Lecture 3: Localization & chaos in multi-tone driven systems.

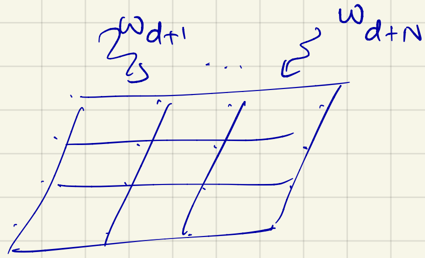
July 2023

Boulder summer school

Lecture 3 :

Reminder about freq lattice with spatial dimensions

Setup : d- dimensional tight-binding model driven by N incommensurate tones.



Looking for solutions to

$$i\frac{d|\psi\rangle}{dt} = H(\vec{\theta}_t)|\psi\rangle \quad — \quad \text{I}$$

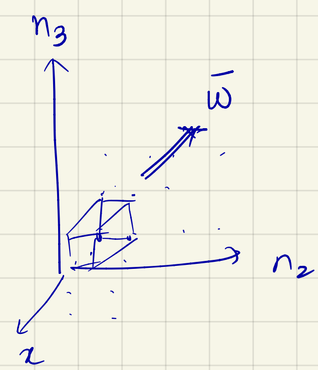
Generalized Floquet solutions

$$\text{II} \quad — \quad |\psi_\alpha(t)\rangle = e^{-i\varepsilon_\alpha t} \underbrace{|\phi_\alpha(\bar{\theta}_t)\rangle}_{}$$

smooth on $\bar{\theta}_t$ - torus (N-dimensional torus)

$\Downarrow$ Plug into I + Fourier transform

$$\tilde{K}|\tilde{\phi}_\alpha\rangle = \varepsilon_\alpha |\tilde{\phi}_\alpha\rangle$$



$$|\tilde{\phi}_\alpha\rangle = \sum_x \sum_{\bar{n}} |\tilde{\phi}_{\alpha x \bar{n}}\rangle |x\rangle |\bar{n}\rangle$$

$$|\tilde{\phi}_\alpha(\vec{\theta}_t)\rangle = \sum_x \sum_{\bar{n}} |\tilde{\phi}_{\alpha x \bar{n}}\rangle |x\rangle e^{-i\bar{n}\cdot\bar{\theta}_t}$$

II $\Rightarrow$ localization along frequency dimensions

Proof : II $\Rightarrow$ Fourier expansion of $|\phi_\alpha(\bar{\theta})\rangle$ exists

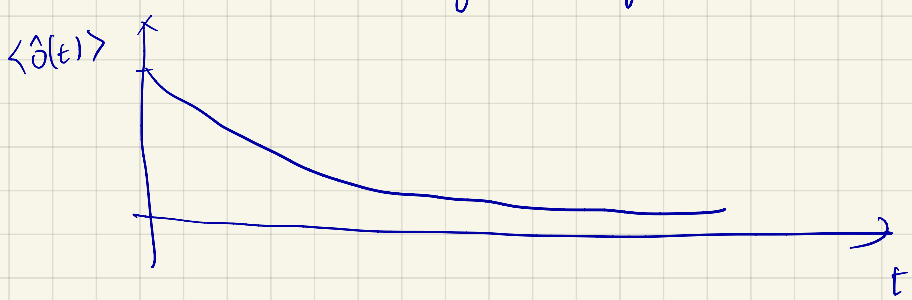
$\Rightarrow \sum_{\bar{n}} \| |\tilde{\phi}_{\alpha x \bar{n}}\rangle \|^2$ is finite.

$\Rightarrow |\tilde{\phi}_\alpha\rangle$ is localized along frequency dimensions.

Stable localization $\equiv$ localization along all dimensions of the frequency lattice.

Immediate dynamical consequence of stable localization on the freq. lattice
= quasi-periodic time dependence of local observables.
Mathematically the spectrum is pure-point.

In contrast, if the quasi-energy states are delocalized in frequency dimensions, the spectrum of a generic operator $\hat{O}$ is absolutely continuous.



" CHAOS " in multi-tone driven systems.

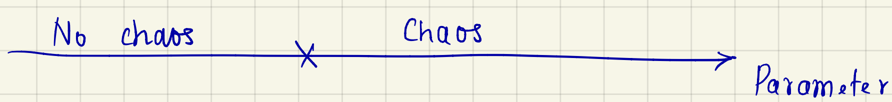
<u>No</u> generalized Floquet solutions for chaotic systems.

Punchlines of the rest of the lecture. $(N > 1)$

1)

	$\bar{\omega}_1$: localized	$\bar{\omega}_1$: delocalized
$\bar{x}$: localized	✓	✗
$\bar{x}$: delocalized	✗	✓

2) $D_{syn} = d + N \geq 3$; can have a transition to chaos.

