Lectures on Bosonization

C.L. Kane
Dept. of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104

This document contains lecture notes from my first two blackboard lectures given at the Boulder summer school in July 2005. The second two lectures are available as power point presentations. These lectures are meant to be a pedagogical introduction to interacting electrons in one dimension, with an emphasis on the technique of bosonization and on the Luttinger liquid.

I. A HEURISTIC INTRODUCTION TO BOSONIZATION

In this lecture we describe the 1D electron gas by examining the extreme limits of strong and weak interactions in order to develop a intuitive picture of the meaning of bosonization.

A. Non Interacting Electrons

Consider non interacting spinless electrons in one dimension, described by the Hamiltonian,

$$\mathcal{H} = \int dx \psi^\dagger(x) \left[-\frac{\hbar^2}{2m} \partial_x^2 \right] \psi(x) \tag{1}$$

$$= \sum_k \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k. \tag{2}$$

In the ground state the electronic states are filled to the Fermi energy E_F, or equivalently the Fermi momentum k_F, as shown in Fig. 1a. In terms of the 1D electron density n_0 we have

$$k_F = \pi n_0. \tag{3}$$

The velocity of the states at the Fermi energy $v_F = (1/\hbar) dE/dk(k_F)$is given by

$$v_F = \hbar k_F / m = \pi \hbar n_0 / m. \tag{4}$$

The density of states at the Fermi energy (or the compressibility) is

$$\partial n / \partial \mu = N(E_F) = 2/(2\pi \hbar v_F). \tag{5}$$

We will be interested in the low energy, long wavelength properties of this system. Consider the spectrum of particle hole excitations arising from states of the form, $c_{k+q}^\dagger c_k |0\rangle$. As shown in Fig. 1b, for small q, ω the particle hole spectrum resembles a sound mode with dispersion $\omega = v_F q$. This says that the low energy excitations of the electron gas are similar to that of a 1D elastic medium. This is also the case even in the presence of interactions. To see this, we will now consider the opposite extreme of strongly interacting electrons.

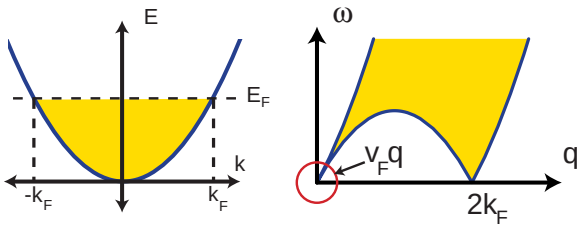

FIG. 1: Non interacting electrons in one dimension. (a) Dispersion, (b) Spectrum of particle-hole excitations.

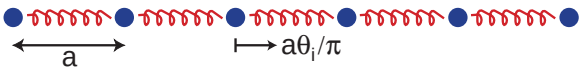

FIG. 2: A Wigner crystal and the phonon displacement coordinate θ_i.

B. Strong Repulsive Interactions: the 1D Wigner Crystal

Non interacting electrons are trivial because all of the energy is kinetic. In a real system, there is a competition between the kinetic energy and the potential energy of interaction. Here we will consider the opposite limit where the potential energy dominates. In this limit, the lowest energy configuration of the electrons will be a crystal, with "lattice constant" $a = 1/n_0$, as shown in Fig. 2. The low energy excitations are then *phonons*. To describe the phonons we introduce a phonon displacement by writing

$$r_i = r_i^0 + \frac{a}{\pi} \theta_i. \tag{6}$$

With this definition, the displacement variable θ_i advances by π when the crystal is displaced by one lattice constant. To describe the phonons, it is useful to construct the Lagrangian:

$$\mathcal{L} = KE - PE, \tag{7}$$

where the kinetic energy is

$$KE = \sum_i \frac{1}{2} m \dot{r}_i^2 = \int dx \frac{ma}{2\pi^2} \dot{\theta}(x)^2 \tag{8}$$

where we assume that for slow variations θ may be treated as a continuous variable. The potential energy

is given by

$$PE = \int dx dx' \frac{1}{2} V(x-x') \delta n(x) \delta n(x') = \int dx \frac{V_0}{2} \delta n(x)^2 \tag{9}$$

