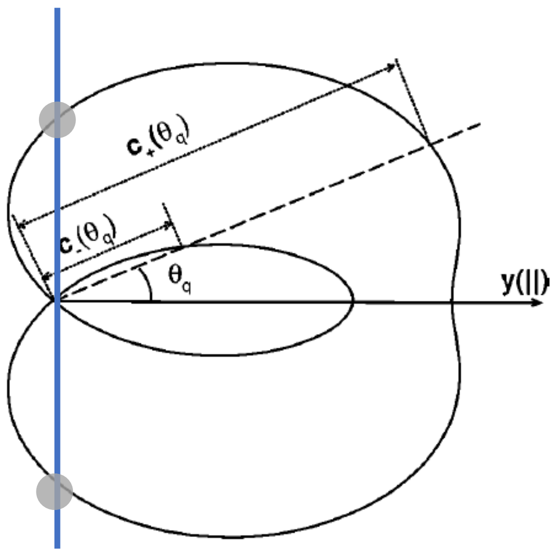


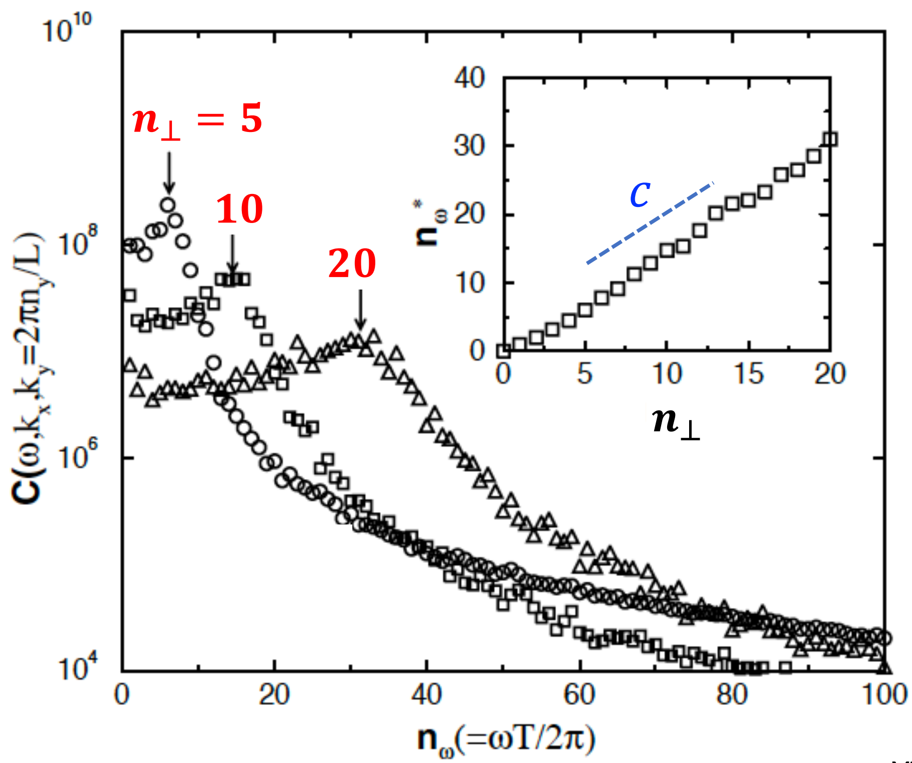
YT, TT, Ulm, PRL, 1998

In the perpendicular direction ($\theta_q = \pi/2$)

$C_{\pm}(\pi/2) = \pm c$ $\quad\quad$ $(c^2 = \sigma_1 \rho_0)$

Pure symmetric sound wave
(*c* is the sound speed)



