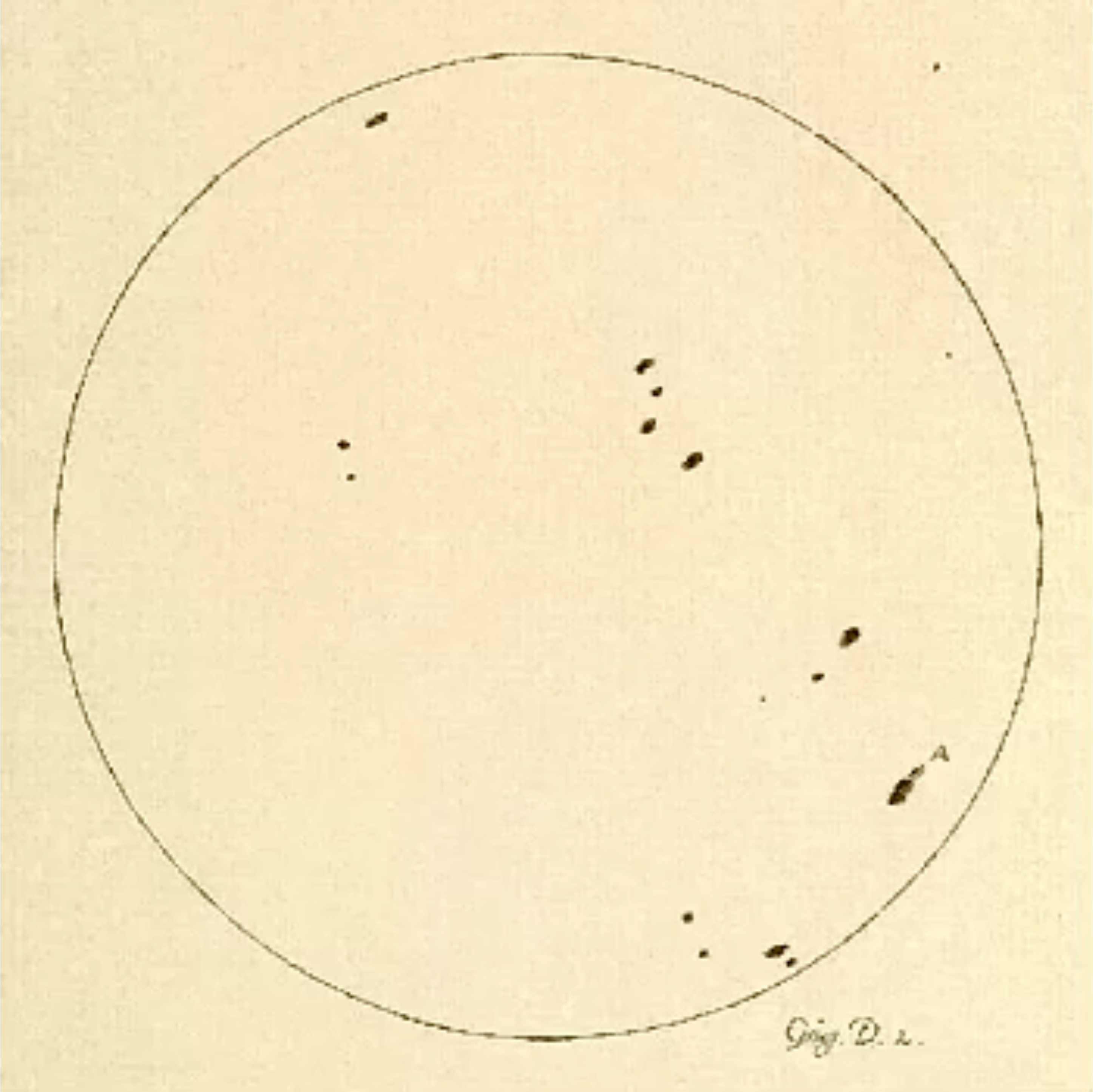
- Lecture I
  - motivate/overview celestial objects where rotation influences buoyancy driven flows
  - discuss energetics, waves, geostrophy

- Lecture II
  - stability theory for rotating convection - what can we glean from it
  - motivates non-hydrostatic quasi-geostrophy
  - derive and investigate semi-analytic solutions (skeleton for strongly NL flows)

Lecture III

- investigate fully NL rotating convection from QG perspective
- assess how the theory holds up c.f. experiments and DNS
- comments on broader view and future outlook

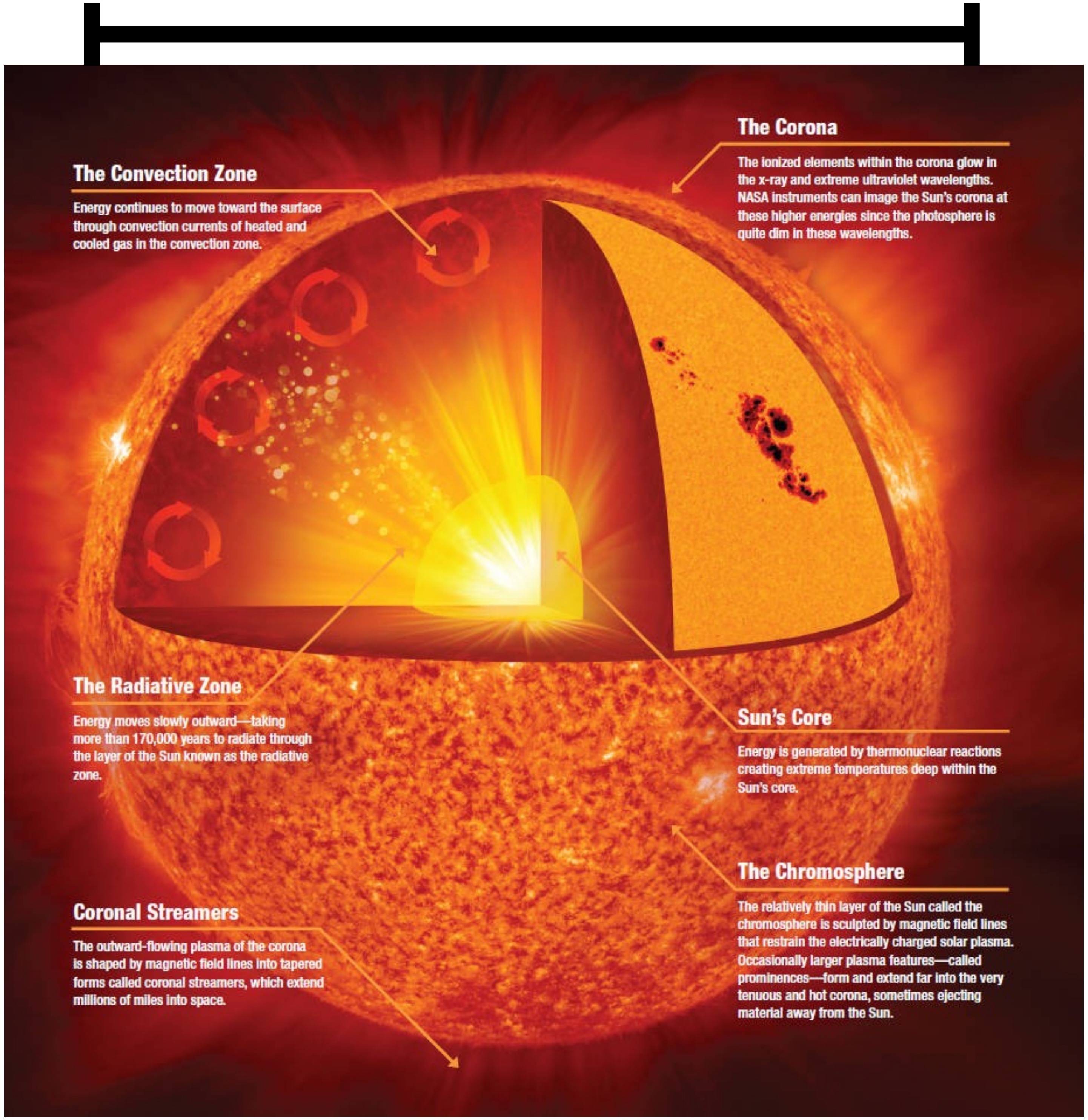
Solar Structure & Dynamics



The Galileo Project, 1995, Galileo 1613

Solar Structure & Dynamics

1400Mm



The Corona

The ionized elements within the corona glow in the x-ray and extreme ultraviolet wavelengths. NASA instruments can image the Sun's corona at these higher energies since the photosphere is quite dim in these wavelengths.

The Convection Zone

Energy continues to move toward the surface through convection currents of heated and cooled gas in the convection zone.

The Radiative Zone

Energy moves slowly outward—taking more than 170,000 years to radiate through the layer of the Sun known as the radiative zone.

Sun's Core

Energy is generated by thermonuclear reactions creating extreme temperatures deep within the Sun's core.