where for the second equality we have assumed a short ranged interaction, with$V_0 = \int dx V(x)$. For a coulomb interaction $V(x) = e^2/x$ screened at large distances by a ground plane at distance R_s, we have $V_0 = 2e^2 \log R_s/a$. Note, that here $\delta n(x)$ is the *deviation* of the density from it's average value n_0. δn can be expressed in terms of θ. To see how, note that if $\theta(L) - \theta(0) = -\pi$, then exactly one extra electron resides between 0 and L. Thus,

$$\delta n(x) = -\frac{\partial_x \theta(x)}{\pi} \tag{10}$$

Combining the two terms we now have

$$\mathcal{L} = \int dx \left[\frac{ma}{2\pi^2} (\partial_t \theta)^2 - \frac{V_0}{2\pi^2} (\partial_x \theta)^2 \right]. \tag{11}$$

This may be rewritten as

$$\mathcal{L} = \frac{\hbar}{2\pi g} \int dx \left[\frac{1}{v_\rho} (\partial_t \theta)^2 - v_\rho (\partial_x \theta)^2 \right], \tag{12}$$

where we have introduced the interaction parameter

$$g = \sqrt{\frac{\pi \hbar v_F}{V_0}} \tag{13}$$

and the phonon velocity

$$v_\rho = \sqrt{\frac{V_0}{ma}} = \frac{v_F}{g}. \tag{14}$$

This Wigner crystal approach to the problem should be valid in the strong interaction limit, $g \ll 1$. However, note the similarity of the low energy elastic theory for the phonons of a Wigner crystal with the "sound mode" for non interacting electrons. We shall see that Eq. 12 remains correct for weaker interactions and even for non interacting electrons. Eq. 13, however, will need to be modified for weaker interactions.

It is known that in low dimensions long range order is destroyed by fluctuations (either thermal or quantum). It is instructive to see how quantum fluctuations destroy the long range crystaline order at zero temperature. To describe the crystaline order, consider a Fourier decomposition of the electron density,

$$n(x) \sim \sum_{q \sim 0} n_q e^{iqx} + \sum_{q \sim 2k_F} n_q e^{iqx} + ... \tag{15}$$

The first term gives the long wavelength fluctuations of the density:

$$\sum_{q \sim 0} n_q e^{iqx} = n_0 - \frac{\partial_x \theta(x)}{\pi}. \tag{16}$$

The second term describes fluctuations at the wavelength of the Wigner crystal $a = 1/n_0 = 2\pi/2k_F$, which may be characterized by a slowly varying complex number $n_{2k_F}(x)$ which gives the amplitude and the phase of the oscillation at $2k_F$,

$$\sum_{q \sim 2k_F} n_q e^{iqx} = n_{2k_F}(x) e^{2ik_F x} + c.c. \tag{17}$$

Since the phase of the $2k_F$ density fluctuations advances by 2π when θ increases by π we have

$$n_{2k_F}(x) \sim e^{2i\theta(x)}. \tag{18}$$

For our present purposes, the constant of proportionality is unimportant.

For a perfect crystal

$$\langle n_{2k_F} \rangle \neq 0, \tag{19}$$

or equivalently

$$\lim_{x \to \infty} \langle n_{2k_F}(x) n_{-2k_F}(0) \rangle \neq 0. \tag{20}$$

This leads to a delta function Bragg peak for scattering. For a classical crystal, thermal fluctuations of the phonons give rise to the Debye Waller factor, which reduced the amplitude of the Bragg peak. We will now see that quantum fluctuations are similar, and give rise to a logarithmically divergent Debye Waller factor, which (barely) destroys the crystaline order.

There are many techniques for computing (19) and (20). A powerful method is to use the imaginary time path integral. This is done by writing the partition function as an integral over all trajectories $\theta(x,\tau)$, where $\tau = it$ is imaginary time. For zero temperature (which we will focus on here) τ runs from $-\infty$ to ∞. We thus write,

$$Z = \int D[\theta(x,\tau)] e^{-S[\theta(x,\tau)]/\hbar} \tag{21}$$

where the action S is given by

$$S = \frac{1}{\hbar} \int d\tau \mathcal{L}[\theta(x,\tau)] = \frac{1}{2\pi g} \int dx d\tau \left[\frac{1}{v_\rho} (\partial_\tau \theta)^2 + v_\rho (\partial_x \theta)^2 \right]. \tag{22}$$

The action is particularly simple in Fourier transform because the Fourier components decouple,

$$S = \frac{1}{2\pi g} \sum_{q,\omega} \left(\frac{\omega^2}{v_\rho} + v_\rho q^2 \right) |\theta(q,\omega)|^2. \tag{23}$$

