KANE

Electrons in 1 Dimension

The study of 1D interacting systems is useful because

1. Solvable models $\Rightarrow$ Paradigm for interacting system
2. 1D physics is relevant in many experimental contexts

Reading

Original Articles:

- Haldane: classic on bosonization, luttinger liquid.
- Kane & Fisher: single impurity problem

Reviews:

- Fisher & Glazman: more physical, less rigorous
- Senechal: more mathematical, field theory oriented.

Applications:

Carbon Nanotubes: 2 original articles

- Kane & Mele: Effective Mass model, curvature gaps
- Kane, Balents & Fisher: Luttinger Liquid theory

Quantum Hall Edge States: 2 reviews

- Kane & Fisher
- Chang

Lecture #1: Bosonization

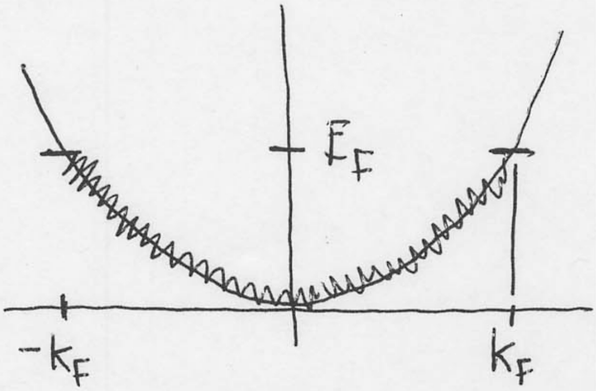
Heuristic Introduction

Spinless free electrons in $d=1$:

$$\mathcal{H} = \int dx\, \psi^\dagger(x)\left[-\frac{\hbar^2}{2m}\partial_x^2\right]\psi(x)$$

$$= \sum_k \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k$$

Fill states to E_F:



$$n_0 = \frac{2k_F}{2\pi} \Rightarrow k_F = \pi n_0$$

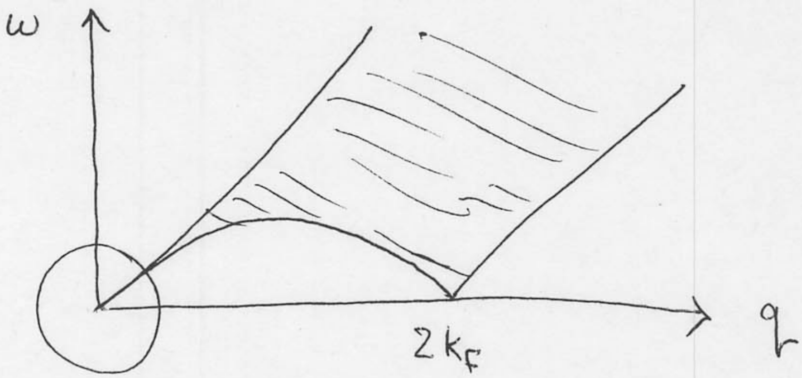
$$v_F^0 = \frac{\hbar k_F}{m} = \pi\frac{\hbar n_0}{m}$$

Density of states $\frac{\partial n}{\partial \mu} = N(E_F) = 2\cdot\frac{1}{2\pi\hbar v_F}$

We will be interested in the low energy, long wavelength properties.

Consider the spectrum of particle hole excitations:

$$c^{\dagger}_{k+q}\, c_k \,|0\rangle$$



~ "sound mode" $\omega = v_F\, q$

Of course non interacting electrons are trivial. All of the energy is kinetic. In a real system there is competition between kinetic and potential energy.

Opposite limit: Strong Repulsive Interactions
Potential energy dominates

Groundstate = Wigner Crystal

$a = n_0^{-1}$ $\qquad \mapsto u_i = \dfrac{a}{\pi}\theta_i$

The low energy excitations are <u>phonons</u>. Let's describe them:

$$r_i = r_i^0 + \frac{a}{\pi}\theta_i$$

Consider Lagrangian:

$$\mathcal{L} = KE - PE$$

$$KE = \sum_i \frac{1}{2} m \dot{r}_i^2$$

$$= \int dx \, \frac{1}{2} \frac{ma}{\pi^2} \dot{\theta}^2$$

$$PE = \frac{1}{2}\int dx\, dx' \, V(x-x')\, n(x)\, n(x')$$

$$= \frac{1}{2}\int dx \, V_0 \, n(x)^2$$

Short range int: $V_0 = \int dx \, V(x-x')$

$$n(x) = \frac{1}{\pi} d_x \theta$$

$$\mathcal{L} = \int dx \left[\frac{1}{2} \frac{ma}{\pi^2} \dot{\theta}^2 - \frac{1}{2} \frac{V_0}{\pi^2} (d_x \theta)^2 \right]$$

