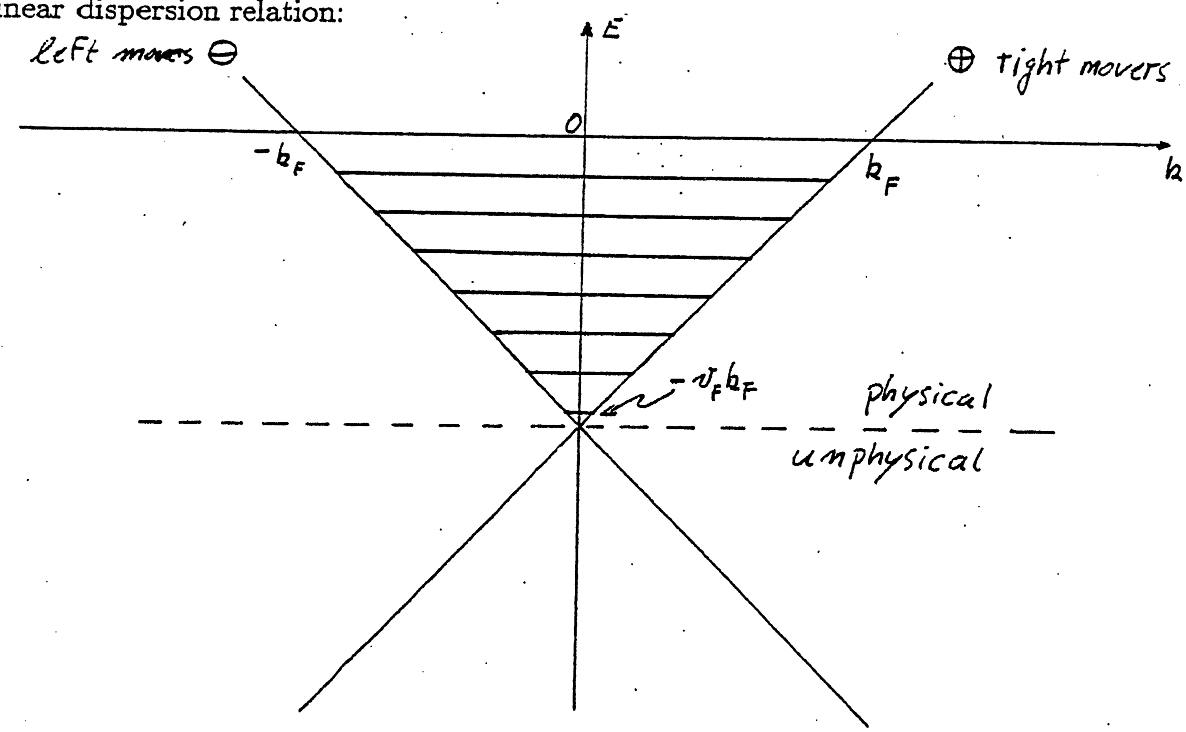
# The Luttinger Liquid Concept via Bosonization

**Abstract**

It is shown how the Hamiltonian of the 1-dimensional spinless Luttinger model can be reexpressed in terms of boson operators. Even when one allows for interactions, the resulting boson Hamiltonian will be that of a free boson field. An example is given that shows how bosonization can simplify calculations of correlation functions. The additional features appearing when one allows for spin such as spin-charge separation are stated.

## 1 The Luttinger Model

The Luttinger model describes electrons of a 1-dimensional system with the following linear dispersion relation:




The states of the lower triangle have been introduced for mathematical reasons as will be apparant later and do not correspond to physical states. Electrons on the right branch are called right movers and those on the left branch left movers. Notation:

- $\oplus$: right-movers $c^\dagger_{\oplus,k}$
- $\ominus$: left-movers $c^\dagger_{\ominus,k}$

These are the creation operators for the corresponing branches. In order to refer to right and left movers by a variable index the letter $r$ will be used. A value of 1 corrersponds to $\oplus$ and $-1$ corresponds to $\ominus$.