It is now straightforward to compute (19) and (20). First, we have

$$\langle n_{2k_F}(x, \tau = 0) \rangle \sim \langle e^{2i\theta(0,0)} \rangle = \frac{1}{Z} \int D[\theta] e^{2i\theta} e^{-S}. \tag{24}$$

This can be evaluated by doing a simple Gaussian integral over $\theta(q,\omega)$. However, there is a very useful trick for evaluating such expectation values, which is to write

$$\langle e^{i(2\theta)} \rangle = e^{-\frac{1}{2}\langle (2\theta)^2 \rangle}. \tag{25}$$

This can be easily remembered by expanding to second order in θ and noting that the term linear in θ has zero expectation value. For a harmonic theory (quadratic in θ), this relation is exact.

The expectation value in the exponent can then be found by noting that for a Gaussian $\exp[-Cx^2/2]$ we have $\langle x^2\rangle = 1/C$. Thus,

$$\langle\theta(x=\tau=0)^2\rangle = \sum_{q,\omega}\langle|\theta(q,\omega)|^2\rangle = \int \frac{dqd\omega}{(2\pi)^2}\frac{\pi g v_\rho}{\omega^2+v_\rho^2 q^2}. \tag{26}$$

Changing variables to $(q_0, q_1) = (\omega/v_\rho, q)$ the integral becomes isotropic and may be written

$$\int_0^\infty \frac{2\pi q dq}{(2\pi)^2}\frac{\pi g}{q^2} = \frac{g}{2}\log\frac{L}{a}. \tag{27}$$

Thus we see that the Debye Waller factor due to quantum fluctuations in one dimension is log divergent. We have cut off the divergence with the system size L and the (electron) lattice constant a. We thus conclude that

$$\langle n_{2k_F}\rangle \sim e^{-2\langle\theta^2\rangle} \sim (a/L)^g \to 0 \tag{28}$$

A similar calculation for the correlation function (20) gives

$$\langle n_{2k_F}(x) n_{-2k_F}(0)\rangle \sim e^{-\frac{1}{2}\langle(2\theta(x)-2\theta(0))^2\rangle} \sim (a/x)^{2g} \tag{29}$$

Thus, the crystaline correlations decay at long distances, but only as a power law. For strong interactions $g \ll 1$ the exponent is very close to zero, so that the system is "almost" a Wigner crystal.

It is instructive to compare this result with the corresponding quantity for non interacting electrons. In that case it is straightforward to show that

$$\langle n_{2k_F}(x) n_{-2k_F}(0)\rangle \sim 1/x^2. \tag{30}$$

In this case the long range correlations are a consequence of Fourier transforming the occupation number $n(k)$ which has a sharp step at k_F. This leads to the familiar Friedel oscillations. These oscillations can also be seen in the pair correlation function

$$\langle\psi^\dagger(x)\psi^\dagger(0)\psi(0)\psi(x)\rangle = n_0^2(1-\sin^2 k_F x/x^2) \tag{31}$$

shown in Fig. 3, which describes the probability of finding two electrons separated by a distance x. The power law decay of the oscillations suggests that non interacting electrons are also described by (12) with $g = 1$. (12) is indeed more general than the Wigner crystal limit, however, our formula (13) for g will have to be modified. For a Wigner crystal $g \ll 1$, the peaks at $x = na$ in the pair correlation function have a similar power law decay, but with a smaller exponent.

Before describing a more rigorous approach to bosonization, we will desribe another simple limit which will contribute to our intuitive understanding of bosonization.

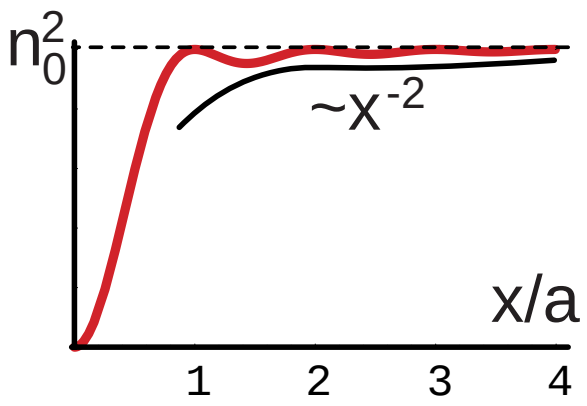

FIG. 3: Pair correlation function for non interacting electrons. The oscillations indicate power law crystaline order.

