Notes on Landau level energy spectrum graphene

Yesterday Nick Read reviewed the basics of Landau quantization for electrons described by the Schrödinger equation.

\[ H_{\text{Sch}} = \frac{\vec{p}^2}{2m^*} \quad \rightarrow \quad H_{\text{Sch}}^* = \frac{(\vec{p} + \frac{e}{c} \vec{A})^2}{2m^*} \]

**Solution:** harmonic oscillator \[ \sum_{n} \text{Landau level index} \]

**Energy spectrum:** \[ E_n = \hbar \omega_c (n + \frac{1}{2}) \]

where \[ \omega_c = \frac{|e| B}{m^* c} = \text{cyclotron frequency} \]

The Hamiltonian for massless Dirac electrons is:

\[ H_D = \mu \vec{p} \cdot \vec{\sigma} \quad \rightarrow \quad H_D^* = \mu \left[ \vec{p} + \frac{|e| B}{c} \vec{A} \right] \cdot \vec{\sigma} \]

where \[ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \quad \text{Pauli matrices} \]

For convenience let's call

\[ \vec{\pi} = \vec{p} + \frac{|e| B}{c} \vec{A} \quad \text{(Kinetic momentum)} \]

Then:

\[ H_D^* = \mu \vec{\pi} \cdot \vec{\sigma} \]

Note that, unlike \( \vec{p} \), the different components of \( \vec{\pi} \) do not commute with each other:
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\[ [\Pi_x, \Pi_y] = \left[ p_x + \frac{ie}{c} A_x, p_y + \frac{ie}{c} A_y \right] = \]
\[ = \frac{ie}{c} \left( [p_x, A_y] - [p_y, A_x] \right) \]
\[ \uparrow \]
\[ \text{Landau gauge } \vec{A} = (0, B \cdot x, 0) \]
\[ = -i \frac{e}{c} B = -i \frac{e}{c} \frac{1}{\ell_B^2} \quad ; \quad \ell_B = \frac{hc}{leB} \equiv \text{magnetic length} \]

Note: \([\Pi_x, \Pi_y]\) is gauge invariant (we just chose Landau gauge as a simple example) as expected, since \(\Pi_x\) and \(\Pi_y\) are gauge invariant (this is because \(\vec{p}\) also transforms under gauge transformations).

Now, recall ladder operators for Q.H.O.:
\[ \hat{a}_{H_0} = \frac{1}{\sqrt{2}} \left( \frac{x}{x_0} - i \frac{p}{p_0} \right); \quad \hat{a}_{H_0}^+ = \frac{1}{\sqrt{2}} \left( \frac{x}{x_0} + i \frac{p}{p_0} \right) \]
where \(x_0 = \sqrt{\frac{\hbar}{\mu m}}\) and \(p_0 = \sqrt{\hbar \mu m}\); \([\hat{a}, \hat{a}^+] = 1\).

Similarly we define the following ladder operators:
\[ \hat{a} = \frac{\ell_B}{\sqrt{2} \hbar} \left[ \Pi_x - i \Pi_y \right]; \quad \hat{a}^+ = \frac{\ell_B}{\sqrt{2} \hbar} \left[ \Pi_x + i \Pi_y \right] \]

It's easy to see that: \([\hat{a}, \hat{a}^+] = 1\)
\[ \Pi_x = \frac{\hbar}{\sqrt{2} \ell_B} (\hat{a}^+ + \hat{a}); \quad \Pi_y = \frac{\hbar}{\sqrt{2} \ell_B} (\hat{a}^+ - \hat{a}) \]
Note: the components of the gauge invariant momentum $\vec{p}$ are mutually conjugate, in the same manner as $x$ and $p_x$, or $y$ and $p_y$. That's why this new ladder operators are defined like that. Notice also that one cannot diagonalize at the same time $\Pi_x$ and $\Pi_y$ (since they do not commute).

The Dirac Hamiltonian at finite $B$ can be written in terms of $\hat{a}$ and $\hat{a}^+$:

$$H_D^B = U_F \vec{\pi} \cdot \vec{\tau} = U_F \begin{pmatrix} 0 & \Pi_x - i\Pi_y \\ \Pi_x + i\Pi_y & 0 \end{pmatrix} =$$

$$= \sqrt{2} \frac{\hbar U_F}{l_B} \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^+ & 0 \end{pmatrix} = \hbar \omega' \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^+ & 0 \end{pmatrix}$$

where $\omega' = \sqrt{2} \frac{U_F}{l_B} = \sqrt{2} \frac{U_F}{l_B} \sqrt{\frac{10eB}{\hbar c}}$ is analogous to the cyclotron frequency. [But note $\omega' \neq \frac{10eB}{m'c}$ because $m' = \frac{\hbar^2}{(2\pi\hbar)^2} = \frac{\hbar}{2\pi\hbar} = \frac{\hbar}{2\pi}$.]

$H_D^B$ is a $2 \times 2$ matrix, so the eigenstates are 2-component spinors:

$$\psi_n = \begin{pmatrix} U_n \\ J_n \end{pmatrix}; \quad H_D^B \psi_n = E_n \psi_n$$
This means we must solve the system of eqs:
\[ \hbar \omega' \hat{a}^\dagger \hat{a}^\dagger \hat{a} \psi_n = \varepsilon_n \hat{a} \psi_n \quad \land \quad \hbar \omega' \hat{a}^+ \psi_n = \varepsilon_n \psi_n \]  
(1)
(2)

Multiplying (1) by \( \hat{a}^+ \) (from left):
\[ \hbar \omega' \hat{a}^+ \hat{a} \psi_n = \varepsilon_n \hat{a}^+ \hat{a} \psi_n = \left( \frac{\varepsilon_n}{\hbar \omega'} \right)^2 \psi_n \Rightarrow \]

\[ \hat{a}^+ \hat{a} \psi_n = \left( \frac{\varepsilon_n}{\hbar \omega'} \right)^2 \psi_n \]

This means that \( \psi_n \) is an eigenstate of the number operator \( (\hat{a}^+ \hat{a} \psi_n) = n \psi_n \), i.e., up to a numerical factor, \( \psi_n \sim n \psi_n \)

Notice the spectrum:
\[ \varepsilon_n^2 = n \left( \frac{\hbar \omega'}{\hbar c} \right)^2 \Rightarrow \]

\[ \varepsilon_n = \pm \hbar \omega' \sqrt{n} = \pm \hbar v_F \sqrt{\frac{2 \varepsilon_n}{\hbar c}} n \cdot B' \]

Note the big difference in the energy spectrum:
\[ \varepsilon_n^\text{Sch} = \hbar \omega_c (n + \frac{1}{2}) = \frac{1}{c} \frac{1}{m^*} \beta (n + \frac{1}{2}) \]
\[ \varepsilon_n^\text{D} = \pm \hbar v_F \sqrt{\frac{2 \varepsilon_n}{\hbar c}} \sqrt{n} \cdot V_B \]
Consequences/differences:

* $E_n^D$ has a Landau level at zero energy whereas $E_n^{s.c.}$ doesn't → most crucial!

* $E_n^D$ has Landau levels both for positive & negative energies, while $E_n^{s.c.}$ only for positive energies (although if we would consider valence band, we would also have negative energy LL's for non-relativistic case).

* $E_n^D$ disperses as $\sqrt{nB}$, whereas $E_n^{s.c.}$ disperses as $(nB)$. This means $E_n^D$ is not a harmonic spectrum (LL's are not equidistant in energy).