The corresponding continuous functions are defined as:

$$\psi^\dagger_\oplus(x) := \frac{1}{\sqrt{L}}\sum_k c^\dagger_{\oplus,k}e^{ikx} \tag{1}$$

$$\psi^\dagger_\ominus(x) := \frac{1}{\sqrt{L}}\sum_k c^\dagger_{\ominus,k}e^{ikx} \tag{2}$$

The ground state $|\phi_0\rangle$ is characterized by

$$\langle n_{r,k}\rangle_0 = \Theta(k_F - rk) \tag{3}$$

Normal ordering with respect to ground state:

$$:A :\equiv A - \langle A\rangle_0 \tag{4}$$

The free Luttinger Hamiltonian is then given by

$$H_0 = v_F : \sum_{k,r}(rk - k_F)(c^\dagger_{r,k}c_{r,k}) : \tag{5}$$

Equivalently

$$H_0 = v_F \int_0^L \mathrm{d}x \ : \psi^\dagger_\oplus(i\partial_x - k_F)\psi_\oplus - \psi^\dagger_\ominus(i\partial_x + k_F)\psi_\ominus : \tag{6}$$

We define the density operators:

$$\psi^\dagger_\oplus(x)\psi_\oplus(x) =: \rho_\oplus(x) =: \frac{1}{L}\sum_q \rho_{\oplus,q}e^{iqx} \tag{7}$$

$$\psi^\dagger_\ominus(x)\psi_\ominus(x) =: \rho_\ominus(x) =: \frac{1}{L}\sum_q \rho_{\ominus,q}e^{iqx} \tag{8}$$

The free Hamiltonian can also be expressed in terms of density operators:

$$H_0 = \frac{\pi v_F}{L} : \sum_q \{\rho_{\oplus,q}\rho_{\oplus,-q} + \rho_{\ominus,-q}\rho_{\ominus,q}\} : \tag{9}$$

The interaction part in the Luttinger model is given by

$$H_I = \frac{1}{2L} : \sum_{q\neq 0} 2g_2\rho_{\oplus,q}\rho_{\ominus,-q} + g_4(\rho_{\oplus,q}\rho_{\oplus,-q} + \rho_{\ominus,-q}\rho_{\ominus,q}) : \tag{10}$$

$g_2$ and $g_4$ are free paramters of the model.

## 2 Link to a lattice model

By means of numerous approximations one can map a 1-dimensional Hubbard model with nearest neighbour interaction onto a Luttinger model.

- one-dimensional lattice of length $L$, $M$ lattice sites,
- lattice point spacing $s$,
- periodic boundary conditions,
- half filling
- spinless fermions
- Hubbard-Hamiltonian with nearest neighbour hopping in the kinetic part,
- repulsive fermion-fermion interaction

$$H = H_0 + H_I \tag{11}$$

where

$$H_0 := -\varepsilon \sum_j \psi_j^\dagger \psi_{j+1} + \psi_{j+1}^\dagger \psi_j \tag{12}$$

$$H_I := \frac{U}{2} \sum_j n_j n_{j+1}, \qquad U > 0 \tag{13}$$

$\psi_i^\dagger$ is a fermionic creation operator that creates a fermion in a state whose wavefunction is localized around lattice point $i$; $\psi_i$ is it's corresponding annihilation operator. They satisfy the common anti-commutator relations:

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij} \tag{14}$$

$$\{\psi_i, \psi_j\} = 0 \tag{15}$$

One can think of the wavefunctions as *Wannier-functions*, which are defined as superpositions of the exact (non interacting fermions) Bloch-wavefunctions:

$$\Phi_k = e^{ikx} u_k(x), \qquad H_0 \Phi_k = \varepsilon(k) \Phi_k \quad \text{Bloch} \tag{16}$$

$$\omega(x - js) := \frac{1}{M} \sum_k e^{-ik(js)} \Phi_k(x) \quad \text{Wannier} \tag{17}$$

Where $k$ is of the form

$$k = \frac{2\pi}{L} n, \qquad n \in \mathbb{N} \tag{18}$$

Wannier-functions are localized around their corresponding lattice site.