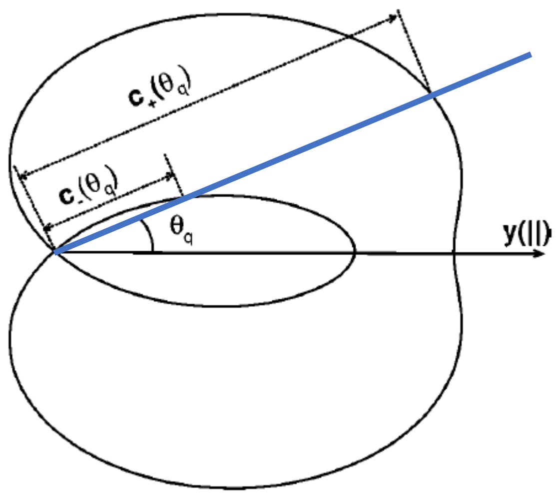


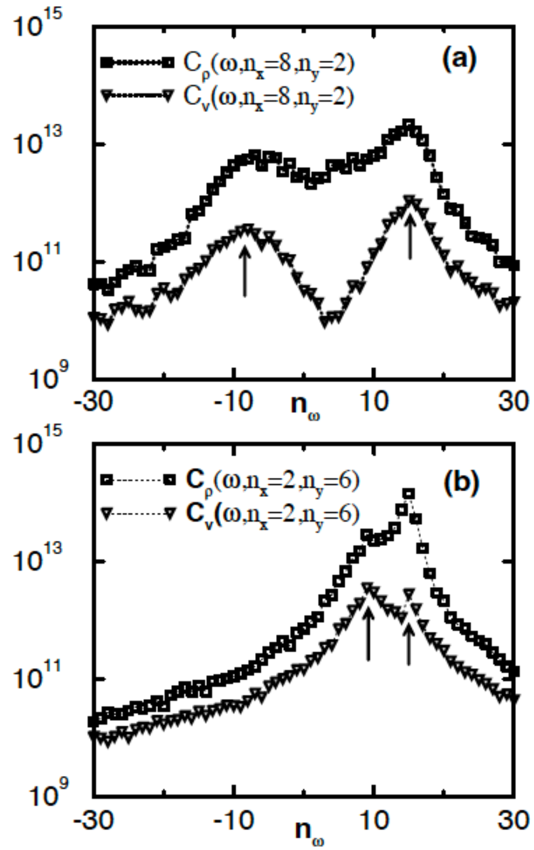
YT, TT, Ulm, PRL, 1998

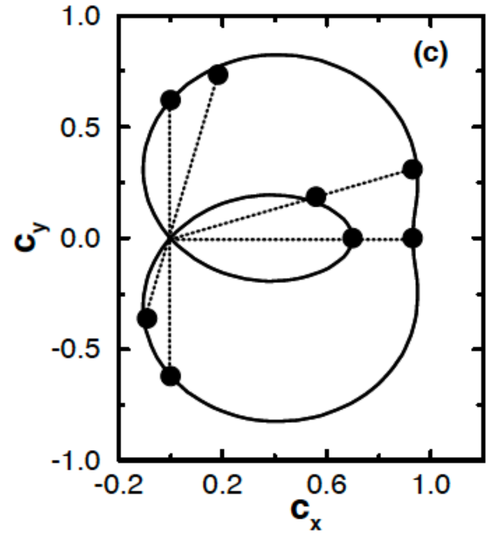
In the other directions $(0 < \theta_q < \pi/2)$

$$C_{\pm}(\theta_q) = \frac{1}{2}(1+\lambda)v_0\cos(\theta_q) \pm [\frac{1}{4}(1-\lambda)^2 v_0^2 \cos^2(\theta_q) + c^2\sin^2(\theta_q)]^{\frac{1}{2}}$$

The pure sound mode and the advection mode are mixed







YT, TT, Ulm, PRL, 1998

The dream of two theorists

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Flocks, herds, and schools: A quantitative theory of flocking

