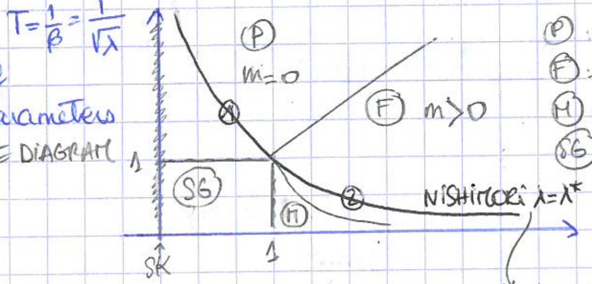
diagram of model parameters

PHASE DIAGRAM



Ⓟ: paramagnetic

Ⓕ: ferromagnetic

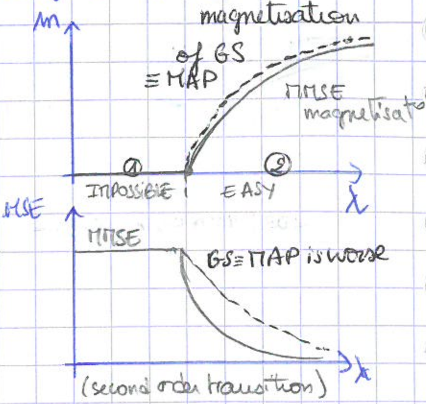
Ⓜ: mixed: m>0, full RSB

ⓈⒼ: spin glass; full RSB

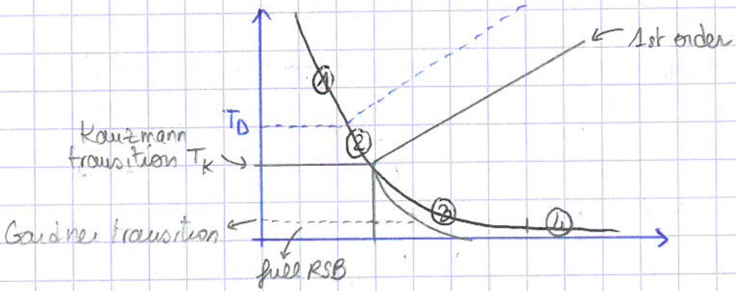
$J=\sqrt{\lambda^*}$

where the Nishimori relations hold.

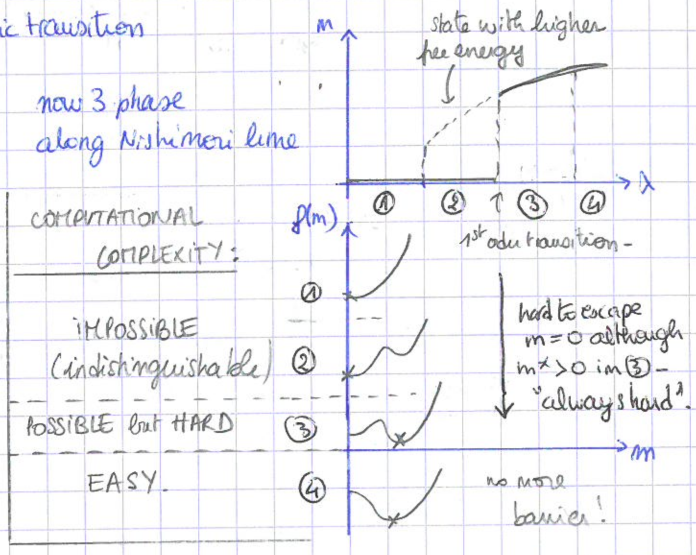
along the Nishimori line



Now in the case of a tensor of order 3:



now 3 phase along Nishimori line



↳ This two cases are representative of the different scenario: → 1st order with impossible → easy
↘ 2nd order with impossible + hard → easy.

We'll see how to make the necessary computations on the Nishimori line.