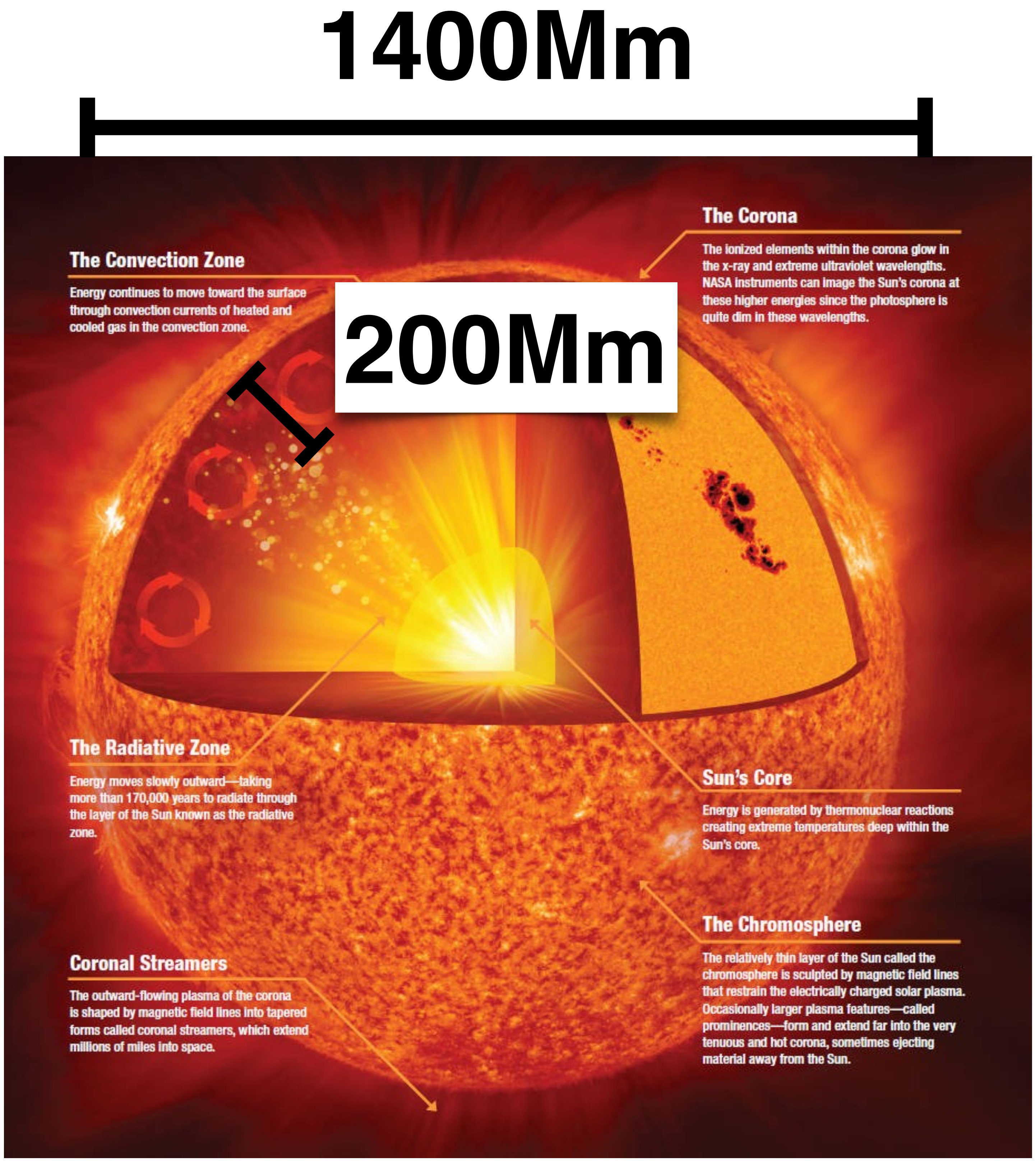
The Chromosphere

The relatively thin layer of the Sun called the chromosphere is sculpted by magnetic field lines that restrain the electrically charged solar plasma. Occasionally larger plasma features—called prominences—form and extend far into the very tenuous and hot corona, sometimes ejecting material away from the Sun.

Coronal Streamers

The outward-flowing plasma of the corona is shaped by magnetic field lines into tapered forms called coronal streamers, which extend millions of miles into space.

Solar Structure & Dynamics



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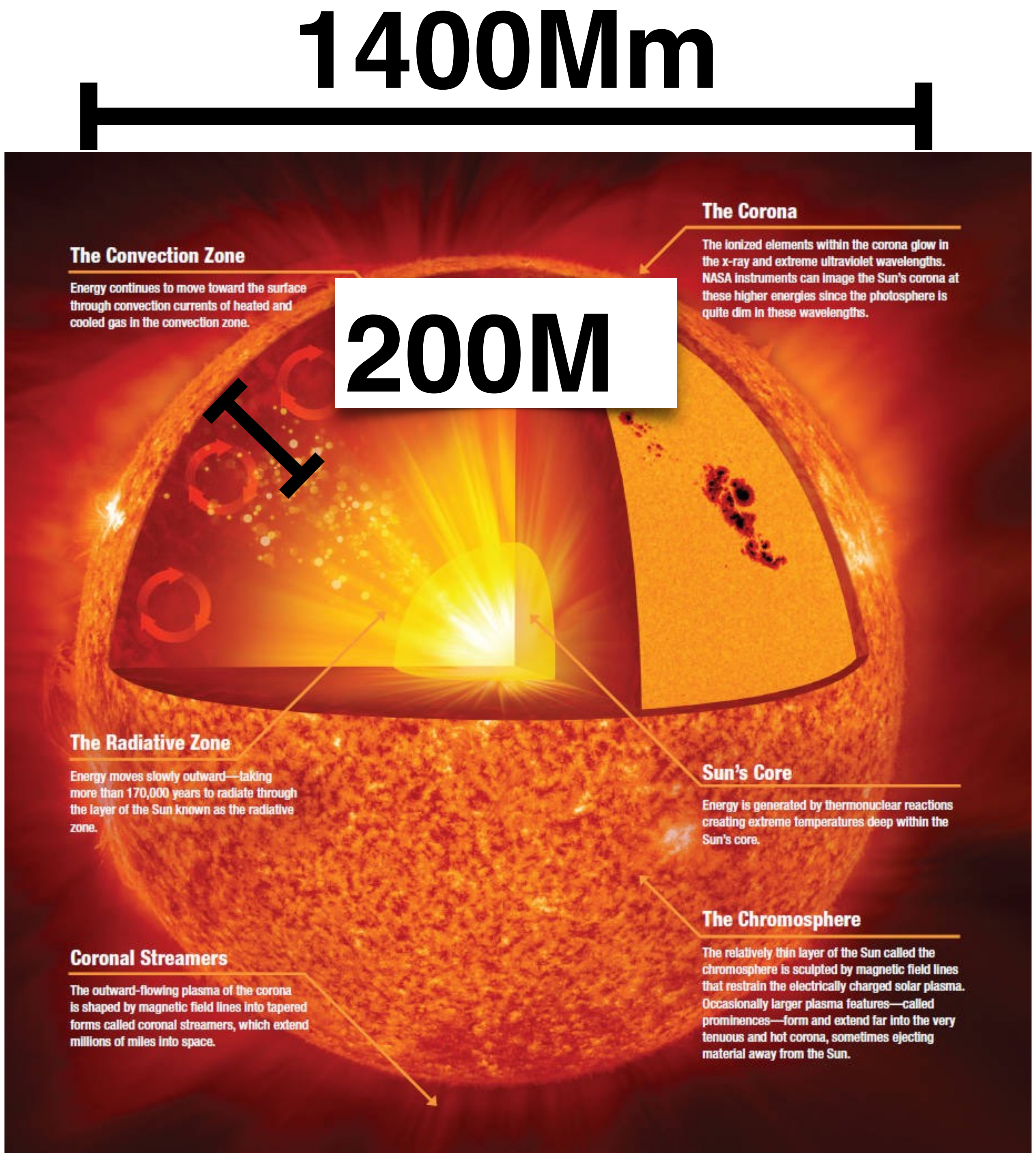

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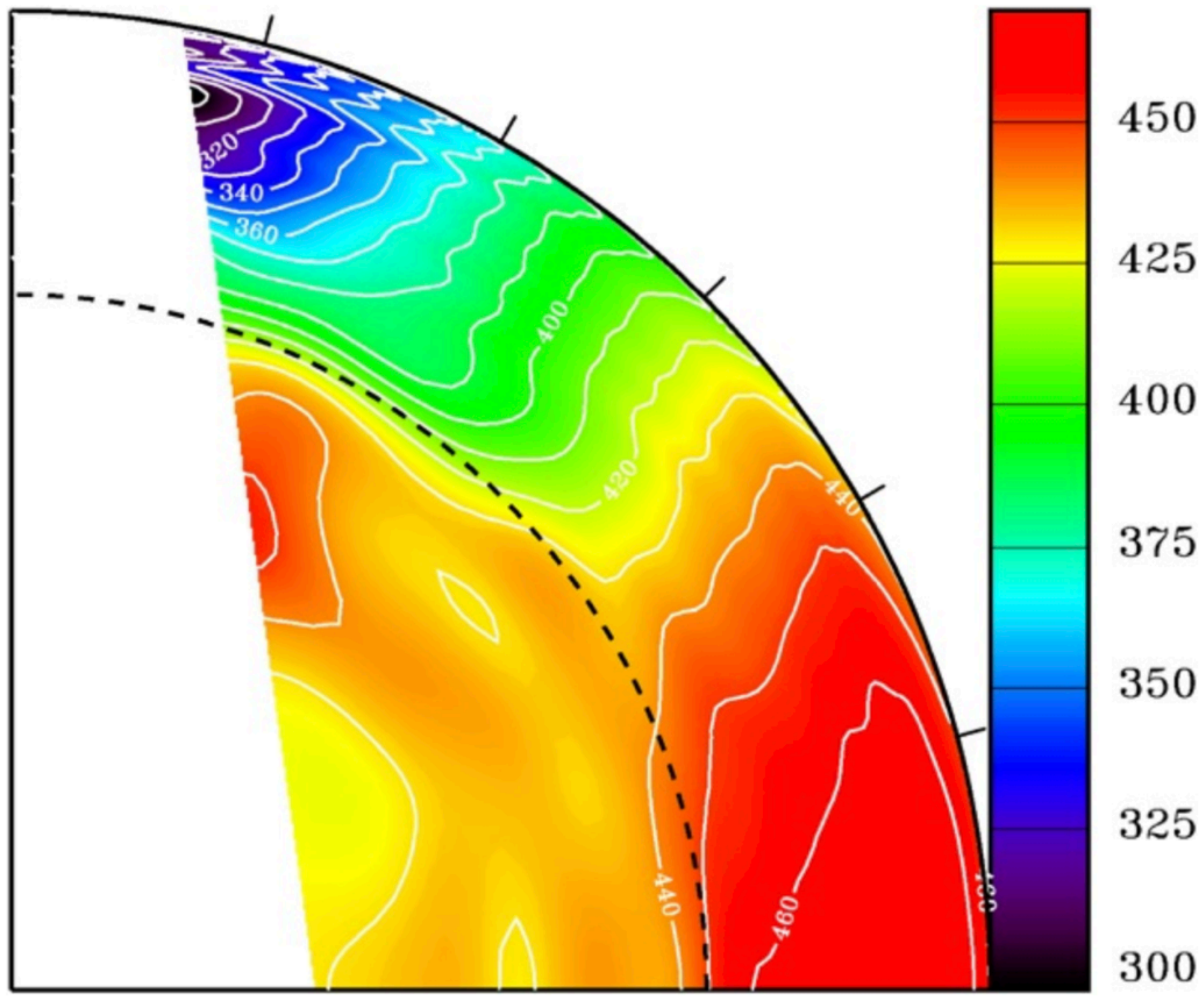
Solar differential rotation