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VII. TESTING THE THEORY IN SIMULATIONS AND EXPERIMENTS

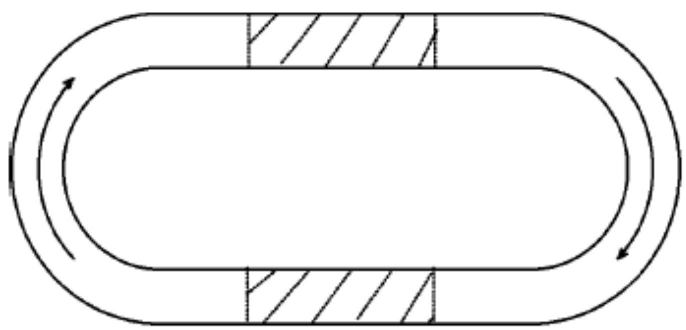
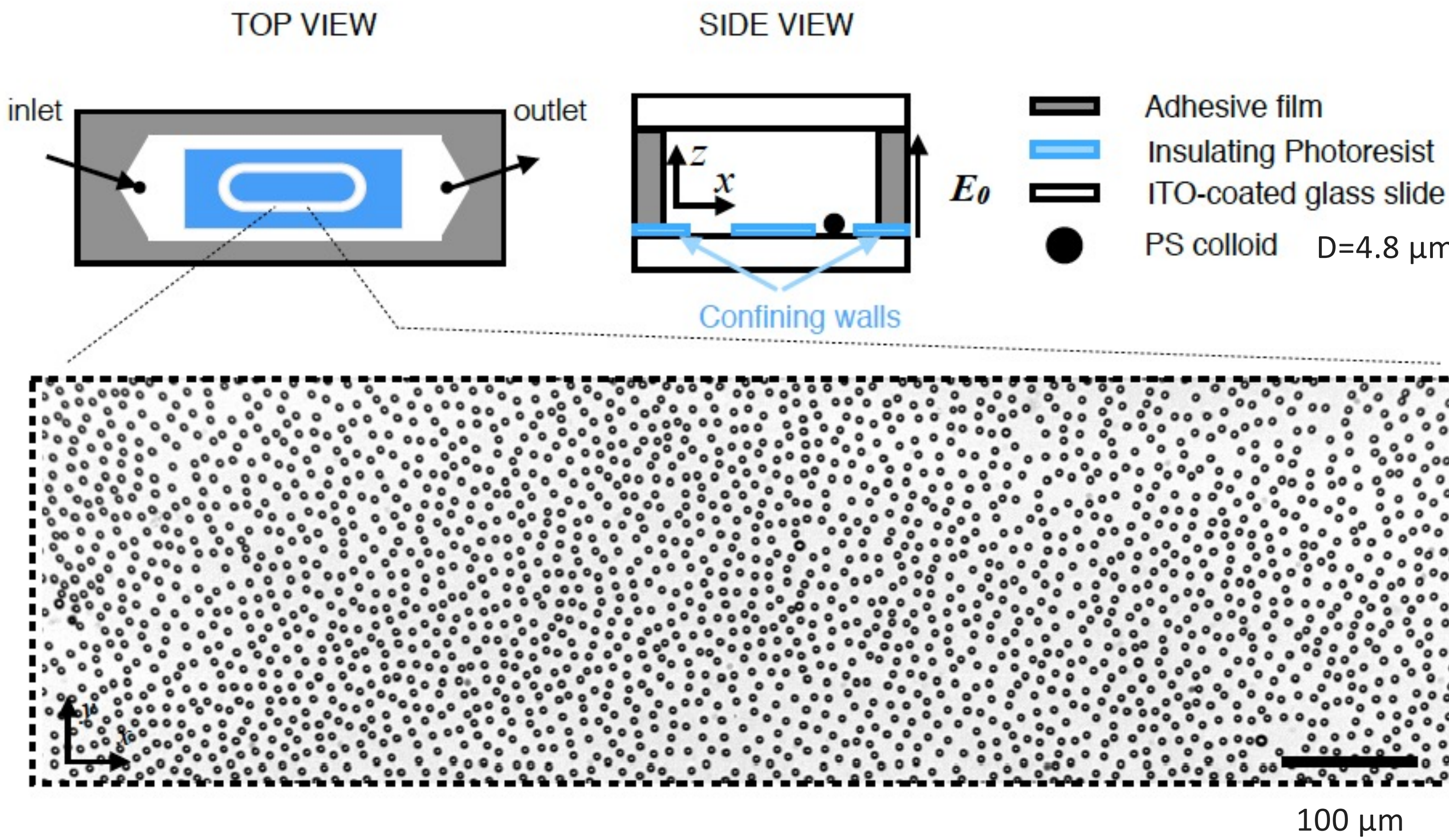


FIG. 10. More practical "track" geometry for experiments on real flocks. Data should only be taken from the cross-hatched region centered on the middle of the "straightaway."