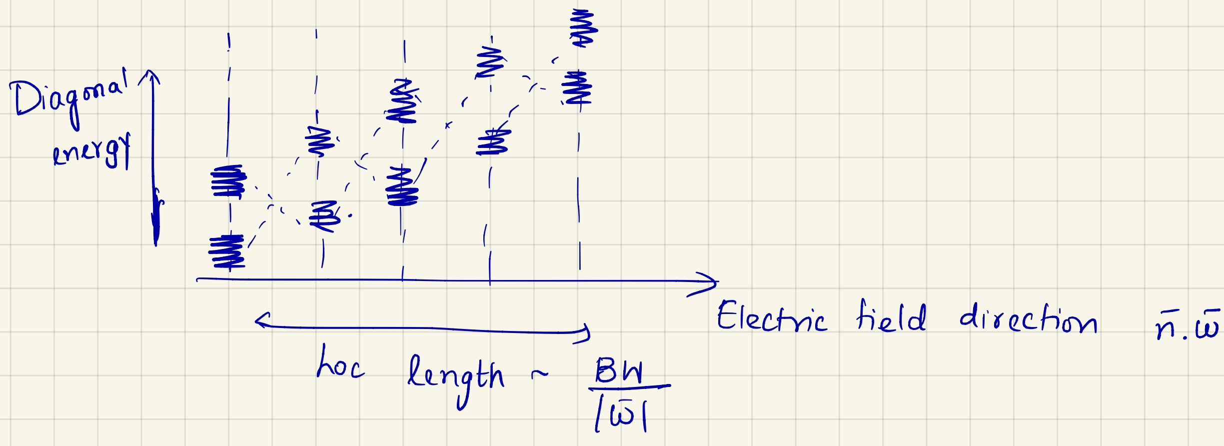


3) Stably localized hopping models are classified by integers when $D_{syn} = d + N =$ odd. This topological classification is stable to interactions for $d = 1$, $N = 2$ with sufficient inhomogeneity (due to many-body localization). Interacting driven phases of matter with no equilibrium analog!

Punchline 1 :

a) QE states cannot be localized along space-like dimensions & delocalized along frequency dimensions.

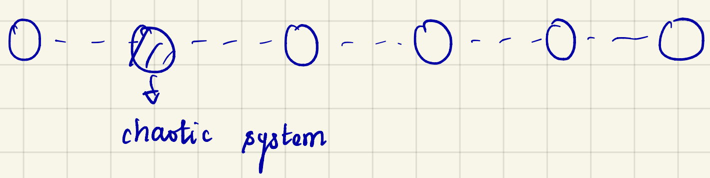
Comment : Systems are always localized $\parallel$el to the $\bar{E}$-field due to Stark localization. When we say delocalized along frequency dimensions, we mean the $(N-1)$ dimensions $\perp$r to the $\bar{E}$-field.



Let's assume that (a) is wrong. Then we could have a steady state array of chaotic systems (because the states are localized in space).

If we now perturb the system weakly, we expect the chaotic systems to become weakly connected.

Eg $d=1$



No matter the spatial inhomogeneity, such a system cannot be spatially localized. (In the language of lectures this week, the level spacing on each dot vanishes as $t\to\infty$ while the FGR decay rate to decay into the continuum of a dot a finite distance away stays finite).

We arrive at a contradiction.

Punchline 1 :

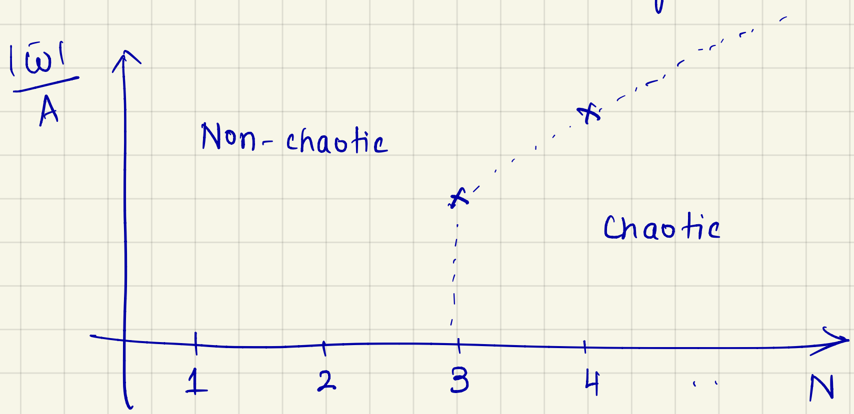
(b) Delocalized along spatial dimensions $\Rightarrow$ delocalization along freq dimensions.