Sound velocity

$$V_\rho = \sqrt{\frac{V_0}{m a}} = V_F^0 \sqrt{\frac{V_0}{\pi \hbar V_F^0}} \equiv \frac{V_F^0}{g}$$

$$g = \sqrt{\frac{\pi \hbar V_F^0}{V_0}}$$

This approach is valid for $g \ll 1$ (V_0 large)

Expressed in these variables we may write

$$\mathcal{L} = \frac{\hbar}{2\pi g} \int dx \left[\frac{1}{V_\rho} \dot{\theta}^2 - V_\rho (\partial_x \theta)^2 \right]$$

Effect of Quantum fluctuations on crystaline order:

$$n(x) \sim \sum_{q \sim 0} n_q e^{iqx} + \sum_{q \sim 2k_F} n_q e^{iqx} + \cdots$$

$$\sim \left(n_0 + \frac{1}{\pi} \partial_x \theta \right) + \left(n_{2k_F}(x) e^{i 2 k_F x} + c.c. \right) + \cdots$$

$$n_{2k_F}(x) \sim e^{i 2\theta}$$

2θ describes phase of $2k_F$ density wave.

For a perfect crystal:

$$\langle \hat{\rho}_{2k_F} \rangle \neq 0$$

or

$$\lim_{x \to \infty} \langle \hat{\rho}_{2k_F}(x)\, \hat{\rho}_{-2k_F}(0) \rangle \neq 0$$

Quantum fluctuations $\Rightarrow$ "Debye Waller factor"

Many ways to compute $\langle \hat{\rho}_{2k_F} \rangle$. A powerful method is to use imaginary time path integral.

$$Z = \int D[\theta(x,\tau)]\, e^{-\frac{i}{\hbar} S[\theta(x,\tau)]}$$

τ ← it

$$S = \frac{1}{\hbar}\int d\tau\, \mathcal{L} = \frac{1}{2\pi g}\int dx\, d\tau \left[\frac{1}{v_\rho}(\partial_\tau \theta)^2 + v_\rho (\partial_x \theta)^2 \right]$$

$$= \frac{1}{2\pi g} \sum_{q,\omega} \left(\frac{\omega^2}{v_\rho} + v_\rho q^2 \right) |\theta(q,\omega)|^2$$

$$\langle \hat{\rho}_{2k_F} \rangle = \langle e^{2i\theta} \rangle = \frac{1}{Z}\int D\theta\, e^{i2\theta}\, e^{-S}$$

$$= e^{-\frac{1}{2}\langle (2\theta)^2 \rangle}$$

Can see by expanding in powers of θ

For harmonic theory it is exact!

$$\left\langle \theta(x,\tau)^2 \right\rangle = \sum_{q\omega} \left\langle |\theta(q,\omega)|^2 \right\rangle$$

Log divergent Debye Waller factor

$$= \sum_{q\omega} \frac{\pi g \, v_\rho}{\omega^2 + v_\rho^2 q^2}$$

$$= \int \frac{\overbrace{d^2q}^{2\pi q\,dq}}{(2\pi)^2} \frac{\pi g}{q^2} = \frac{g}{2} \log \frac{L}{a}$$

$$\left\langle \overset{n}{\rho}_{2k_F} \right\rangle \sim e^{-2\langle \theta^2 \rangle} \sim \left(\frac{a}{L} \right)^g \to 0$$

Quantum fluctuations destroy crystaline order, but just barely.

Similar to spin correlations in 2D XY model.