Comes true (only after 20 years)

(Geyer et al, Nat. Material, 2018)



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LETTERS

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2018

Sounds and hydrodynamics of polar active fluids

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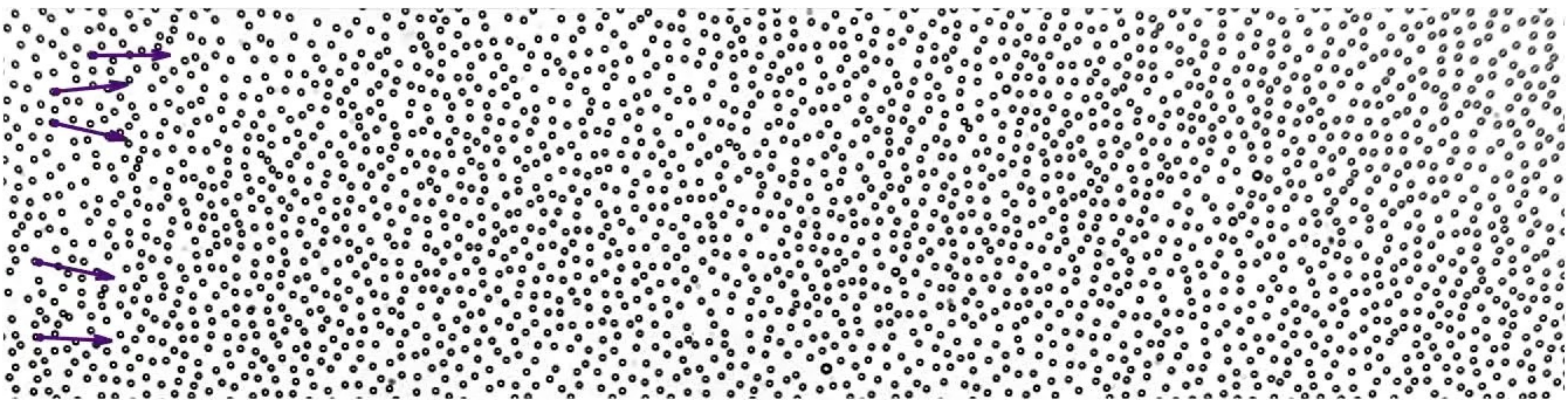
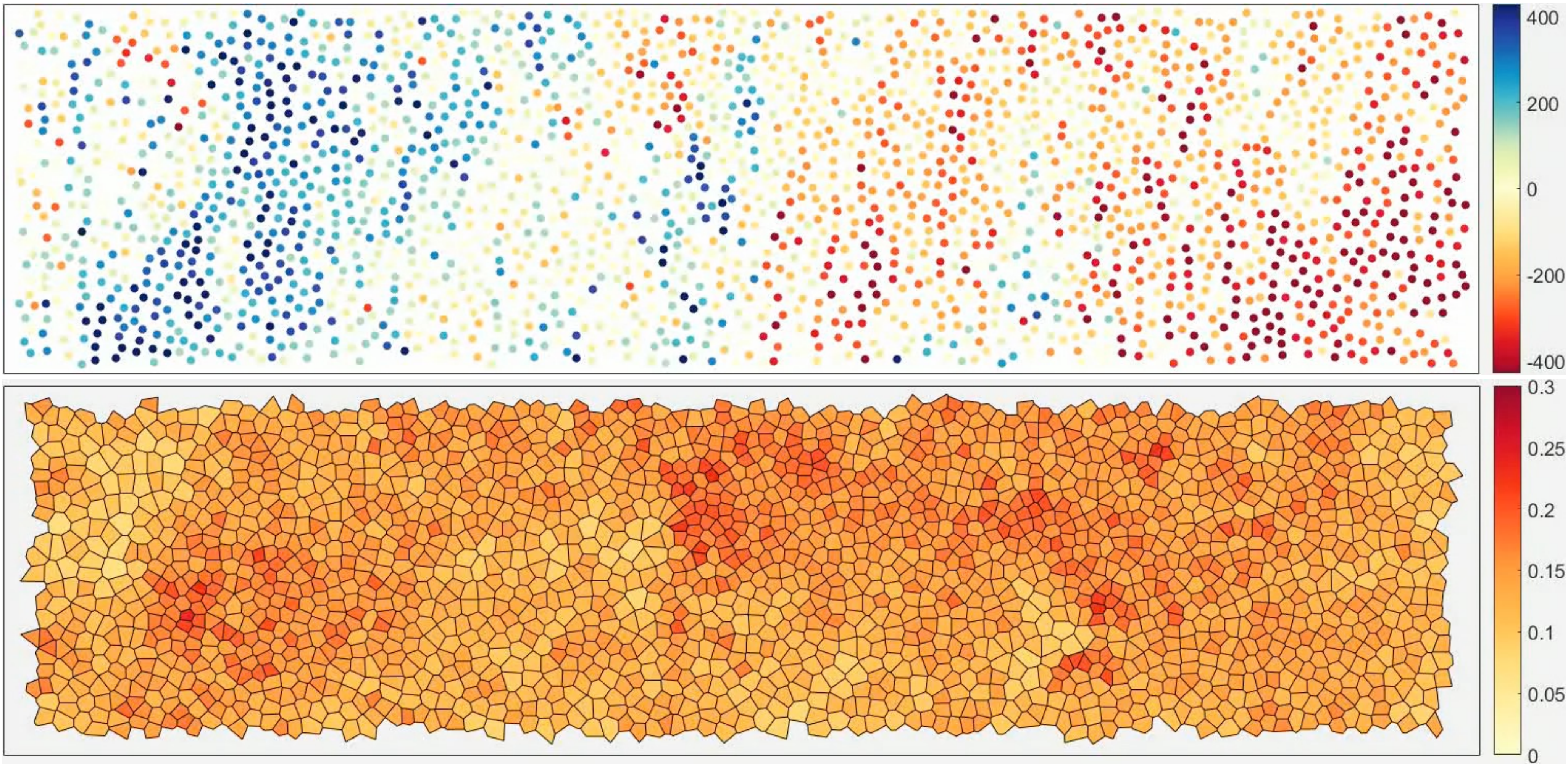