In terms of the Fourier coefficients $c_k$ of $\psi$ definded by

$$\psi_j = \frac{1}{\sqrt{M}} \sum_k c_k e^{ikjs} \tag{19}$$

the kinetic Hamiltonian $H_0$ yields:

$$H_0 = -2\varepsilon \sum_k \cos(ks) c_k^\dagger c_k \tag{20}$$

Thus the Energy spectrum of the noninteracting fermions is given by

$$\varepsilon(k) = -2\varepsilon \cos(ks) \tag{21}$$

We assume half filling. Then the Fermimomentum $k_F$ yields

$$k_F = \frac{M\pi}{2L} = \frac{\pi}{2s} \tag{22}$$

Which lies half way from the 1. Brillouin zone Boundary.
For the fermi-velocity $v_F$ we obtain:

$$v_F = \left. \frac{\partial \varepsilon(k)}{\partial k} \right|_{k=k_F} = 2\varepsilon s \sin(k_F s) = 2\varepsilon s \tag{23}$$

In order to proceed we will perform a continuum limit based on the assumption:

$$\frac{s}{L} \to 0 \tag{24}$$

- lattice spacing $s$ is assumed to stay *finite*.
- continuum limit considers
  - macroscopic $L$
  - atomic $s$
- compared to $L$ the lattice points $ns$ can be regarded as values of a continuous variable $x$.

We are interested in the behaviour near $k_F$ and linearize the energy spectrum in the continuum limit around $\pm k_F$:

$+k_F$

$$\begin{aligned} \varepsilon(k) &= \varepsilon(k_F) + 2\varepsilon s \sin(k_F s)(k - k_F) + O(s^3(k-k_F)^3) \\ &\approx v_F(k - k_F) \end{aligned}$$

$-k_F$

$$\begin{aligned} \varepsilon(k) &= \varepsilon(-k_F) + 2s\sin(-k_F s)(k+k_F) + O(s^3(k+k_F)^3) \\ &\approx v_F(-k-k_F) \end{aligned}$$

We work now with two major simplifications: !

- Linear spectrum $v_F(\pm k - k_F)$ extended over the whole range of $k$.
- Addition of "unphysical" states with a spectrum that is
  - linear
  - unbounded below

These approximations restrict the validity of the model to

- *LOW TEMPERATURES*

We have arrived at the assumptions of the free Luttinger Hamiltonian.
The mapping of the interaction part is not shown here.

# 3 Bosonization

We first consider the free case without interaction.
In terms of the Fourier-components $c^\dagger_{r,k}, c_{r,k}$ of the fermi field the density operator is:

$$\rho_{r,q} = \begin{cases} \sum_k c^\dagger_{r,k+q} c_{r,k} & (q \neq 0) \\ :\sum_k n_{r,k}: & (q = 0) \end{cases} \tag{25}$$

The hermitian conjugate is given by:

$$\rho^\dagger_{r,q} = \rho_{r,-q} \tag{26}$$

as shows the following computation ($r$ omitted)

$$\begin{aligned} \rho^\dagger_p &= \left(\sum_k c^\dagger_{k+p} c_k\right)^\dagger \\ &= \sum_k c^\dagger_k c_{k+p} \\ &= \sum_k c^\dagger_{k-p} c_k \\ &= \rho_{-p} \end{aligned}$$

A straightforward calculation gives the following commutator

$$[\rho_{r,q}, \rho_{r',-q'}] = -\delta_{r,r'}\delta_{q,q'} r \frac{Lq}{2\pi} \tag{27}$$

However this result that can only be obtained if the unphysical states are present: Let's assume $q > 0$. Routine commutator manipulation leads to

$$[\rho_{\oplus,q}, \rho_{\oplus,-q'}] = \delta_{q,q'} \sum_k \langle n_{\oplus,k}\rangle_0 - \langle n_{\oplus,k-q}\rangle_0$$

If the number of occupied states were finite a shift in the summation index would turn the sum into 0.