Density Correlations:

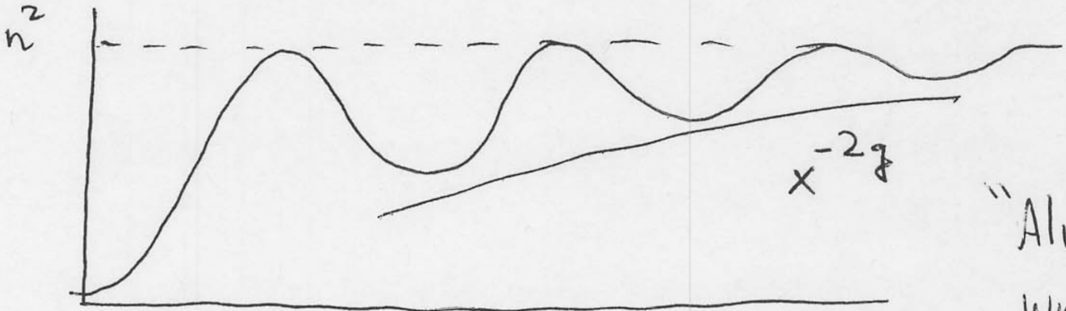
$$\left\langle \overset{n}{\rho}_{2k_F}(x) \, \overset{n}{\rho}_{-2k_F}(0) \right\rangle \sim \left\langle e^{i(2\theta(x) - 2\theta(0))} \right\rangle$$

$$= e^{-\frac{1}{2}\left\langle (2\theta(x) - 2\theta(0))^2 \right\rangle}$$

$$= \left(\frac{a}{x} \right)^{2g}$$

Pair correlation function

$$\langle \psi^\dagger(x)\psi^\dagger(0)\,\psi(0)\psi(x)\rangle$$



"Almost a Wigner crystal"

Non interacting electrons:

$$D(x) \sim n^2 - \frac{\cos 2k_F x}{x^2} \; : \quad \text{Friedel Oscillations}$$

$$\Rightarrow \text{"} g = 1 \text{"}$$

The elastic Lagrangian is more general than the strong interaction limit, though our formula for g will have to be modified.

The Wigner crystal picture is very useful for intuition.

There is another simple limit: a superfluid

Consider bosons:

Bose condensate: $\langle b^{+} \rangle = \sqrt{n}\, e^{i\varphi}$

Phase fluctuations:

$$b^{+}(x) = \sqrt{n}\; e^{i\varphi(x)}$$

$$\left[n(x),\ \varphi(x') \right] = i\,\delta(x-x')$$

Or, if $n = n_0 + \frac{1}{\pi}\partial_x \theta$

$$\left[\partial_x \theta,\ \varphi \right] = i\pi\,\delta(x-x') :$$

θ, φ are "dual" variables

Lagrangian for phase fluctuations

$$\mathcal{L}_{eff} = \frac{g}{2\pi}\int dx \left[\frac{1}{v_\rho}\dot{\varphi}^{2} - v_\rho (\partial_x \varphi)^2 \right]$$

$\Rightarrow$ Power law superfluid order

$$\langle b^{+}(x)\, b(0) \rangle \sim \frac{1}{x^{1/2g}}$$

Now consider fermions:

Jordan Wigner Transformation

$$\psi^{+}(x) = e^{\pm i\pi \int_{-\infty}^{x} n(x)} \, b^{+}(x)$$

fermion

"Jordan Wigner String: $(-1)^{N_L}$

boson

$$\psi^{+} = e^{i(\pm\theta + \varphi)}$$

We will return to this.

A Luttinger liquid is:

- Almost a Wigner Crystal
- Almost a superfluid

In order to generalize these intuitive notions we need a (slightly) more rigorous formulation of bosonization

Low energy theory of free electrons

Linearize :
$$\mathcal{H} = \sum_q v_F q \left[C^{\dagger}_{k_F+q} C_{k_F+q} - C^{\dagger}_{-k_F+q} C_{-k_F+q} \right]$$

$$\psi_{R/L}(x) = \sum_q e^{iqx} C_{\pm k_F + q}$$

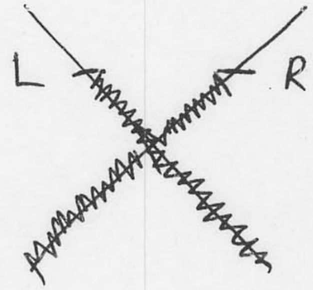
$$\mathcal{H} = \int dx \; -i v_F \left[\psi_R^{\dagger} \partial_x \psi_R - \psi_L^{\dagger} \partial_x \psi_L \right]$$

This is valid for energies << cutoff

distance >> a

Luttinger Model :

(like 1+1 D Dirac Equation)



Chiral Symmetry :

$$\left.\begin{aligned} N_L &= \int dx \; \psi_L^{\dagger} \psi_L \\ N_R &= \int dx \; \psi_R^{\dagger} \psi_R \end{aligned}\right\} \text{Independently Conserved.}$$

Chiral Fermions ψ_R

- "Half" of 1DEG
- Relavent to QH edgestates
- "Building Blocks" for 1DEG's w/ spin, etc.