Similar argument as in (a), but along lattice planes "most perpendicular" to $\bar{E}$-field.

Leave this as informal homework.

Comment : Can formulate an ETH ansatz for $(\text{deloc}_{\bar{x}} + \text{deloc}_{\bar{\omega}})$ on the frequency lattice. A bit subtle because the frequency lattice is formally infinite!

Punchline 2 : Let's stick to m-level systems for concreteness.

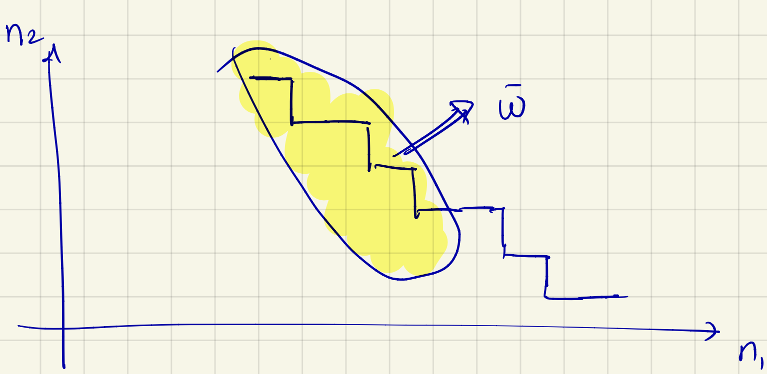


* Few-level systems can be chaotic!
* Need $N \geqslant 3$ tones
* Chaotic region grows with N.
* Always non-chaotic at large frequencies

Where do these results come from? By analyzing the inhomogeneous hopping problem $\perp$ to the $\bar{E}$-field on the freq. lattice.

N=1 : no dimensions $\perp$ to $\bar{E}$-field, no chaos (lecture 1).

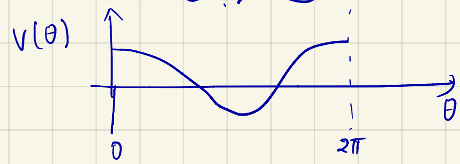
N=2



- Stark localization along $\bar{\omega}$
- Effective 1d quasi-periodic model $\perp$ to $\bar{\omega}$
- 1d model localized.

<u>Aside</u> : 1d quasi-periodic models need not be localized. Prominent example that is not

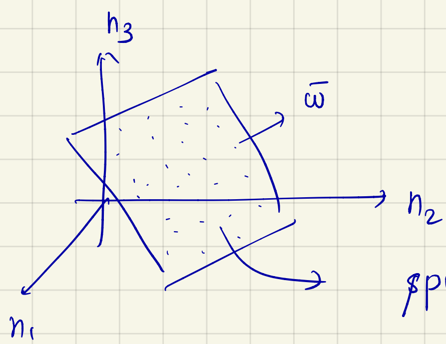
$$H_{AA} = -t \sum_i \left(c_i^+ c_{i+1} + h.c. \right) + \sum_i \underbrace{V \cos(\alpha i)} c_i^+ c_i$$



sample with period α.

In our model, $V(\theta)$ has a jump discontinuity. These models are generically localized.

<u>N=3</u> :



spin-orbit coupled + non-time-reversal invariant 2d model.

Should have a delocalized phase.

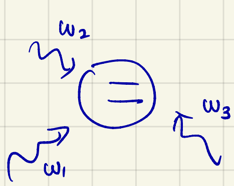
Numerically, even a 3-tone qubit appears to exhibit chaos.

<u>Comments</u>

1) We may expect that the transition is in the Anderson class. <u>Research topic</u>!

2) NISQ systems / NV-centers could see this transition. Finite observation time would function as finite size.

3) In kicked systems, this physics has been studied since the 80's. Experiments in cold atomic gases.

4) 

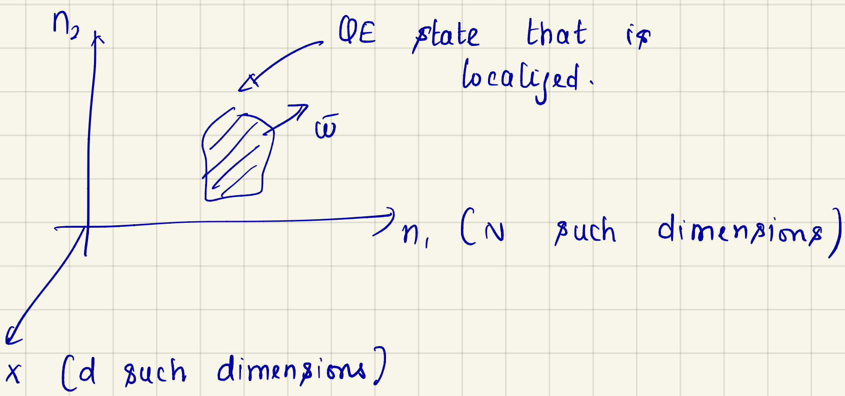
Interesting way to design baths with controllable Fourier spectra.