Let us assume that there is a $k_0 < k_F$ with all levels below occupied. Above $k_0$ arbitrarily many particle-hole states may be excited:

$$\begin{aligned}
\sum_k \langle n_{\oplus,k}\rangle_0 - \langle n_{\oplus,k-q}\rangle_0 &= -\left(\sum_{k\geq k_0} + \sum_{k<k_0}\right)\left(\langle n_{\oplus,k-q}\rangle_0 - \langle n_{\oplus,k}\rangle_0\right) \\
&= -\sum_{k\geq k_0}\left(\langle n_{\oplus,k-q}\rangle_0 - \langle n_{\oplus,k}\rangle_0\right) \\
&= -\left(\sum_{k\geq k_0-q} \langle n_{\oplus,k}\rangle_0 - \sum_{k\geq k_0} \langle n_{\oplus,k}\rangle_0\right) \\
&= -\sum_{k_0-q\leq k<k_0} \langle n_{\oplus,k}\rangle_0 \\
&= -\frac{L}{2\pi} q
\end{aligned}$$

The following operators turn out to be bosonic:

$$b_q^\dagger := \sqrt{\frac{2\pi}{L\,|q|}} \sum_r \Theta(rq)\rho_{r,q} \tag{28}$$

$$b_q := \sqrt{\frac{2\pi}{L\,|q|}} \sum_r \Theta(rp)\rho_{r,-q} \tag{29}$$

Reversively:

$$\rho_{r,q} = \delta_{q,0} n_r + \sqrt{\frac{L\,|q|}{2\pi}} \{\Theta(rq) b_q^\dagger + \Theta(-rq) b_{-q}\} \tag{30}$$

The bosonic commutation relations are satisfied:

$$\left[b_q, b_{q'}^\dagger\right] = \delta_{q,q'} \qquad \left[b_q^\dagger, b_{q'}^\dagger\right] = 0 \tag{31}$$

Proof:

$$\begin{aligned}\left[b_q, b^\dagger_{q'}\right] &= \frac{2\pi}{L}\frac{1}{\sqrt{|q|\,|q'|}}\sum_{r,r'}\Theta(rq)\Theta(r'q')\left[\rho_{r,-q},\rho_{r',q'}\right] \\ &= \delta_{q,q'}\sum_r \frac{rq}{|q|}\Theta(rq) \\ &= \delta_{q,q'}\end{aligned}$$

$$\begin{aligned}\left[b^\dagger_q, b^\dagger_{q'}\right] &= \frac{2\pi}{L}\frac{1}{\sqrt{|q|\,|q'|}}\sum_{r,r'}\Theta(r,q)\Theta(r'q')\left[\rho_{r,q},\rho_{r',q'}\right] \\ &= \frac{-1}{|q|}\delta_{q,-q'}\sum_r rq\Theta(rq)\Theta(-rq) \\ &= 0\end{aligned}$$

- $b_q$ annihilates the ground state

$$\begin{aligned}b_q\,|\phi_0\rangle &= \sqrt{\frac{2\pi}{L\,|q|}}\sum_r \Theta(rq)\rho_{r,-q}\,|\phi_0\rangle \\ &= \sqrt{(*)}\sum_{r,k}\Theta(rq)c^\dagger_{r,k-q}c_{r,k}\,|\phi_0\rangle \\ &\overset{q\geq 0}{=} \sqrt{(*)}\sum_k c^\dagger_{\oplus,k-q}c_{\oplus,k}\,|\phi_0\rangle = 0 \\ &\overset{q\leq 0}{=} \sqrt{(*)}\sum_k c^\dagger_{\ominus,k+|q|}c_{\ominus,k}\,|\phi_0\rangle = 0\end{aligned}$$

- $b^\dagger_q$ creates exited states but does *not raise the fermion number.*

The commutator $[b_q, H_0]$ is given by

$$[b_q, H_0] = v_F\,|q|\,b_q \tag{32}$$

The commutator of the corresponding construction operator is now simply

$$\left[H_0, b^\dagger_q\right] = [b_q, H_0]^\dagger = |q|\,v_F b^\dagger_q \tag{33}$$

We thus have the algebra of harmonic oscillators with $\omega(q) = |q|\,v_F$ and consequently the free Hamiltonian $H_0$ on the space $\mathcal{H}_B$ spanned by the excited states with respect to $b^\dagger_q$ is given by:

$$: H_{0B} := v_F \sum_q |q| \, b_q^\dagger b_q \tag{34}$$

It can be shown, that $\mathcal{H}_B = \mathcal{H}$, so that the boson states form a complete set. Then $H_0 \equiv H_{0B}$ and the fermonic system can completley be represented in terms of bosonic states.

So far we have not gained any advantages by using boson operators but merely got acquainted with them. This will change when we introduce the interaction of the model. $H = H_0 + H_I$ still corresponds to a free, massless bosonic field.