C. Bosons: The 1D superfluid

In our discussion of the Wigner crystal limit we made no use of the fact that electrons are fermions. Since in that limit the electrons never exchange the statistics are in fact irrelevant. Let us now suppose, for the moment, that we have bosons instead of fermions. Then, in addition to the Wigner crystal limit, one may consider the limit of Bose condensation.

For a Bose condensate, the ground state boson creation operator acquires a finite expectation value,

$$\langle b^\dagger\rangle = \sqrt{n}e^{i\phi} \tag{32}$$

Long wavelength fluctuations of the phase of the bose condensate are described by

$$\mathcal{L} = \frac{g}{2\pi}\int dx\left[\frac{1}{v_\rho}(\partial_t\varphi)^2 - v_\rho(\partial_x\varphi)^2\right]. \tag{33}$$

This describes similar elastic distortions to Eq. 12, which propagate at velocity v_ρ. We will see later that the g in Eq. 33 is in fact the same as the g in (12), so that the two Lagrangians are *dual* to each other.

From (33) it can immediately be seen that quantum fluctuations in φ (barely) destroy the superfluid order,

$$\langle b^\dagger(x) b(0)\rangle \sim 1/x^{1/2g}. \tag{34}$$

Thus, there is power law superfluid order. In the Wigner crystal limit $g << 1$ the phase fluctuations are very strong and the superfluid correlations die away quickly. On the other hand in the opposite limit $g \to \infty$ the phase correlations are nearly long range, but the crystaline correlations decay quickly.

The duality between θ and φ can further be explored by noting that the boson creation operator increases the density by one unit, so that $[n(x), \sqrt{n}\exp i\varphi(x')] = \delta(x-x')$. This implies the canonical commutation relation between number and phase:

$$[n(x), \varphi(x')] = i\delta(x-x') \tag{35}$$

If we write $n = n_0 + \partial_x\theta/\pi$, then this may be rewritten

$$[\partial_x\theta(x)/\pi, \varphi(x')] = i\delta(x - x'). \tag{36}$$

In this sense, θ and φ are dual variables.

The above analysis may be related to fermions by performing a Jordan Wigner transformation, which converts bosons to fermions in 1 dimension. We thus write a fermion creation operator $\psi^\dagger$ as

$$\psi^\dagger(x) = e^{\pm i\pi \int_{-\infty}^{x} n(x)} b^\dagger(x). \tag{37}$$

The term with the exponent is the "Jordan Wigner string" which counts the number of particles N_L to the *left* of x and mulitiplies the operator by $(-1)^{N_L}$. It is clear that this converts the boson commutation relation among $b(x)$ into fermion anticommutation relations for $\psi(x)$.

Noting that $n(x) \sim n_0 + \partial_x\theta/\pi$ we may write the *fermion* creation operator as

$$\psi^\dagger(x) \sim e^{i(\pm\theta(x)+\varphi(x))}. \tag{38}$$

The fermion operator thus includes both the dual "superfluid" and "crystaline" variables.

Equations (12, 33, 38), which we have "derived" by following our intuition are the hallmarks of bosonization and the Luttinger liquid. In order develop a (slightly) more rigorous understanding, which will correctly predict g and allow for generalizations to include spin and other degrees of freedom it will be necessary to return to non interacting electrons, where the exact bosonization transformations can be cleanly formulated.

II. BOSONIZATION OF NON INTERACTING ELECTRONS

In this lecture we return to non interacting electrons and describe the exact bosonization mapping for chiral fermions.

A. The Luttinger Model

Now we return to non interacting electrons and focus on the low energy excitations. For this purpose we linearize the dispersion (1) about the Fermi energy E_F, which we define to be 0.

$$\mathcal{H} = \sum_q v_F q \left[c^\dagger_{k_F+q} c_{k_F+q} - c^\dagger_{-k_F+q} c_{-k_F+q} \right]. \tag{39}$$

Defining continuum electron operators

$$\psi_{R,L}(x) = \sum_q e^{iqx} c_{\pm k_F+q} \tag{40}$$

the Hamiltonian density becomes

$$\mathcal{H} = -iv_F \left[\psi^\dagger_R \partial_x \psi_R - \psi^\dagger_L \partial_x \psi_L \right] \tag{41}$$