"Chiral" Density operator:

$$n_R(x) = :\psi_R^\dagger(x)\psi_R(x): \quad \text{"Normal Ordering}$$

$$= \psi_R^\dagger(x)\psi_R(x) - \langle\psi_R^\dagger\psi_R\rangle_0$$

$$= \sum_q e^{iqx} n_{Rq}$$

$$n_{Rq} = \sum_k C^\dagger_{Rk+q} C_{Rk} \qquad (q \neq 0)$$

Kac Moody Commutation Relation:

$$\left[n_{R-q}, n_{Rq} \right] = \frac{q}{2\pi}$$

You can check this by working out the commutator explicitly. However, you have to be careful with the infinite sums on k that appear.

Check it as an operator acting on the groundstate:

$$n_{Rq} |0\rangle = \sum_k$$ 

$$n_{R-q} |0\rangle = 0$$

$$n_{R-q} n_{Rq} = \int_{-q}^{0} \frac{dk}{2\pi} |0\rangle = \frac{q}{2\pi} |0\rangle$$

$$n_{Rq} = \sqrt{\frac{q}{2\pi}}\, b_q^{\dagger}$$
$$n_{R-q} = \sqrt{\frac{q}{2\pi}}\, b_q \Rightarrow [b_q, b_q^{\dagger}] = 1$$

Equivalently:

$$[n_R(x), n_R(x')] = -\frac{i}{2\pi} \partial_x \delta(x-x')$$

$$n_R = \partial_x \phi_R / 2\pi$$

Then,

$$\Big[\underbrace{\tfrac{1}{2\pi} \partial_x \phi_R}_{\text{"}p\text{"}}, \underbrace{\phi_R}_{\text{"}x\text{"}}\Big] = i\, \delta(x-x')$$

Hamiltonian:

$$E = \frac{1}{2} \frac{1}{dn/d\mu} \int dx \; n_R(x)^2$$

$$= \frac{1}{2} (2\pi v_F) \int dx \left[\frac{\partial_x \phi_R}{2\pi} \right]^2$$

$$= \frac{v_F}{4\pi} \int dx \; (\partial_x \phi_R)^2$$

Lagrangian

$$\mathcal{L} = \text{“} \, p\dot{q} - \mathcal{H}(p,q) \, \text{”}$$

$$= -\frac{1}{4\pi} \partial_x \phi_R \left[\partial_t \phi_R + v_F \partial_x \phi_R \right]$$

Electron Operator:

$$\psi_R^{\dagger}(x) \propto e^{i \phi_R(x)}$$

Technicalities:

1. Prefactor (cutoff dependent)

Exponential cutoff: $\sum_k \Rightarrow \sum_k e^{-kx_c}$

$$C = \frac{1}{\sqrt{2\pi x_c}}$$

2. Klein Factor

One must treat $q=0$ mode for finite system carefully $\Rightarrow$ even in infinite size limit.

$$K|N\rangle_0 = |N+1\rangle_0$$

Often Klein factors can be ignored, though they are necessary when there is more than one band:

$$K_i K_j = -K_j K_i$$

Thus, a more complete expression is

$$\psi_R^\dagger = \frac{K}{\sqrt{2\pi x_c}} e^{i\phi_R(x)}$$

We now have an exact mapping between Fermi and Boson representations. The Hilbert spaces are identical.

Single electron Green's Function

$$G(x,\tau) = \left\langle T_\tau \left[\psi(x,\tau)\, \psi^\dagger(0,0) \right] \right\rangle$$

$$= \frac{1}{2\pi x_c}\, e^{-\frac{1}{2}\left\langle \left(\phi(x,\tau) - \phi(0,0)\right)^2 \right\rangle}$$

$$= \frac{1}{2\pi x_c} \left[\frac{x_c}{x + i v_F \tau + x_c} \right]$$ (Using exponential cutoff)

$$= \frac{1}{2\pi}\, \frac{1}{x + i v_F \tau + x_c}$$

Same as you would get doing the calculation with fermions.