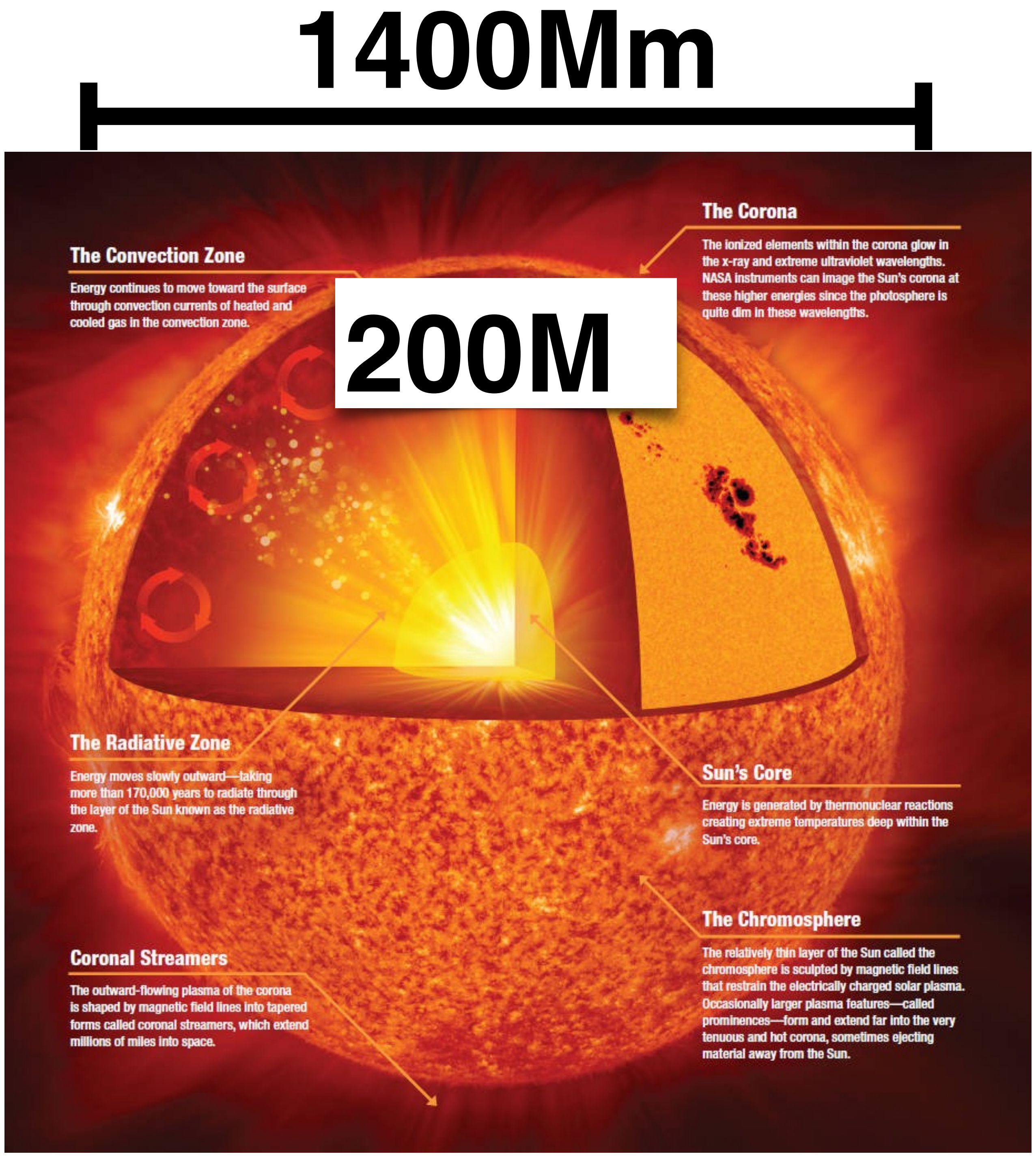


Differential rotation in the Sun. Rotation rate is higher at the equator than at the poles. The frequency is measured in nHz and the dashed line represents the base of the convection zone. Taken from Schou et al. (1998)

Helioseimology - acoustic inferences observe rotation rates constant along radial lines

- convective redistribution of angular mtm.

Solar Structure & Dynamics



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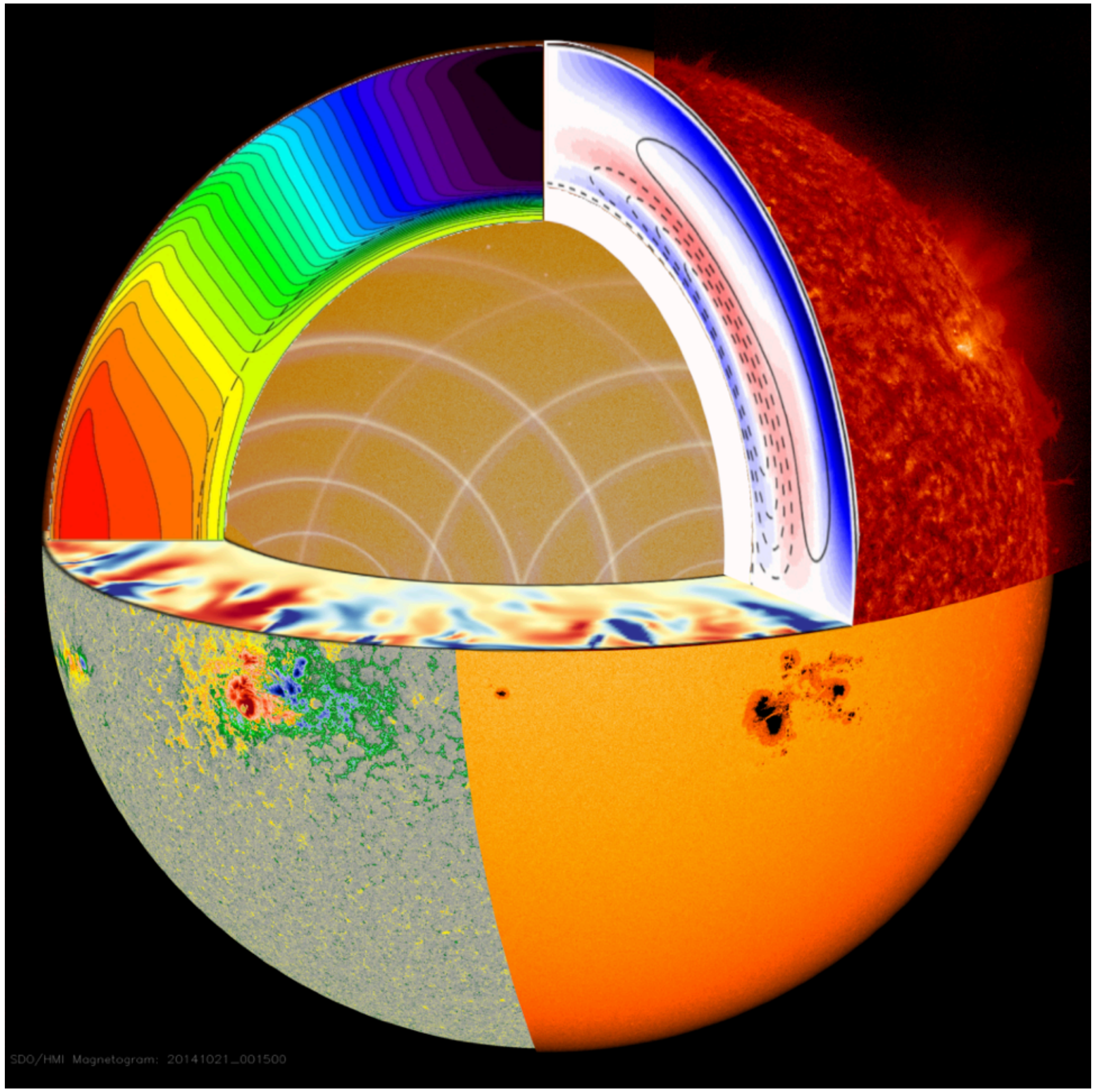
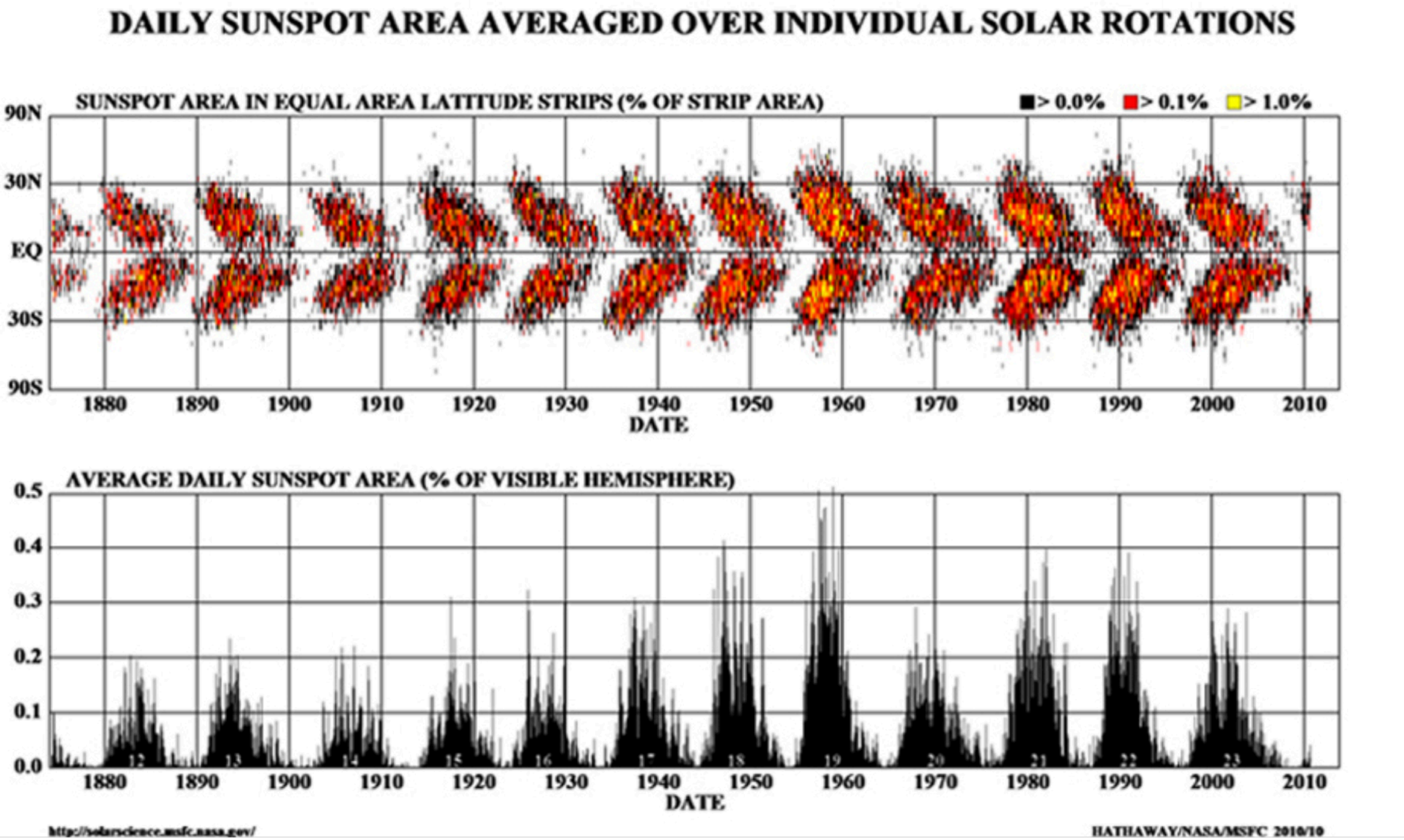