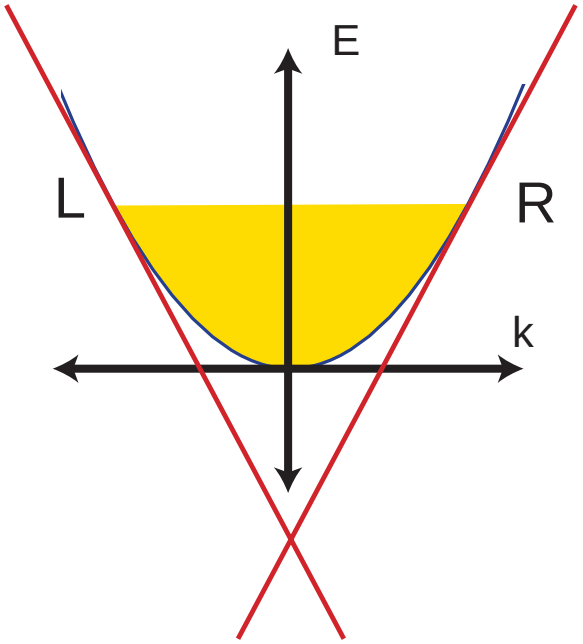

FIG. 4: The Luttinger model, with linear bands extending to $-\infty$.

This Hamiltonian is appropriate for describing low energy ($E \ll E_F$) and long wavelength $q \ll 1/a$ properties, which are independent of the states deep below the Fermi level.

It is useful to consider a model for which the linear representation (39) is exact. This is known as the Luttinger model, and as described in Fig. 4, the linear dispersion extends between energies $\pm\infty$. Since the states deep below the Fermi level are inert, the Luttinger model is expected to have the same low energy behavior as the real electron gas. The bosonization tranformation is *exact* for the Luttinger model.

The Hamiltonian (41) is the 1+1 dimensional analog of the massless Dirac equation, familiar from particle physics. As in the 3+1 dimensional case, (41) exhibits *chiral symmetry*, which is manifested by the independent conservation of the number of right and left moving electrons, N_R, N_L, given by

$$N_a = \int dx \psi^\dagger_a \psi_a \tag{42}$$

Thus in addition to the total charge $N_R + N_L$, the "chiral" charge $N_R - N_L$ is also conserved.

B. Bosonization of Chiral Fermions

We will now consider a single branch of chiral fermions ψ_R (or equivalently ψ_L. This will describe "half" of the one dimensional electron gas. In addition to providing a building block with which one can construct the one dimensional electron gas (also allowing for generalizations

to include spin, etc.) chiral fermions are directly relevant to edge states in the quantum Hall effect.

We begin by defining the chiral density operator,

$$n_R(x) =: \psi_R^\dagger(x)\psi_R(x) : . \tag{43}$$

Since the electronic states in the Luttinger model extend to energy $-\infty$, the number of electrons is infinite. Therefore, it only makes sense to talk about *deviations* in the density from the density of the ground state. This is the purpose of the colons, which indicate that the operator is to be *normal ordered*, such that the operators which annihilate the ground state are always placed on the right side. Equivalently, normal ordering subtracts off the (infinite) vacuum expectation value,

$$: \psi_R^\dagger(x)\psi_R(x) := \psi_R^\dagger(x)\psi_R(x) - \langle\psi_R^\dagger(x)\psi_R(x)\rangle_0. \tag{44}$$

It is useful to consider the Fourier transformed density operator,

$$n_{Rq} = \int dx e^{-iqx} n_R(x) = \sum_k : c_{Rk+q}^\dagger c_{Rk} : \tag{45}$$

where for $q \neq 0$ the normal ordering is unimportant. Where necessary, we will assume below that we are in a system of length L with periodic boundary conditions, so that the discrete values of q are $q_n = 2\pi n/L$.

A fundamental relation obeyed by the chiral density operator is known as the Kac Moody commutation relation

$$[n_{R-q}, n_{Rq}] = \frac{qL}{2\pi}. \tag{46}$$

This can be derived by using (45) and the fermion commutation relations to write the commutator as $\sum_k c_{Rk}^\dagger c_{Rk} - c_{Rk+q}^\dagger c_{Rk+q}$. There is a subtlety here, though, because each of the terms is formally infinite. This prevents one from naively changing variables in the first term to get zero. The correct approach is to rewrite $c_{Rk}^\dagger c_{Rk} =: c_{Rk}^\dagger c_{Rk} : + \langle c_{Rk}^\dagger c_{Rk}\rangle_0$. The normal ordered terms are then perfectly finite, and the variable change can be safely done, so the two terms cancel. The difference of the vacuum expectation values, however, gives $qL/(2\pi)$.