Combine left & right movers

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_L$$

$$= -\frac{1}{4\pi}\partial_x\phi_R\left(\partial_t\phi_R + v_F\partial_x\phi_R\right)$$

$$-\frac{1}{4\pi}\partial_x\phi_L\left(-\partial_t\phi_L + v_F\partial_x\phi_R\right)$$

New variables:

$$\phi_R = \theta + \varphi \qquad\qquad \rho = \frac{1}{2\pi}\left(\partial_x\phi_R - \partial_x\phi_L\right)$$

$$\phi_L = -\theta + \varphi \quad \Longrightarrow \quad = \frac{1}{\pi}\partial_x\theta$$

$$\mathcal{L} = \int dx \; \underbrace{\frac{1}{\pi}\partial_x\theta\,\partial_t\varphi}_{\text{“}p\dot{q}\text{”}} - \frac{v_F}{2\pi}\underbrace{\left((\partial_x\theta)^2 + (\partial_x\varphi)^2\right)}_{\text{“}\mathcal{H}(p,q)\text{”}}$$

$$\left[\frac{1}{\pi}\partial_x\theta, \varphi\right] = i$$

$$\psi_R = \frac{K}{\sqrt{2\pi x_c}}\, e^{i\varphi + \theta}$$

$$\psi_L = \frac{K}{\sqrt{2\pi x_c}}\, e^{i\varphi - \theta}$$

"Integrate out" θ or φ (Equivalently use equation of motion)

$$\mathcal{L}[\theta] = \frac{1}{2\pi}\int dx \left[\frac{1}{v_F}(\partial_t\theta)^2 - v_F(\partial_x\theta)^2\right]$$

$$\mathcal{L}[\varphi] = \frac{1}{2\pi}\int dx \left[\frac{1}{v_F}(\partial_t\varphi)^2 - v_F(\partial_x\varphi)^2\right]$$

Free electrons are "self dual"

Forward Scattering Interactions

$$\mathcal{H}_{int} = \frac{1}{2} V_0 \int dx \, n(x)^2$$

$n_L + n_R$

$$= \frac{V_0}{2\pi^2} \int dx \, (\partial_x \theta)^2$$

$$\mathcal{L}[\theta] = \frac{1}{2\pi} \int dx \left\{ \frac{1}{v_F} (\partial_t \theta)^2 - \left[v_F + \frac{V_0}{\pi} \right] (\partial_x \theta)^2 \right\}$$

$$= \frac{1}{2\pi g} \int dx \left\{ \frac{1}{v_\rho} (\partial_t \theta)^2 - v_\rho (\partial_x \theta)^2 \right\}$$

$$g = \left[1 + \frac{V_0}{\pi v_F} \right]^{-1/2}$$

$g < 1$ Repulsive Int.

$g = 1$ Non Interacting

$g > 1$ Attractive

$$v_\rho = v_F / g$$

This gives the <u>exact</u> solution of the Luttinger model.

Properties of a Luttinger Liquid

1. $2k_F$ Density Correlations

$$\langle \psi_L^\dagger \psi_R(x)\, \psi_R^\dagger \psi_L(0) \rangle \sim \langle e^{i(2\theta(x)-2\theta(0))} \rangle \sim \frac{1}{x^{2g}}$$

"Almost a Wigner Crystal"

2. Pair Correlations

$$\langle \psi_L \psi_R(x)\, \psi_R^\dagger \psi_L^\dagger(0) \rangle \sim \langle e^{i(2\varphi(x)-2\varphi(0))} \rangle \sim \frac{1}{x^{2/g}}$$

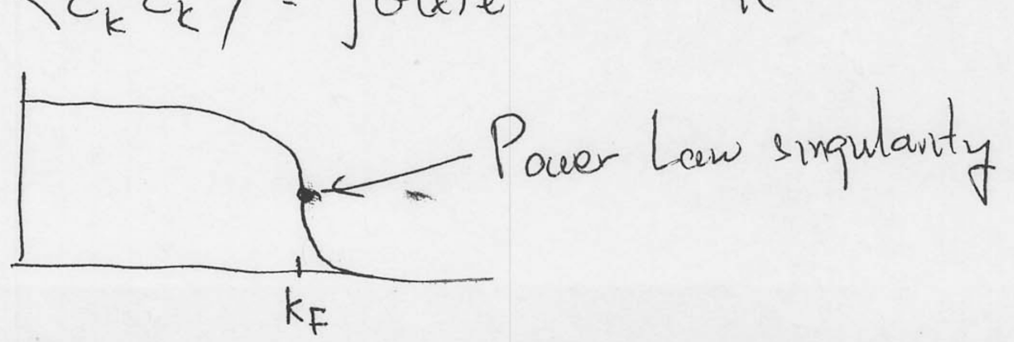
"Almost a superfluid"