↳ REPLICA COMPUTATION ON THE EXERCISE SHEET.

$$\lim_{N\to+\infty}\frac{F}{N}=\underset{\substack{m\\ \downarrow\\ m_{eq}}}{\min}\left[\Psi_{den}(\lambda m,\lambda m)+\frac{\lambda m^2}{4}\right]$$

→ replicas lengthy (crazy as well...)
↘ rigorous proof

Lecture III Free energies and algorithms

III.1 Replica free energy

Recall the hamiltonian of the spike matrix model:

$$\mathcal{H}_{SM}=-\sum_{i\leq j}-\frac{\lambda}{2N}x_i^2x_j^2 \cdot \frac{\lambda}{N}x_ix_jx_i^*x_j^*+\sqrt{\frac{\lambda}{N}}x_ix_jW_{ij}$$

↳ exercise shows:

$$\begin{cases} f=\lim_{N\to+\infty}\frac{\mathbb{E}_{W,x^*}(-\log Z)}{N}=\min_m\underbrace{\left[\Psi_{den}(\lambda m,\lambda m)+\frac{\lambda m^2}{4}\right]}_{f_{RS}(m)} \\ m_{eq}=\underset{m}{\operatorname{argmin}}\,f_{RS}(m)\end{cases}$$

(1) for computation later.

In some cases, the crazy replica method can be actually rigourously proven. The main technique to do so is the Guerra interpolation.

In our problem we have no replica symmetry breaking → pretty simple setting.

↳ idea: interpolate between our problem and the simple model of 1-spin in a field.

simple → $\mathcal{H}_{DEN}=-\sum_{i=1}^{N}-\frac{\lambda x_i^2}{2}+x_ix_i^*\lambda+x_i\overset{\leftarrow\text{noise}}{\xi}\sqrt{\lambda}$ → independent spins $y_i=\sqrt{\lambda}x_i+\xi_i$.

Francesco guerra trick: introduce a time dependent snr λt for $\mathcal{H}_{SK}$
$\lambda m(1-t)$ for $\mathcal{H}_{DEN}$

$\mathcal{H} = \mathcal{H}_{SK}(\lambda t) + \mathcal{H}_{DEN}(\lambda m(1-t))$ $t=0 \rightarrow$ easy denoising problem
$t=1 \rightarrow$ pb we would like to solve

basic theorem of calculus = aka integration
↳ defining the time dependent free energy: $F(1) = F(0) + \int_0^1 dt \, \frac{\partial F(t)}{\partial t}$

$$\Rightarrow F_{SK} = N\varphi_{DEN}(\lambda m, \lambda m) + \int_0^1 dt \; \underset{w, x^*, \xi}{\mathbb{E}} \underbrace{\left\langle \frac{\partial \mathcal{H}}{\partial t} \right\rangle}_{} dt$$

$$\sum_{i<j}\left(\frac{\lambda}{N}\frac{\langle x_i^2 x_j^2\rangle}{2} - \frac{\lambda}{N}\langle x_i x_j x_i^* x_j^* \rangle - \sqrt{\frac{\lambda}{N}}\, w_{ij}\, \langle x_i x_j\rangle \frac{1}{2\sqrt{t}}\right) + \sum_i \left(\frac{-\lambda}{2}\langle x_i^2\rangle - \lambda m \langle x_i x_i^*\rangle + \frac{x_i \xi_i \sqrt{\lambda m}}{\sqrt{1-t}}\right)$$

gaussian noise → stein lemma $\mathbb{E}_w(w f(w)) = \mathbb{E}_w(f'(w))$

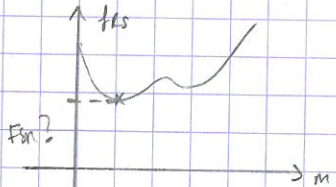
$$\mathbb{E}\left\langle\frac{\partial \mathcal{H}}{\partial t}\right\rangle = \mathbb{E}\left[-\frac{\lambda}{N}\sum_{i\leq j}\langle x_i x_j x_i^* x_j^*\rangle_t + \frac{\lambda}{2N}\sum_{i<j}\langle x_i x_j\rangle_t^2 + \sum_i \langle x_i x_i^*\rangle_t m\lambda - \sum_i \langle x_i\rangle_t^2 \frac{m\lambda}{2}\right]$$