An alternative way of seeing this, which emphasizes the physical meaning of the chiral density operators is to consider how the operators act on the ground state. $n_{Rq}|0\rangle$ (for $q > 0$) is a superposition of states with a single particle hole excitation at k and $k+q$ where the hole has momentum k between $-q$ and 0 (see Fig. 5). On the other hand, $n_{R-q}|0\rangle = 0$ because in the ground state there are no empty states with momentum less than an occupied state. Moreover, when n_{R-q} acts on $n_{Rq}|0\rangle$ the only thing it can do is put the excited particle back into the hole. Thus we conclude that

$$[n_{R-q}, n_{Rq}]|0\rangle = \int_{-q}^{0} \frac{Ldq}{2\pi}|0\rangle = \frac{qL}{2\pi}|0\rangle. \tag{47}$$

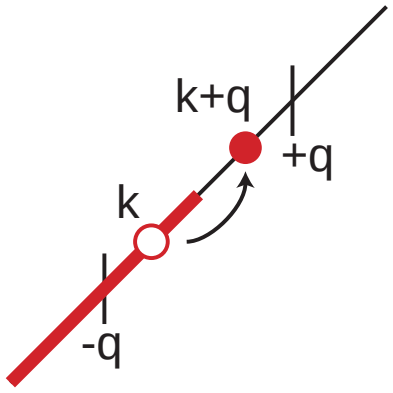

FIG. 5: A particle hole excitation created by the chiral density operator.

With a bit more work one can convince oneself that this argument works when acting on excited states as well as the ground state. In addition, it is clear that $[n_{Rq}, n_{Rq'}] = 0$ when $q + q' \neq 0$.

The above properties of the chiral density operators invite the following transformation to boson creation and annihilation operators for $q > 0$:

$$b_{Rq}^\dagger = (2\pi/qL)^{1/2} n_{Rq} \tag{48}$$

$$b_{Rq} = (2\pi/qL)^{1/2} n_{R-q}. \tag{49}$$

These obey the Bose commutation relations, $[b_{Rq}, b_{Rq}^\dagger] = 1$. Remarkably, the Hilbert space generated by the boson operators $b_{Rq}^\dagger$ is *identical* to the Hilbert space of all excited states of the Luttinger model for fixed particle number. To complete the description we must also include a $q = 0$ fermion number operator N_R which has integer eigevalues between $-\infty$ and ∞.

Using the fact that the compressibility of the chiral Fermi gas is $\partial n/\partial\mu = 1/(2\pi\hbar v_F)$ we may write the energy as a function of density as

$$\mathcal{H}_\mathcal{R} = \frac{1}{2\partial n/\partial\mu}\int dx n_R(x)^2 \tag{50}$$

$$= \frac{\pi\hbar v_F}{L}\left[N_R^2 + \sum_{q>0} b_{Rq}^\dagger b_{Rq}\right] \tag{51}$$

As discussed above the Hilbert spaces are identical in the fermion or boson representations. Within this Hilbert space the boson Hamiltonian (51) is *identical* to the fermion Hamiltonian,

$$\mathcal{H}_\mathcal{R} = \sum_k \hbar v_F k c_{Rk}^\dagger c_{Rk}. \tag{52}$$

It is often useful to express the chiral density in terms of a chiral phase operator,

$$n_R(x) = \partial_x \phi_R(x)/(2\pi) \tag{53}$$

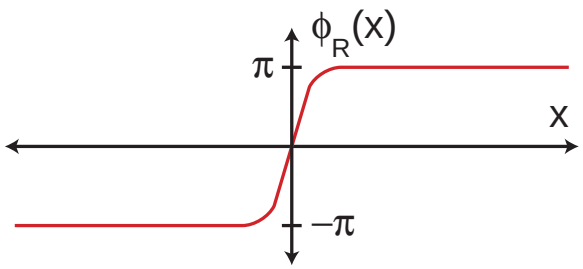

FIG. 6: $e^{i\phi_R(x')}$ introduces a 2π kink in $\phi_R(x)$ at $x = x'$.