Convective dynamo action - 22 yr magnetic cycle.

Solar Structure & Dynamics

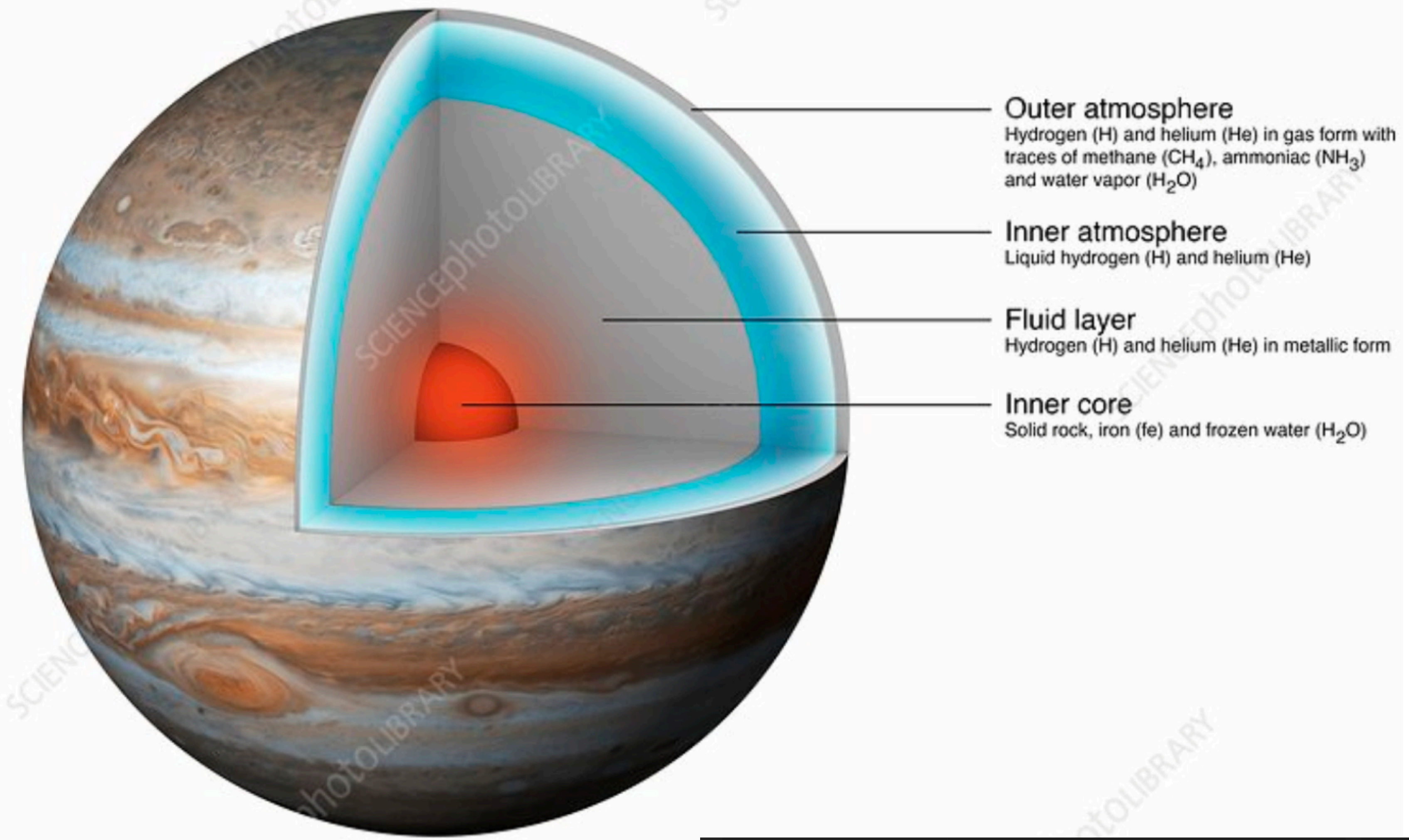


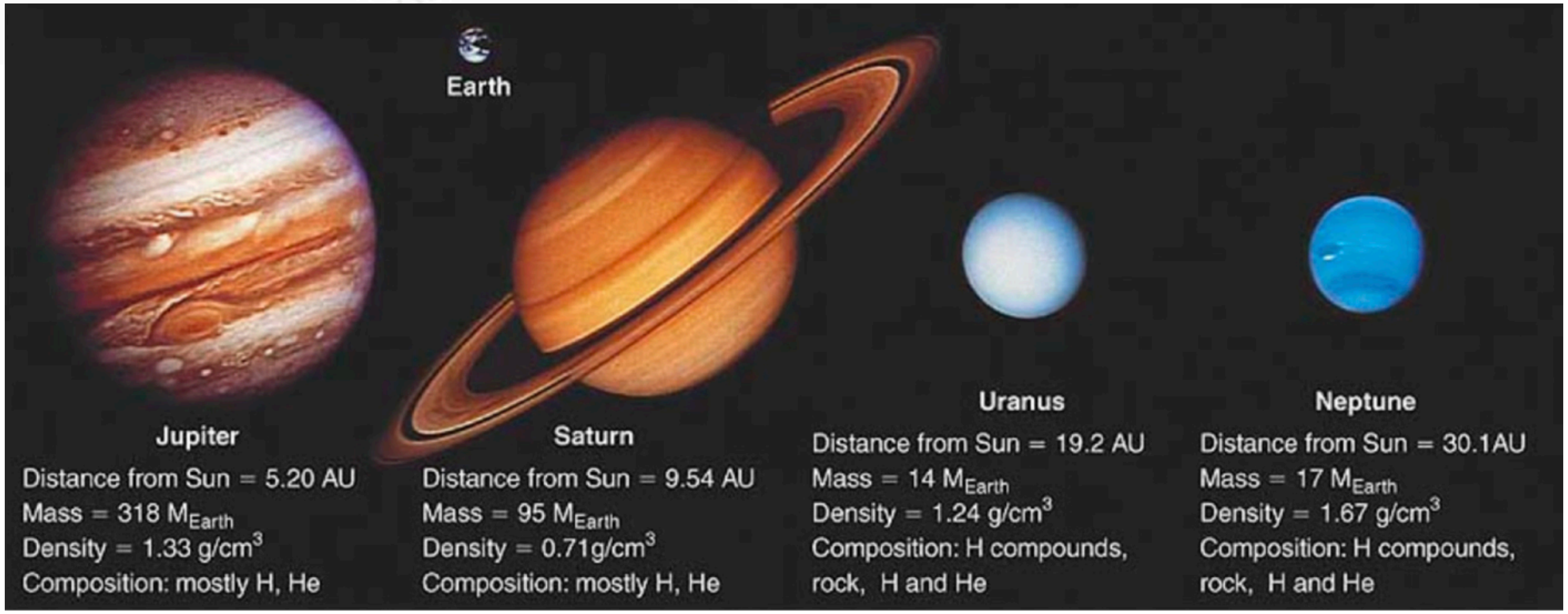
Convective dynamo action - 22 yr magnetic cycle