↳ the explicit dependence on t has vanished, but recall that $\langle\,\rangle_t$.

$$\frac{F_{SK}}{N} = \varphi_{DEN}(\lambda m, \lambda m) + \int dt \, \frac{\lambda}{4}\,\mathbb{E}\left\langle\left(\frac{x^{(1)}x^{T(2)}}{N} - m\right)^2\right\rangle_t - \frac{\lambda}{2}\,\mathbb{E}\left[\left\langle\left(\frac{x\, x^{*T}}{N} - m\right)^2\right\rangle_t\right] - \frac{\lambda m^2}{4} + \frac{\lambda m^2}{2}$$

$$\overset{(1)}{=} f_{RS}(m) + \int dt \, \frac{\lambda}{4}\,\mathbb{E}\left[\left\langle\left(\frac{x^{(1)}x^{(2)}}{N} - m\right)^2\right\rangle_t\right] - \frac{\lambda}{2}\,\mathbb{E}\left[\left\langle\left(\frac{x x^{*T}}{N} - m\right)^2\right\rangle_t\right]$$

$$\underset{\text{Nishimori}}{=} f_{RS}(m) - \int dt \, \frac{\lambda}{4}\,\mathbb{E}\left[\left\langle\left(\frac{x x^*}{N} - m\right)^2\right\rangle\right]$$



⇒ which gives us a bound: $F_{SK}/N \leq f_{RS}(m) \quad \forall m \geq 0$.

To find the converse bound:
we consider $\mathcal{H}_{DEN}(q, m, \lambda, t) = -\sum_i -\frac{\lambda}{2}q(1-t)x_i^2 + x_i x_i^* \lambda m(1-t) + x_i \xi_i \sqrt{\lambda q(1-t)}$

$$\mathcal{H}_{DEN}^t + \mathcal{H}_{SK}^t \leftarrow \frac{F'(t=1)}{N} = \varphi_{DEN}(\lambda q, \lambda m) + \int dt \, \frac{\lambda}{4}\,\underset{(a)}{\mathbb{E}\left[\left\langle\left(\frac{x^{(1)}x^{(2)T}}{N} - q\right)^2\right\rangle\right]} - \frac{\lambda}{2}\,\underset{(b)}{\mathbb{E}\left[\left\langle\left(\frac{x x^{*T}}{N} - m\right)^2\right\rangle\right]} - \frac{\lambda q^2}{4} + \frac{\lambda m^2}{2}$$

↳ consider the restricted free at fixed value of overlap $M = \frac{x^T x^*}{N}$: $F(M) = -\mathbb{E}\log Z(M)$
with $Z(M) = \sum_{x/M} e^{-\mathcal{H}}$ $\leq \underset{\uparrow}{Z}$
all config.

$$↳ \frac{F}{N}[M=m] = \varphi_{DEN}(\lambda q, \lambda m) + \overset{(a)}{\geq 0} + \overset{(b)}{0} - \frac{\lambda q^2}{4} + \frac{\lambda m^2}{2}$$

$$\Rightarrow f(M=m) \geq f_{RS}(m,q) \quad \forall q \underset{m=q}{\Rightarrow} f(M=m) \geq f_{RS}(m)$$

↓
now, what is the true free energy?
$f(m_{eq})$ / $m_{eq} = \underset{m}{\mathrm{argmin}}\, f(m)$!

must coincide at the minimum given the inequalities derived.

$$\rightarrow \quad \boxed{f = \min_m f_{RS}(m)}$$

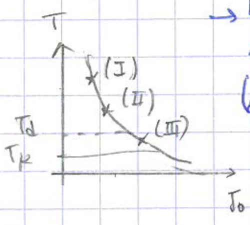
$f(M)$ = fixed overlap with truth free energy
≡ $f(q)$ = fixed overlap with equilibrium configuration
≡ FRANZ PARISI POTENTIAL.