In terms of ϕ_R the Kac Moody commutation relation is

$$[\frac{\partial_x \phi_R(x)}{2\pi}, \phi_R(x')] = i\delta(x - x'). \tag{54}$$

This suggests that $\partial_x\phi_R(x)$ and $\phi_R(x)$ are canonically conjugate variables similar to x and p in elementary quantum mechanics. This indicates how to write the Lagrangian, which is usually written as $L = p\dot{q} - H(p, q)$. In the present case we find (using Eq. (50)) that

$$\mathcal{H} = \frac{v_F}{4\pi}\int dx(\partial_x\phi_R)^2 \tag{55}$$

and

$$\mathcal{L} = -\frac{1}{4\pi}\partial_x\phi\left[\partial_t\phi_R + v_F\partial_x\phi_R\right] \tag{56}$$

Readers will notice that there is a factor of 2 difference in the first term from the expression from elementary mechanics. This is because unlike p and q, $\partial_x\phi_R$ and ϕ_R are not independent variables. Below we will transform to sums and differences of ϕ_R and ϕ_L (See Eq. 74 below). In terms of those new variables the conjugate variables are independent, and there is no tricky factor of 2.

Finally, we wish to express the fermion creation operator in terms of the boson fields. This can be accomplished by noting that since $[\phi_R(x), \phi_R(x')] = i\pi\mathrm{sgn}(x - x')$ we have $[\phi_R(x), \exp i\phi_R(x')] = \pi\mathrm{sgn}(x - x')\exp i\phi_R(x')$. This means that the operator $\exp i\phi(x')$ displaces $\phi_R(x)$ to introduce a 2π "kink" at $x = x'$ as shown in Fig. 6. This corresponds to an additional charge e at x'.

In order to be more precise, we must address two technicalities. The first has to do with the prefactor of $\exp i\phi_R$. The prefactor is clearly necessary since $\psi^\dagger$ has units $1/\sqrt{L}$. However, it depends on the method for regularizing the theory at large k. The simplest method is to introduce an exponential cutoff, so that sums on k are replaced by

$$\sum_k \to \sum_k e^{-|k|x_c}. \tag{57}$$

The parameter x_c should really be interpreted as an infinitesimal convergence factor which regularizes formally divergent integrals. x_c is often interpreted as a short distance cutoff a of order the inverse electron density. However, this identification is not precise, and it is not possible to define the numerical ratio between x_c and a. Given this regularization scheme, the appropriate prefactor is $(2\pi x_c)^{-1/2}$.

The second subtlety has to do with carefully treating the $q = 0$ part of ϕ. This necessitates the introduction of a Klein factor. The Klein factor κ is the operator which connects the ground states of different charge sectors:

$$\kappa^\pm|N\rangle_0 = |N \pm 1\rangle_0. \tag{58}$$

In some situations the Klein factors can be ignored, and in the literature you will find many instances where they are not included. However, they are necessary when there is more than one species of fermion, since they are responsible for enforcing the anticommutation relation between different species of fermion operators $\kappa_i\kappa_j = -\kappa_j\kappa_i$. Thus, our final expression for the fermion operator is

$$\psi_R^\dagger(x) = \frac{\kappa^+}{\sqrt{2\pi x_c}}e^{i\phi_R(x)}. \tag{59}$$

As an application let us compare the single electron Green's function computed in the fermion and boson representations,

$$G(x, \tau) = \langle T_\tau\left[\psi(x, \tau)\psi^\dagger(0, 0)\right]\rangle \tag{60}$$

Here T_τ indicates time ordering in imaginary time, including the appropriate minus sign for the Fermi operators:

$$T_\tau[\psi(\tau)\psi^\dagger(0)] \equiv \theta(\tau)\psi(\tau)\psi^\dagger(0) - \theta(-\tau)\psi^\dagger(0)\psi(\tau). \tag{61}$$

In the Fermion representation, this can be easily computed by inserting a complete set of states between the fermion operators. One then finds

$$G(x, \tau) = \sum_k e^{k(ix - v_F\tau)}\left(\theta(k)\theta(\tau) - \theta(-k)\theta(-\tau)\right) \tag{62}$$

$$= \frac{1}{2\pi}\frac{1}{v_F\tau - ix}. \tag{63}$$

In the boson representation the Klein factor is unimportant, since $\kappa^+\kappa^- = \kappa^-\kappa^+ = 1$. However, we must explicitly include the minus sign in the definition of the Fermion time ordering. We then have

$$G(x, \tau) = \frac{\mathrm{sgn}(\tau)}{2\pi x_c}\langle T_\tau\left[e^{i(\phi_R(0,0) - \phi_R(x,\tau))}\right]\rangle. \tag{64}$$