- Envelope exhibits chaotic overtones

Giant (Jovian) Planets

Rapid rotators





Jupiter
Distance from Sun = 5.20 AU
Mass = 318 M_{Earth}
Density = 1.33 g/cm^3
Composition: mostly H, He

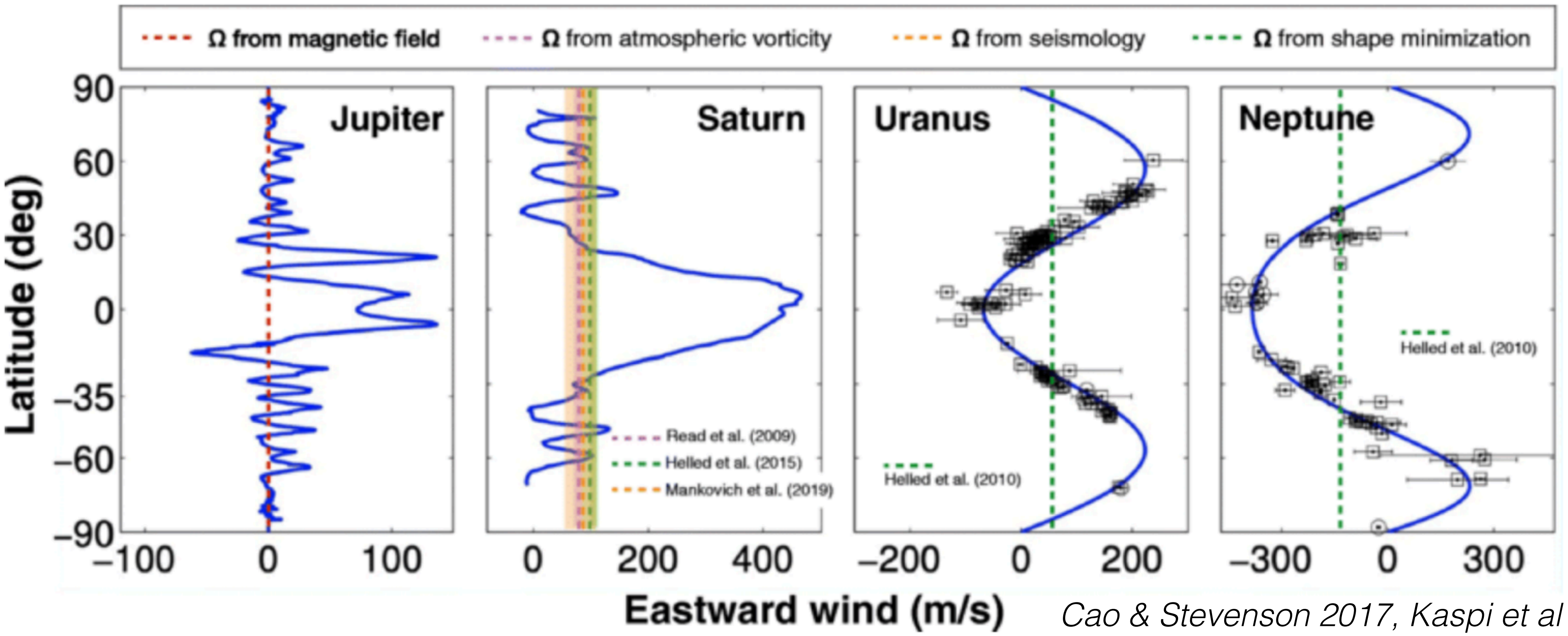
Saturn
Distance from Sun = 9.54 AU
Mass = 95 M_{Earth}
Density = 0.71g/cm^3
Composition: mostly H, He

Uranus
Distance from Sun = 19.2 AU
Mass = 14 M_{Earth}
Density = 1.24 g/cm^3
Composition: H compounds, rock, H and He

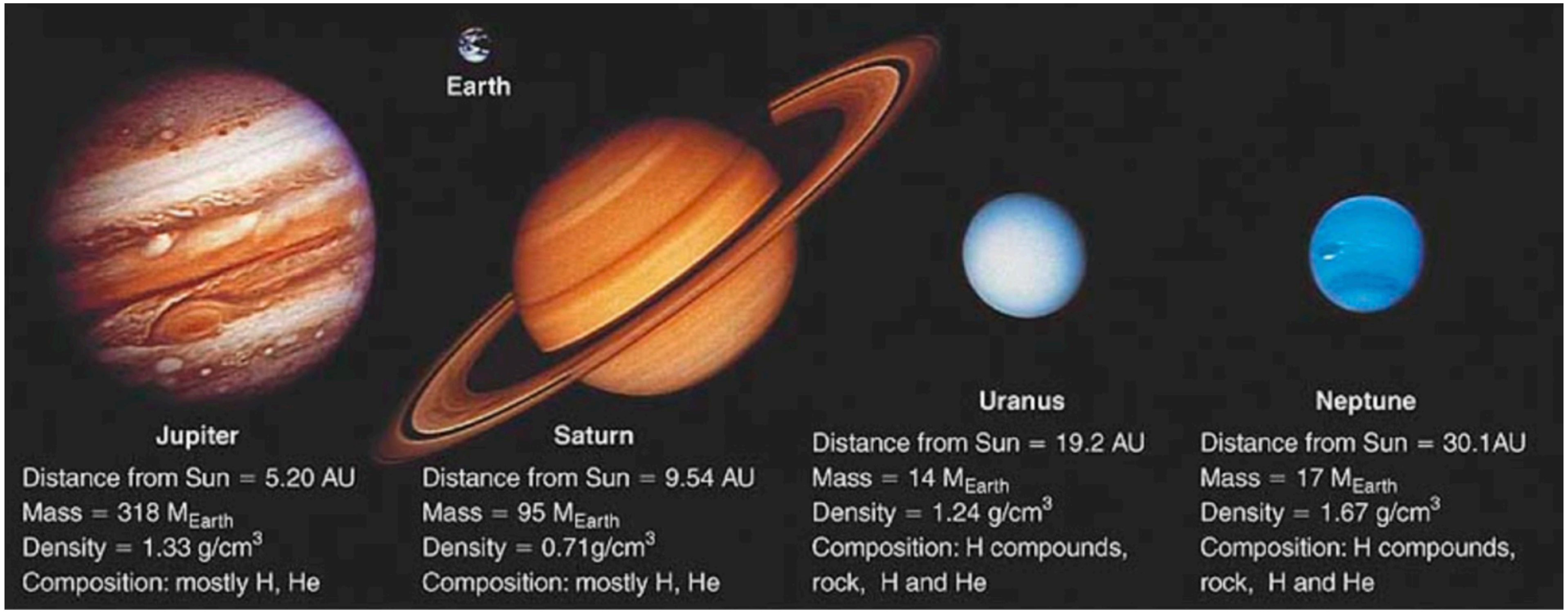
Neptune
Distance from Sun = 30.1AU
Mass = 17 M_{Earth}
Density = 1.67 g/cm^3
Composition: H compounds, rock, H and He

Giant (Jovian) Planets

Convective motions under influence of rotation drive zonal jets.



Cao & Stevenson 2017, Kaspi et al 2018



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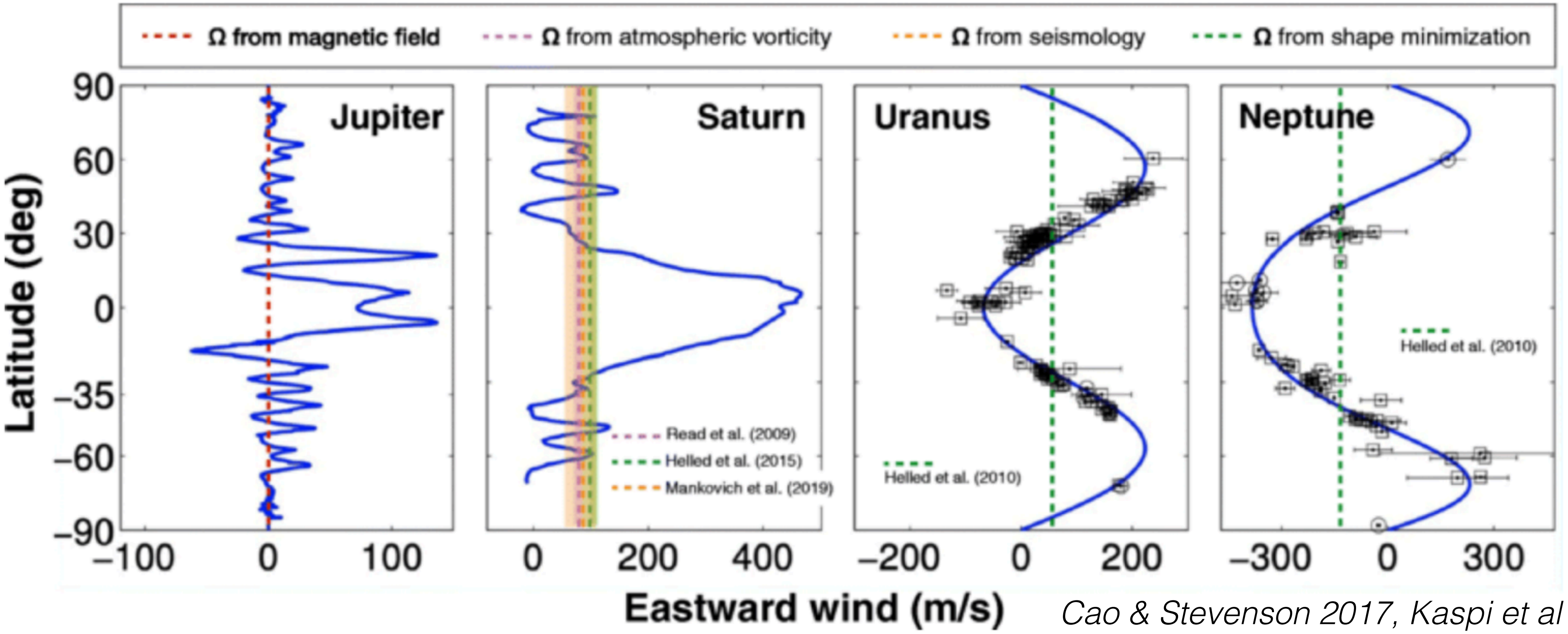
Giant (Jovian) Planets