Useful in thermalization studies.

Punchline 3 :

Stably localized multi-tone driven systems are far from boring.

Classified by a winding number in $D_{syn} = d+N$ dimensions.



Localization along all dimensions $\Rightarrow$ character of dimension lost.

Classification only depends on $D_{syn} = d+N$

Formally, define micromotion operator

$$V(\vec{\Phi}, \vec{\theta}) = \sum_\alpha |\phi_\alpha(\vec{\theta})\rangle\langle\alpha|$$

$\hookrightarrow$ fixed basis

$\Downarrow$

d fluxes twisting the periodic boundary conditions of the spatial dimensions

Topological invariant that characterizes the map V.

$$W[V] \propto \int d^d\vec{\Phi}\, d^N\theta\, \epsilon^{j\dots k}\, \mathrm{Tr}\left[(V^\dagger \partial_j V)\dots(V^\dagger \partial_k V)\right]$$

$$W[V] \propto \begin{cases} \text{integer} & D_{syn} = d+N \text{ odd} \\ 0 & D_{syn} = d+N \text{ even.} \end{cases}$$

A close relationship to class A static classification; we'll come back to it if we have time.

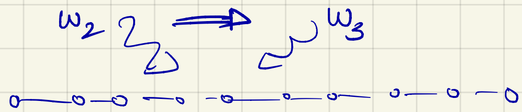
We dubbed the non-trivial phases as anomalous, localized, topological phases (ALTPs)

$\Downarrow$

no equilibrium counterpart

Classification physically interesting because it predicts quantized observables which depend on d.

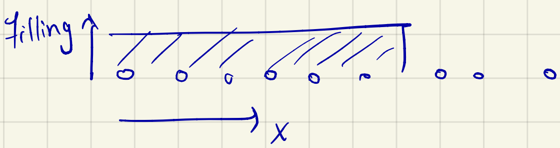
Eg :



Disordered wire

Edge observable

$$\overline{P}_{2\to 3} = \frac{\omega_2 \omega_3}{2\pi} W[v]$$



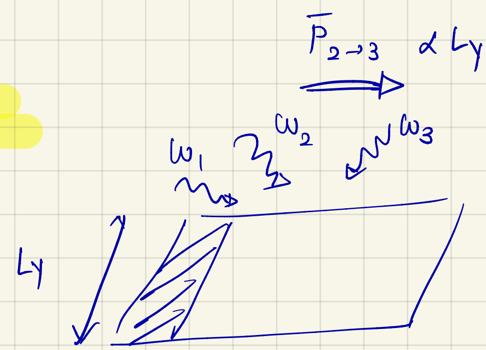
Quantized energy pump.

Bulk observable

$$\overline{\mathrm{Tr}[M]} = \frac{|\bar{\omega}|}{2\pi} W[v]$$

Quantized circulation in the bulk.

Another eg of an ALTP :



2d hopping model

The edges exhibit the 4d Hall effect on the frequency lattice.

What of interactions ?

1) Generically would give new channels to heat. Thermodynamically large systems an extensive number of quanta & heat

2) One way the heating can be extremely slow / stop :

through many-body localization.

We argued that (1+2) - dimensional ALTPs are stable to interactions.

↳ rare regions do not form locally chaotic blocks

↳ rare regions, although more potent than in the static case do not lead to avalanche thresholds for localization lengths below a threshold value.

↳ typical regions can be stable to the formation of long-ranged resonance for $\xi < \xi_{th}$.

3rd punchline : Interacting, disordered & driven wires can exhibit non-equilibrium orders in the steady state with MBL.

	Phase	Observable
$N=1$	Discrete time crystal	Period doubling starting from all initial states
$N=2$	ALTP (fermionic phase)	Quantized energy current mediated by edge states.

Interesting directions

1) In $d=1$, there are interacting $N=2$ phases with an emergent $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. Dumitrescu et al, Else et al have studied these; connection to classification here?

2) With symmetries?

3) There are 2d interacting classification of Floquet phases (Vishwanath, Fidkowski, ...). Are these extended in interesting ways with $N>1$?

4) Ongoing work: classification applies as is to Bloch systems. Edge physics: why is it stable?