Both velocity and density fluctuations are studied



Flocking order and the fluctuations around the ordered state

(Geyer et al, Nat. Material, 2018)

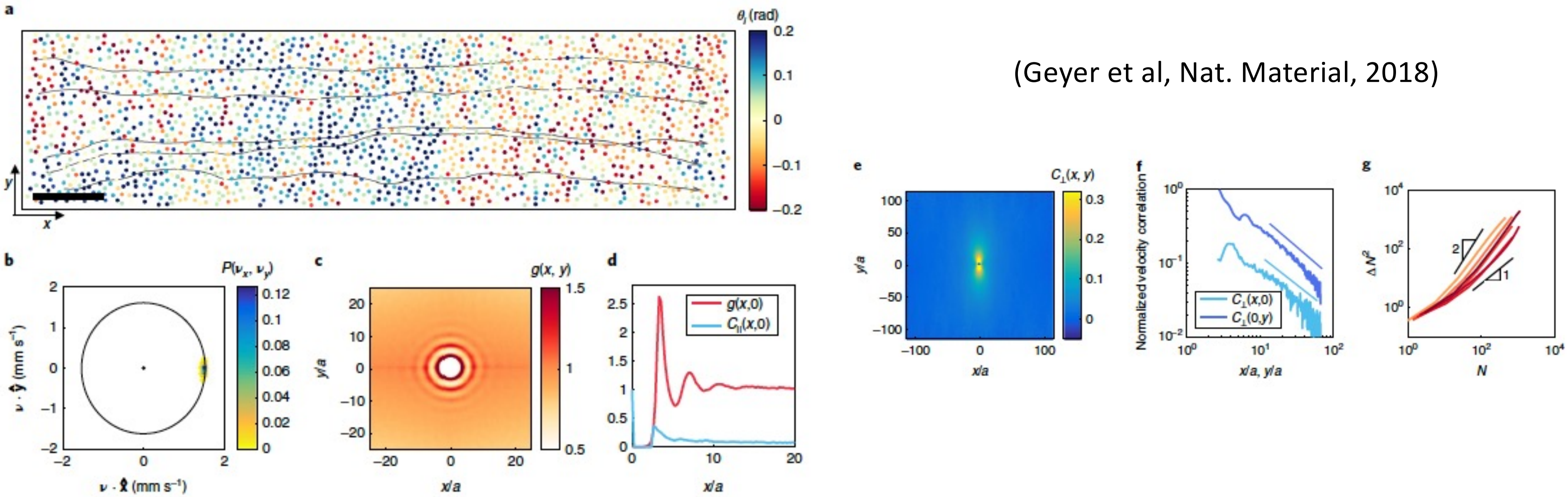
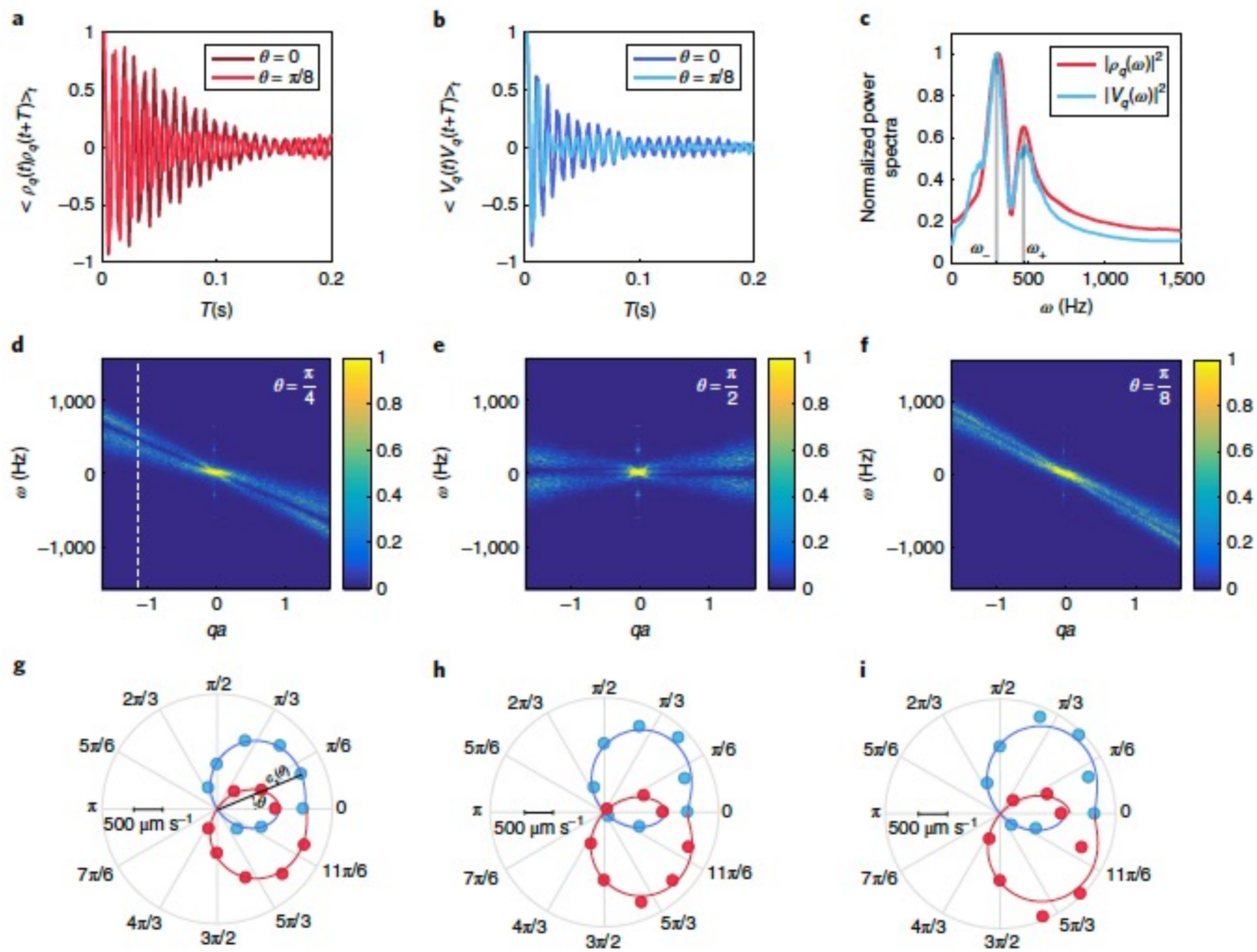