Convective motions under influence of rotation drive zonal jets. JUNO mission- Jets extend O(1000 km) into interior

Kaspi et al , Nature 2018



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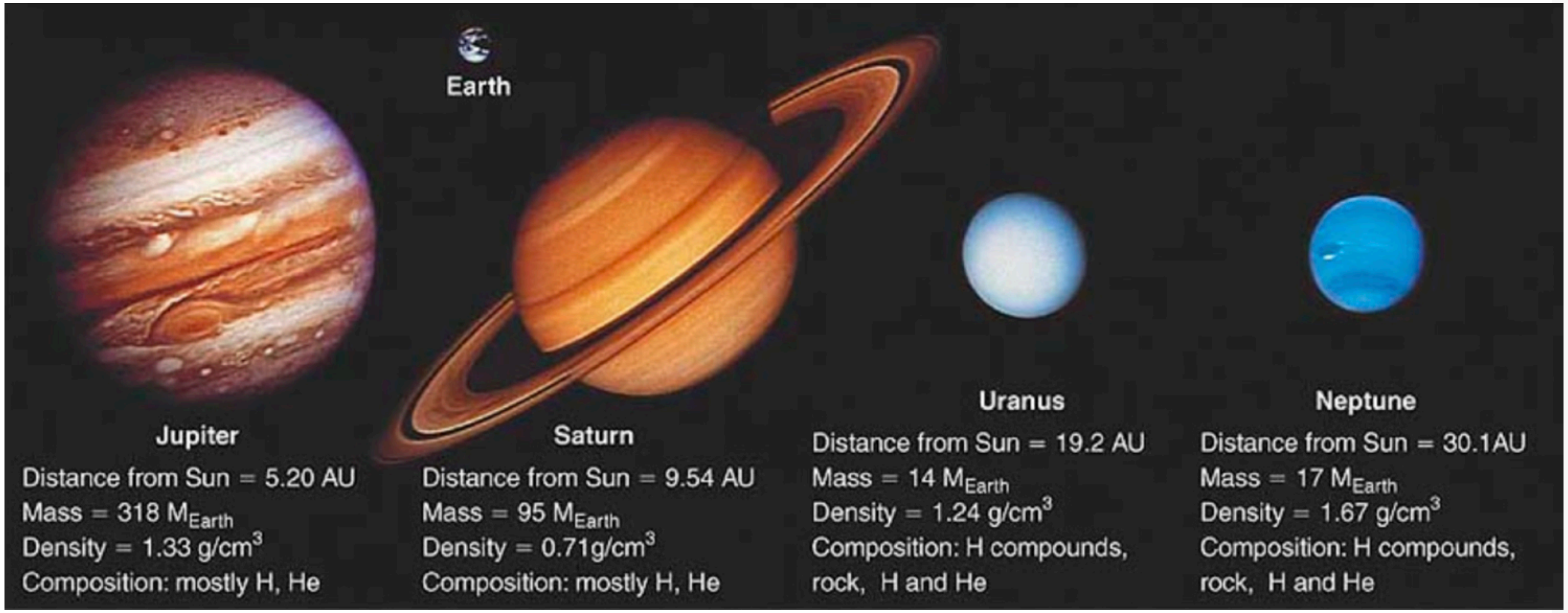

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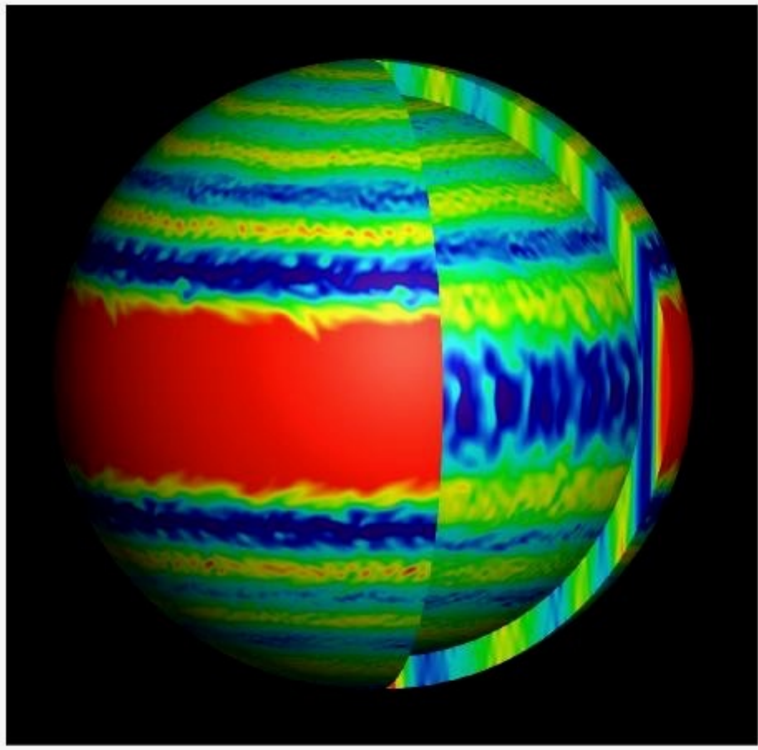
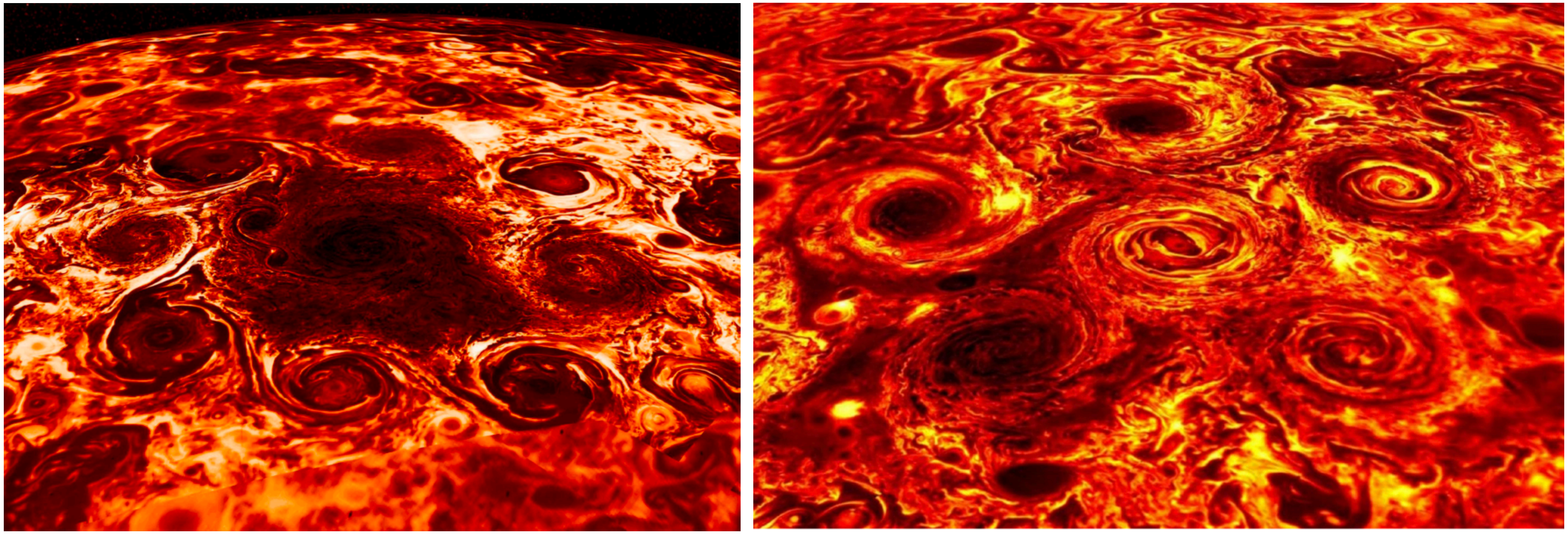
Observed in simulations