The expectation of the boson operators can be written as $\exp[-C]$ with

$$C = \frac{1}{2}\langle T_\tau\left[(\phi(x, \tau) - \phi(0, 0))^2\right]\rangle. \tag{65}$$

Using the Lagrangian (56) with $\partial_t \to i\partial_\tau$ and including the exponential regularization this gives

$$= \int \frac{dqd\omega}{(2\pi)^2}(1 - \cos(\omega\tau - qx))\frac{2\pi}{q(i\omega - v_F q)}e^{-x_c|q|}. \tag{66}$$

The integral over ω can be done by contour integration, and after some manipulation gives

$$\int_0^\infty \frac{dq}{q}\left(1-e^{-q(v_F\tau - ix)\mathrm{sgn}(\tau)}\right)e^{-x_c q} \tag{67}$$

$$= \log\frac{v_F\tau - ix + x_c\mathrm{sgn}(\tau)}{x_c\mathrm{sgn}(\tau)}. \tag{68}$$

Combining the terms in (64) we thus see that the x_c and $\mathrm{sgn}(\tau)$ factors cancel, and the formula (63) is reproduced exactly for $x_c \to 0$.

C. Combine the left and right movers

We now consider both left and right movers, which are described by the Lagrangian density

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_L \tag{69}$$

with

$$\mathcal{L}_R = -\frac{1}{4\pi}\partial_x\phi_R\left(\partial_t\phi_R + v_F\partial_x\phi_R\right) \tag{70}$$

$$\mathcal{L}_L = -\frac{1}{4\pi}\partial_x\phi_L\left(-\partial_t\phi_L + v_F\partial_x\phi_L\right). \tag{71}$$

We now define new variables,

$$\phi_R = \varphi + \theta \tag{72}$$
$$\phi_L = \varphi - \theta \tag{73}$$

In terms of these variables the total density is $n = (\partial_x\phi_R - \partial_x\phi_L)/(2\pi) = \partial_x\theta/\pi$. In terms of these new variables we have

$$\mathcal{L} = \frac{1}{\pi}\partial_x\theta\partial_t\varphi - \frac{v_F}{2\pi}\left((\partial_x\theta)^2 + (\partial_x\varphi)^2\right) \tag{74}$$

This has the form $p\dot{q} - H(p,q)$. The first term shows that $\partial_x\theta/\pi$ and φ are canonically conjugate variables. "Integrating out" either θ or ϕ yields two equivalent dual representations of the non interacting electron gas: the θ representation,

$$\mathcal{L}[\theta] = \frac{1}{2\pi}\left[\frac{1}{v_F}(\partial_t\theta)^2 - v_F(\partial_x\theta)^2\right], \tag{75}$$

and the φ representation,

$$\mathcal{L}[\varphi] = \frac{1}{2\pi}\left[\frac{1}{v_F}(\partial_t\varphi)^2 - v_F(\partial_x\varphi)^2\right]. \tag{76}$$

These equivalent representations in terms of the dual variables θ and φ show that non interacting electrons are "self dual".

The fermion creation operators have the form

$$\psi_R^\dagger = \frac{\kappa^+}{\sqrt{2\pi x_c}}e^{i(\varphi+\theta)}, \tag{77}$$
$$\psi_L^\dagger = \frac{\kappa^+}{\sqrt{2\pi x_c}}e^{i(\varphi-\theta)}. \tag{78}$$

Note the similarity of this description with the description in section 1. The virtue of this bosonized description is that it is now quit simple to incorporate interactions into the theory. This will be done in the next lecture.

III. REFERENCES

The following is a vastly incomplete selection of references for bosonization and the Luttinger liquid.

A few orignal articles:

F.D.M. Haldane, J. Phys. C **14**, 2585 (1981). A classic and very readable article where Haldane coins the term Luttinger liquid.

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A.M. Chang, Rev. Mod. Phys 75 No. 4, 1449 (2003). Review which describes Luttinger liquid theory as well as numerous experiments.