Fig. 1 | Colloidal rollers self-assemble into a spontaneously-flowing liquid. a, Close up on a microfluidic channel including ~3×10^6 colloidal rollers forming a homogeneous polar liquid. The colour of the particles indicates the value of the angle, θ_i, between their instantaneous velocity and the direction of the mean flow. Five trajectories illustrate the typical motion of the rollers. $\rho_0 = 0.11$. Scale bar: 100 µm. **b**, Probability density function of the roller velocities, $v_i(t)$ (ensemble and time integration). All the rollers propel along the same average direction. $\rho_0 = 0.24$, as in all following panels. **c**, The colour indicates the value of the density pair correlation function $g(x, y)$ evaluated at positions (x, y). Structural correlations are short ranged and display only weak anisotropy. **d**, Cuts along the flow direction of the pair distribution functions, $g(x, 0)$ (ref. [34]), and of the longitudinal velocity correlations $C_\parallel(x, 0)$, where $C_\parallel(\mathbf{r}) \equiv \langle v_i^\parallel(t) v_j^\parallel(t)\rangle_{(\mathbf{r}_i-\mathbf{r}_j)=\mathbf{r},t} / \langle\langle (v_i^\parallel)^2(t)\rangle_{i,t}$. Both structural and longitudinal-velocity correlations decay over few particle radii. **e**, Correlations of the transverse velocity fluctuations (ensemble and time average): $C_\perp(\mathbf{r}) \equiv \langle v_i^\perp(t) v_j^\perp(t)\rangle_{(\mathbf{r}_i-\mathbf{r}_j)=\mathbf{r},t} / \langle\langle (v_i^\perp)^2(t)\rangle_{i,t}$. The transverse fluctuations are long ranged and strongly anisotropic. **f**, The correlations of the transverse velocity fluctuations, $C_\perp(\mathbf{r})$, decay algebraically in both directions. The solid lines correspond to best algebraic fits: $C_\perp(x, 0) \sim x^{-0.84}$ and $C_\perp(0, y) \sim y^{-0.76}$. **g**, Giant number fluctuations. Variance, $\Delta N^2(\ell)$, of the number of particles measured in square regions of size ℓ. $\Delta N^2(\ell)$ is plotted as a function of the average number of particles $N(\ell)$ for five different polar active liquids of average area fractions $\rho_0 = 0.12, 0.18, 0.18, 0.24, 0.30, 0.39$, labelled by colours of increasing darkness. Solid lines: scaling $\Delta N^2(\ell) \sim N(\ell)$, corresponding to normal density fluctuations as in equilibrium fluids, and $\Delta N^2(\ell) \sim N^2(\ell)$, the scaling law predicted from linear hydrodynamic theory, see, for example, ref. [26]. Details about number fluctuation measurements and power-law fit values are provided in Supplementary Note 1.