The following fields are defined:

$$\phi(x) := -i\frac{\pi}{L}\sum_q \frac{1}{q} e^{-\alpha\frac{|q|}{2}} e^{-iqx}(\rho_{\oplus,q} + \rho_{\ominus,q}) \tag{35}$$

$$\theta(x) := i\frac{\pi}{L}\sum_q \frac{1}{q} e^{-\alpha\frac{|q|}{2}} e^{-iqx}(\rho_{\oplus,q} - \rho_{\ominus,q}) \tag{36}$$

Commutator:

$$[\phi(x), \theta(y)] = i\frac{\pi}{2}\mathrm{sign}(x-y) \tag{37}$$

The gradient of $\theta$:

$$\nabla\theta = \frac{\pi}{L}\sum_q e^{-\alpha\frac{|q|}{2}} e^{-iqx}(\rho_{\oplus,q} - \rho_{\ominus,q}) \tag{38}$$

Define the field conjugate to $\phi$:

$$\Pi := \frac{\nabla\theta}{\pi} \tag{39}$$

Commutator:

$$[\phi(x), \Pi(y)] = i\delta(x-y) \tag{40}$$

Define:

$$u := \frac{1}{2\pi}\sqrt{(2\pi v_F + g_4)^2 - g_2^2} \tag{41}$$

$$K := \sqrt{\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}} \tag{42}$$

The crucial steps in Bosonization are the following claims:

$$H =: \int_0^L \mathrm{d}x \left\{ \frac{\pi u K}{2}\Pi^2 + \frac{u}{2\pi K}(\partial_x\phi)^2 \right\} : \tag{43}$$

$$\psi_r(x) \propto e^{-irk_F x}\frac{1}{\sqrt{2\pi\alpha}} e^{i[\theta - r\phi]} \tag{44}$$

The expression for the Hamiltonian can be verified by a straightforward calculation just inserting the expressions for the fields $\phi$ and $\Pi$

The Hamiltonian has the form of an elastic string, which in it's standard form looks like:

$$H = \int \mathrm{d}x \left\{ \frac{1}{2\mu}\Pi^2 + \frac{\varepsilon}{2}(\partial_x \phi)^2 \right\} \tag{45}$$

In our case

$$\mu = \frac{1}{\pi u K}, \qquad \varepsilon = \frac{u}{\pi K} \tag{46}$$

The eigenfrequencies of such a string are given by

$$\omega_k = \sqrt{\frac{\varepsilon}{\mu}}\,|k|$$

which in our case evaluates to

$$\omega_k = u\,|k| = \frac{1}{2\pi}\sqrt{(2\pi v_F + g_4)^2 - g_2^2}\,|k| \tag{47}$$

We can imediatley compute the *Specific Heat* out of that result:

With

$$U = \sum_q \varepsilon(k)\frac{1}{e^{\beta\varepsilon(k)} - 1}, \quad \varepsilon(k) = u\,|k|, \quad \beta := \frac{1}{T} \tag{48}$$

$$\begin{aligned} C_V &= \frac{\partial U}{\partial T} = \frac{1}{T^2}\sum_q \varepsilon^2(k)\frac{e^{\beta\varepsilon(k)}}{\left(e^{\beta\varepsilon(k)} - 1\right)^2} \\ &= \frac{u^2}{4T^2}\sum_q \frac{q^2}{\sinh^2\left(\frac{\beta u q}{2}\right)} \\ &\approx \frac{u^2}{4T^2}\frac{L}{2\pi}\left(\frac{2}{\beta u}\right)^3 \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{x^2}{\sinh^2(x)} \\ &= \frac{L\pi}{3}\frac{1}{u}T \end{aligned}$$

$$C_V = \frac{L\pi}{3}\frac{1}{u}T \tag{49}$$

For free fermions $u$ must be replaced by $v_F$ an we get

$$\frac{C_V}{C_{V0}} = \frac{v_F}{u} \tag{50}$$

Since in our model $g_4 = g_2$ and $g_2 \ll v_F$ we can approximate:

$$\frac{C_V}{C_{V0}} \approx 1 - \frac{g_2}{2\pi v_F} \tag{51}$$

Since here $g_2 > 0$ the specific heat decreases.