Rk: We can use this same method for the p spin: $f_{RS}(m) = \varphi_{DEN}(\lambda m, \lambda m) + \frac{\lambda m^p}{2p}(p-1)$
as long as p is even.

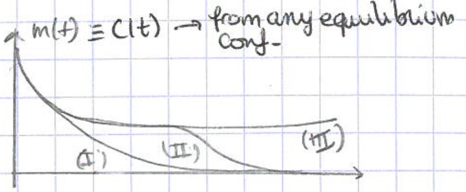
looking back at the phase diagram ≡ spin-tensor factorization.

- For $T > T_K$, on the Nishimori line, paramagnetic phase → indistinguishable from random problem, same as if $J_0 = 0$ ≡ unplanted

→ but if you planted the problem, you knew one equilibrium configuration. which is of great help in many circumstances.



Monte Carlo simulation from large magnetizations. on the Nishimori line
↓
≡ simulation from any equilibrium conf, which would be very costly to obtain otherwise at low Temperature, especially $T < T_d$ → GREAT TRICK !



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III 2 Belief propagation, TAP free energy

We have this situation

to find the solution we can run monte carlo but alternative, faster algorithms → Belief Propagation

Belief propagation was rediscovered many times, but originally Bethe-Peierls in the 30s.

The idea is the following, (see exercise sheet):
hamiltonian: $H = -\sum_{i<j} J_{ij} S_i S_j$

1. Write recursive BP "tree with high connectivity":
$m_{i\to j} = f(\{m_{k\to i}\}) = \tanh\left(\sum_{k\in\partial i\setminus j} \operatorname{atanh}(\tanh(\beta J_{ik})\, m_{k\to i})\right)$

2. Expand in J (small J)

large connectivity assumption, marginals ← 3. Close the equations on the marginals ≡ magnetizations
$m_i = \tanh\left(\beta \sum_j J_{ij} m_j - \beta^2 \sum J_{ij}^2 (1-m_j^2)\, m_i^{t-1}\right)$
↳ time indexing is crucial for convergence !!

This algorithm can be analyzed: it is basically doing a gradient descent along the replica symmetric potential → TAP magnetization get to (local) minima of the RS free energy.
↳ still stuck for the hard phase.
↳ finds optimal solution in easy phase.

Rk: Belief in stat mech community that message passing algorithm are somewhat optimal: there are not polynomial algorithm that can succeed in the hard phase.

Lecture IV: GENERALISED LINEAR MODELS

The situation is still signal recovery from noisy observations.

signal $X^* \in \mathbb{R}^N$ distributed $P_x(x)$ → observation in $\mathbb{R}^M$ $Y = \overset{\text{KNOWN}}{f}[\overset{\text{measurement matrix } \mathbb{R}^{M\times N} \text{ KNOWN!}}{\Phi} X^*]$

component-wise! $y_\mu = f[\Phi_{\mu}. X^*]$

$$\begin{cases} \alpha = M/N \\ \rho = \text{non zeros in } X^*/N \\ \Phi_{i\mu} \sim \mathcal{N}(0, 1/N) \text{ iid.} \end{cases}$$

different situations for different f: GENERALISED LINEAR MODELS !

1) $f(x) = x$ → linear model → statphys community Tanaka 2000 something
2) stochastic version: $f(x) = x + \xi$ ↳ $\mathcal{N}(0,\Delta)$
3) non-linearities: $f(x) = \text{sign}(x+\xi)$ - perceptron. classification problem