Flocking order and the fluctuations around the ordered state



Active-fluid spectroscopy: Key parameters can be determined quantitatively

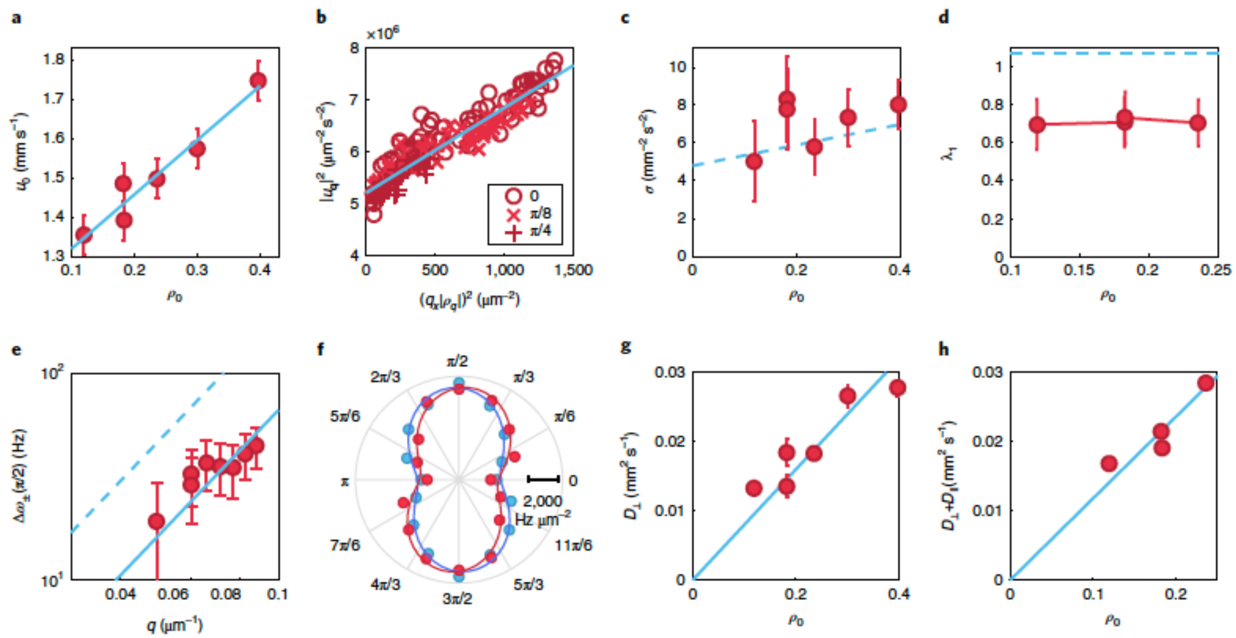


Fig. 3 | Active-fluid spectroscopy. a–h, The hydronamic description of the active fluid is inferred from the plots. In all panels, red dots represent experimental data, blue lines the best linear fit, and dashed lines the theoretical prediction with no free fitting parameter deduced from kinetic theory (see Supplementary Note 3). **a**, Variations of the mean-flow speed with the mean area fraction. Error bar: 100 μm s^{-1}, 1 standard deviation. Denser fluids flow faster. **b**, Parametric plot of the longitudinal velocity fluctuations $|u_q|^2$ varying linearly with $(q_x\ |\rho_q|^2)$ for three propagation angles. The slope gives a measure of $D' = 4\times10^{-6}$ mm^2 s^{-1}. The offset at $q_x = 0$ comes from the noise acting on the u mode (see Supplementary Note 3). **c**,**d**, The compressibility coefficient, σ, and advection coefficient, λ_1, are plotted versus the mean area fraction ρ_0. Both quantities are measured from the best fit of the speed of sound (Fig. 2g–i). The error bars are defined by applying the uncertainty-propagation formula on $\sigma = c_\pm(\pi/2)^2/\rho_0$ and $\lambda_1 = c_+(0)/c_-(0)$. The uncertainties on c and ρ_0 are respectively 100 μm s^{-1} and 0.02. **e**, Spectral width $\Delta\omega_\pm(\pi/2)$ of the modes propagating at $\theta = \pi/2$ plotted versus q (log–log plot). $\Delta\omega_\pm(\pi/2)$ grows quadratically with q. Error bars: 10 Hz, estimated by comparing several Lorentzian fits. Solid line: best quadratic fit. The bare prediction from the simplified kinetic theory overestimates $\Delta\omega_\pm(\pi/2)$ by a factor of three. The possible origins of this overestimate are discussed in Supplementary Note 3. **f**, Polar plot of the spectral width normalized by q^2 and averaged over all wavevectors $\Delta_\pm = \langle \Delta\omega_\pm(\theta)/q^2 \rangle_q$. Red (resp. blue) dots: experimental data corresponding to Δ_+ (resp. Δ_-). Solid lines: best fits using the relation $\Delta_\pm = (1/4)[-(D' + D_\perp + D_\parallel) - (D' + D_\parallel - D_\perp)\cos(2\theta) \pm D_\rho u_0\sqrt{\rho_0/\sigma}\sin(2\theta)]$ (see Supplementary Note 3). **g**, Variations of the elastic constant $D_\perp$ with ρ_0. $D_\perp$ is measured from the quadratic fit shown in **e** (see main text). Error bars defined as the 0.95 confidence interval of the quadratic fit in **e**. The elastic constant increases linearly with the particle density. **h**, Variations of the average elastic constant $D_\parallel + D_\perp$ with ρ_0. $D_\perp + D_\perp$ is measured from the quadratic fit of $\Delta\omega_+(\pi/4) + \Delta\omega_-(\pi/4)$ (see main text). Error bars defined as in **g**.

Concluding Remarks

In conclusion, two decades after the seminal predictions of Toner and Tu, we have experimentally demonstrated that the interplay between motility and soft orientational modes results in sound-wave propagation in colloidal active liquids. We have exploited this counterintuitive phenomenon to lay out a generic spectroscopic method which could give access to the material constants of all active materials undergoing spontaneous flows. Active-sound spectroscopy applies beyond synthetic active materials[32,33], and could be used to quantitatively describe large-scale flocks, schools, and swarms as continuous media[18–21].

--- Geyer et al

Hydrodynamic Theory works (Yeah!) and it provides a general framework to understand collective behaviors of active matter