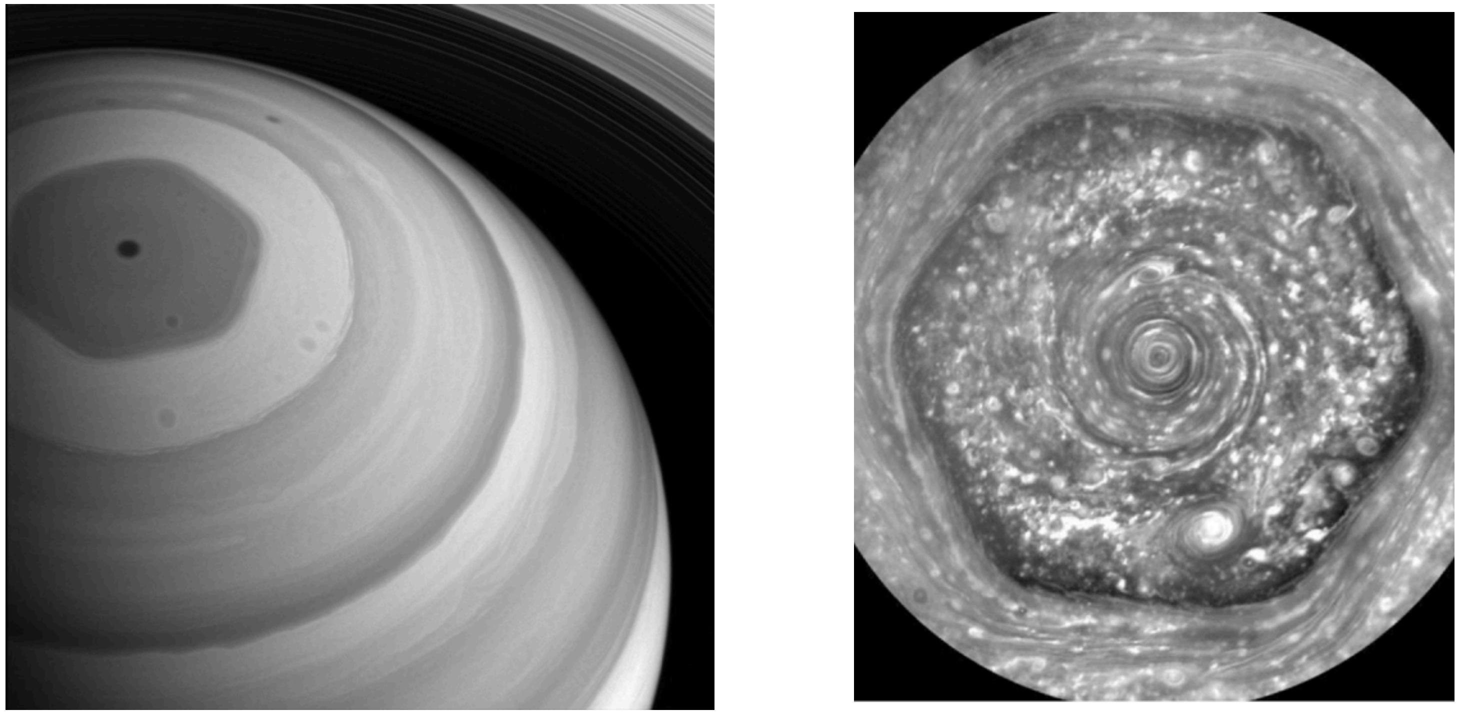
Heimpel, Aurnou & Wicht, Nature 2005

Giant (Jovian) Planets

JUNO observation: cyclonic vortical arrays at north and south poles



NASA/JPL-Caltech/SwRI/ASI/INAF/JIRAM)



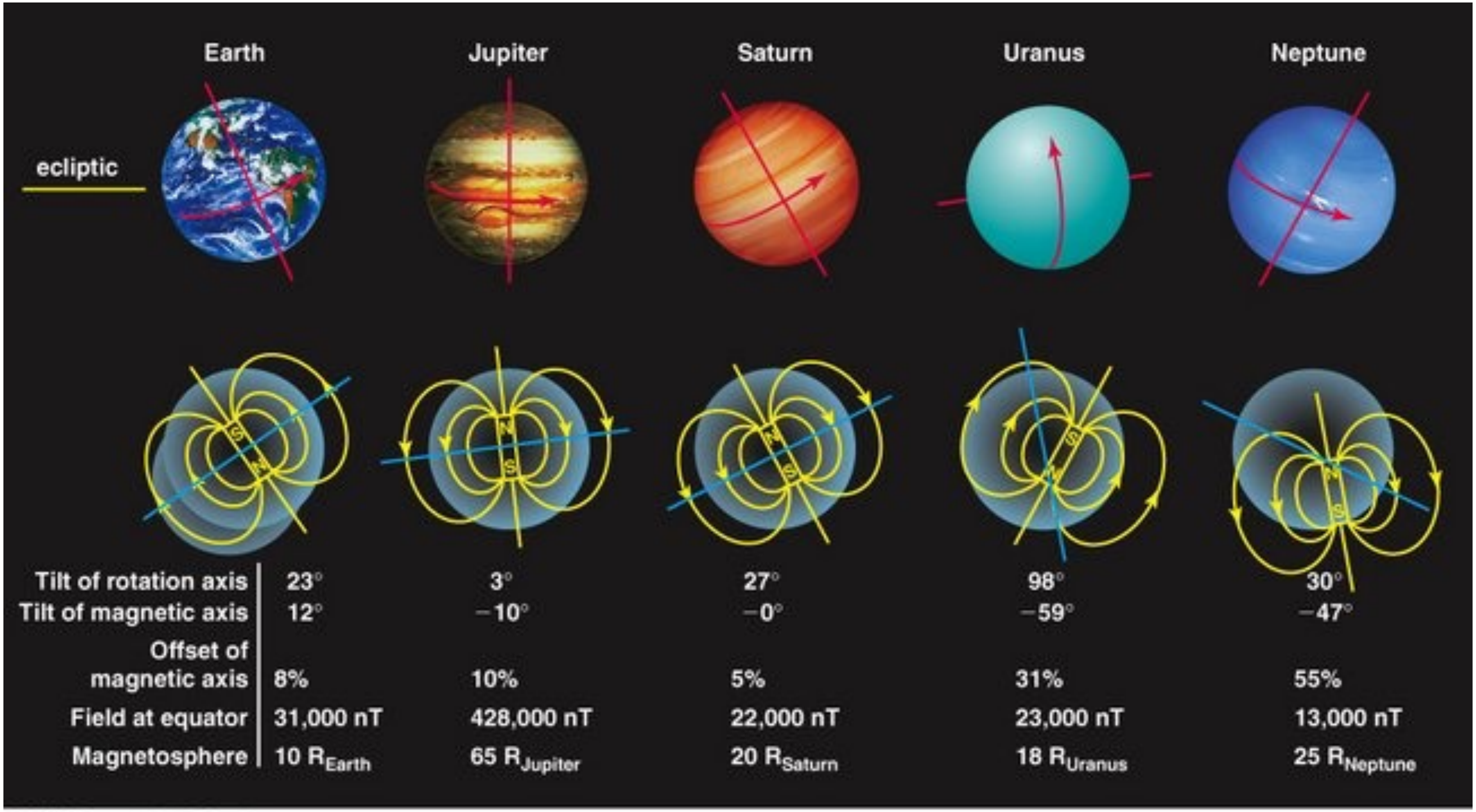
Cassini observation: Hexagonal structure of jets and vortices

Observation and theoretical investigations suggests Convective driving is the source.

Siegelman et al. PNAS, Nature Phys. 2022

Giant (Jovian) Planets

- Observations:
  - rotating convection primary driver for magnetic field generation



	Earth	Jupiter	Saturn	Uranus	Neptune
Tilt of rotation axis	23°	3°	27°	98°	30°
Tilt of magnetic axis	12°	−10°	−0°	−59°	−47°
Offset of magnetic axis	8%	10%	5%	31%	55%
Field at equator	31,000 nT	428,000 nT	22,000 nT	23,000 nT	13,000 nT
Magnetosphere	$10\ R_{Earth}$	$65\ R_{Jupiter}$	$20\ R_{Saturn}$	$18\ R_{Uranus}$	$25\ R_{Neptune}$