3. Single Particle:

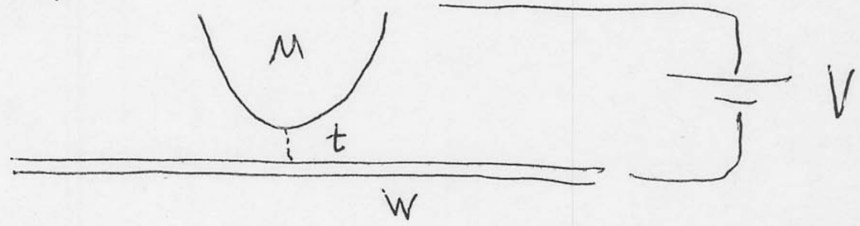
$$G(x) = \langle \psi_R^\dagger(x) \psi_R(0) \rangle \sim \langle e^{i(\theta+\varphi)}\, e^{-i(\varphi+\theta)} \rangle \sim \frac{1}{x^{\frac{1}{2}(g+1/g)}}$$

4. Momentum Distribution

$$n(k) = \langle c_k^\dagger c_k \rangle = \int G(x) e^{-ikx} \sim k^{\frac{1}{2}(g+\frac{1}{g}-1)}$$



Tunneling Density of States



Tunneling rate $\quad I = e/\tau$

$$\frac{1}{\tau} = \frac{2\pi}{\hbar} \sum_N t^2 \left| \langle N | \psi_W^\dagger \psi_M | 0 \rangle \right|^2 \delta(E_N - E_0 - eV)$$

$$\int dt\, e^{it(\cdots)}$$

$$\alpha = \left(\frac{1}{g} + g\right)\frac{1}{2} > 1$$

$$= \frac{2\pi t^2}{\hbar} \int_{-\infty}^{\infty} dt\, e^{ieVt}\, G_W^>(t)\, G_M^<(t)$$

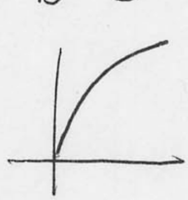
Where $\quad G_W^>(t) = \langle \psi_W(t) \psi_W^\dagger(0) \rangle \sim \dfrac{1}{t^\alpha}$

$\qquad\quad G_M^<(t) = \langle \psi_M^\dagger(t) \psi_M(0) \rangle \sim \dfrac{1}{t}$

$$= \frac{2\pi t^2}{\hbar} \int dE\, G_W^>(E)\, G_M^<(E-V)$$

$$G_W^>(E) = \sum_N \left| \langle N | \psi^\dagger | 0 \rangle \right| \delta(E - E_N) = \text{"Tunneling in DOS"}$$

$$\propto E^{\alpha - 1}\, \theta(E)$$



$$G_M^<(E) \propto \theta(-E)$$

$$I(V) = \frac{e}{\tau} \propto \int_0^V G_W^>(E) \propto V^\alpha \quad (V \gg T)$$

$$\alpha > 1$$

For $V \ll T$

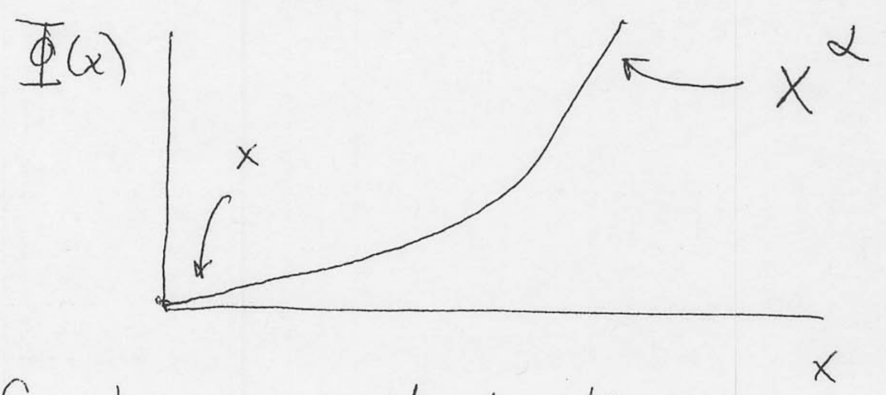
$$I(V) = G(T)V$$

$$G(T) \propto T^{\alpha - 1}$$

For $V \sim T$

$$I(V) \propto T^\alpha \, \Phi\left(\frac{V}{2\pi T}\right)$$

Universal Crossover function



Can be compared directly with experiment.