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Earth

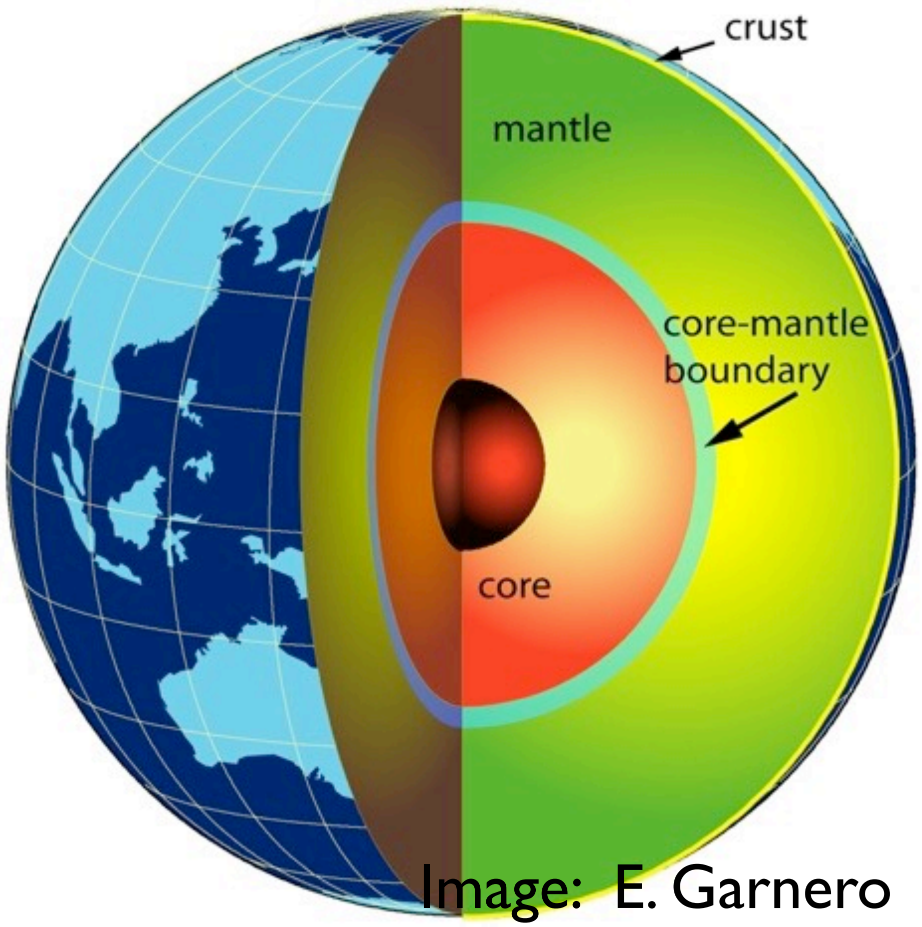
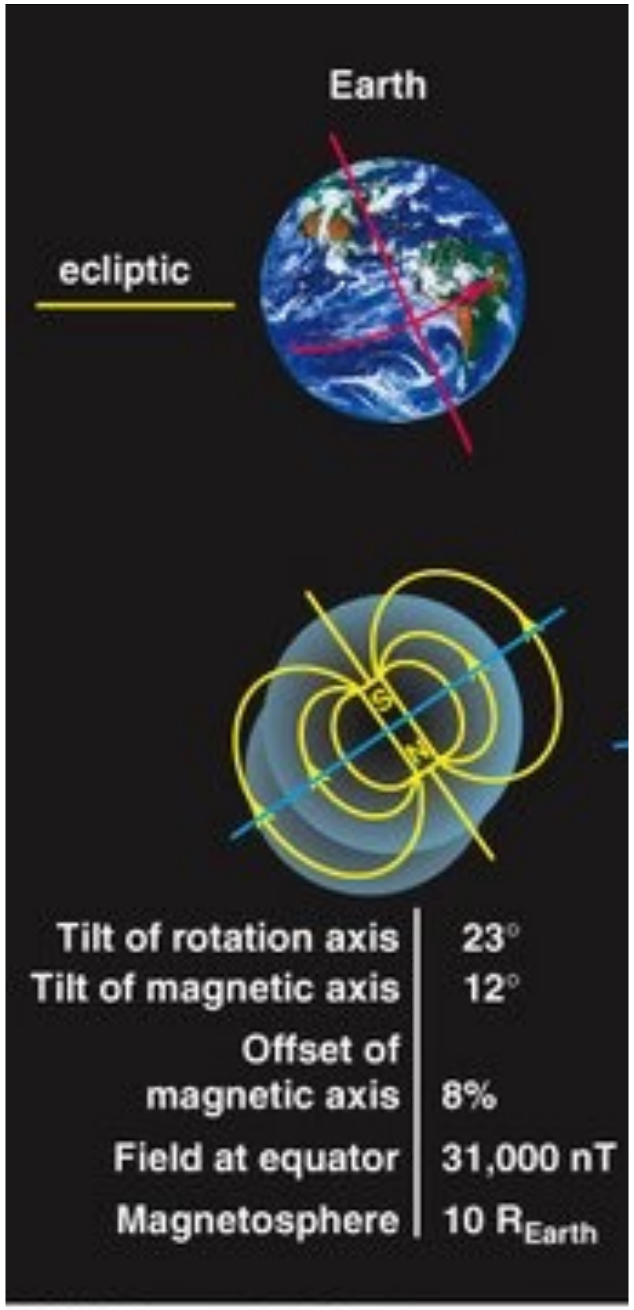


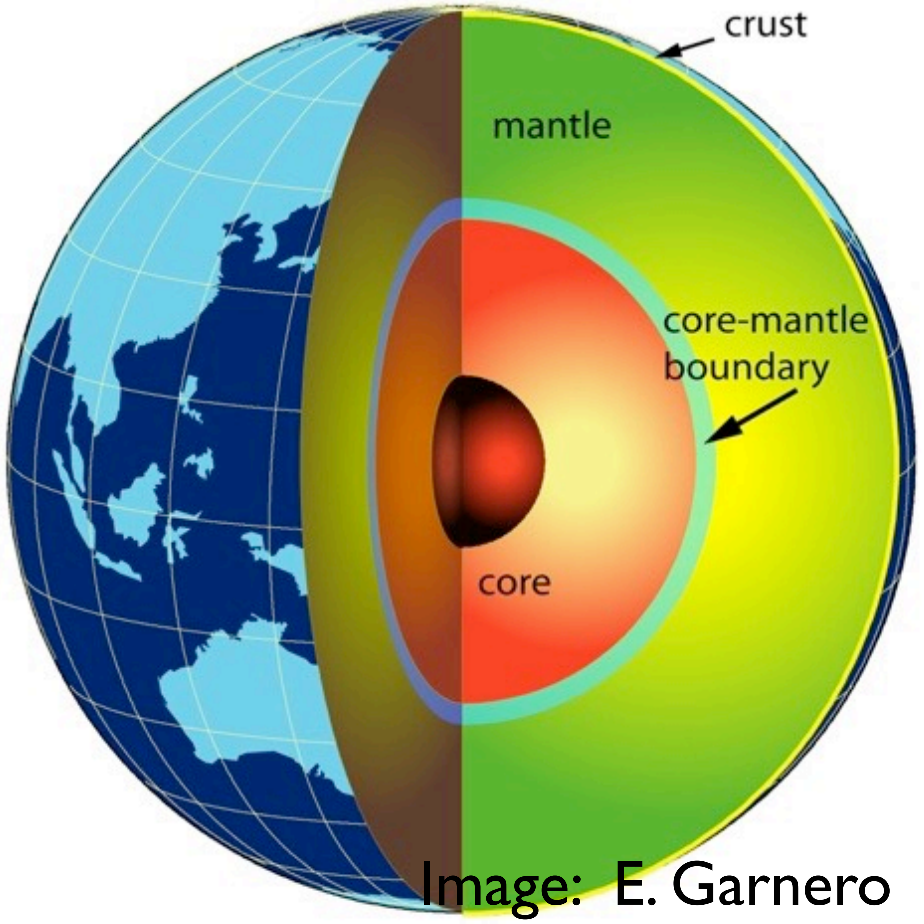
Image: E. Garnero



Tilt of rotation axis	23°
Tilt of magnetic axis	12°
Offset of magnetic axis	8%
Field at equator	31,000 nT
Magnetosphere	10 R_{Earth}

Outer liquid iron core - rotating convective motions sustain dynamo action

Earth



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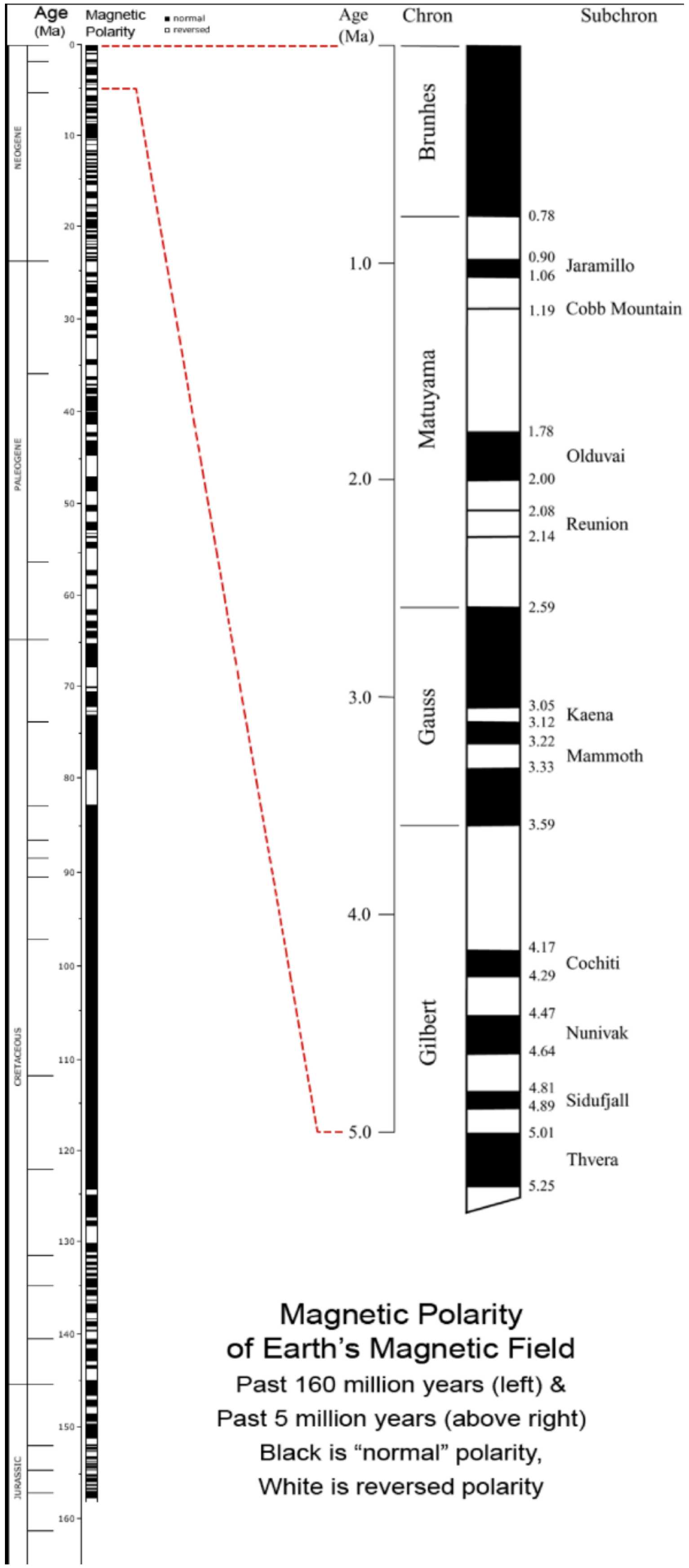
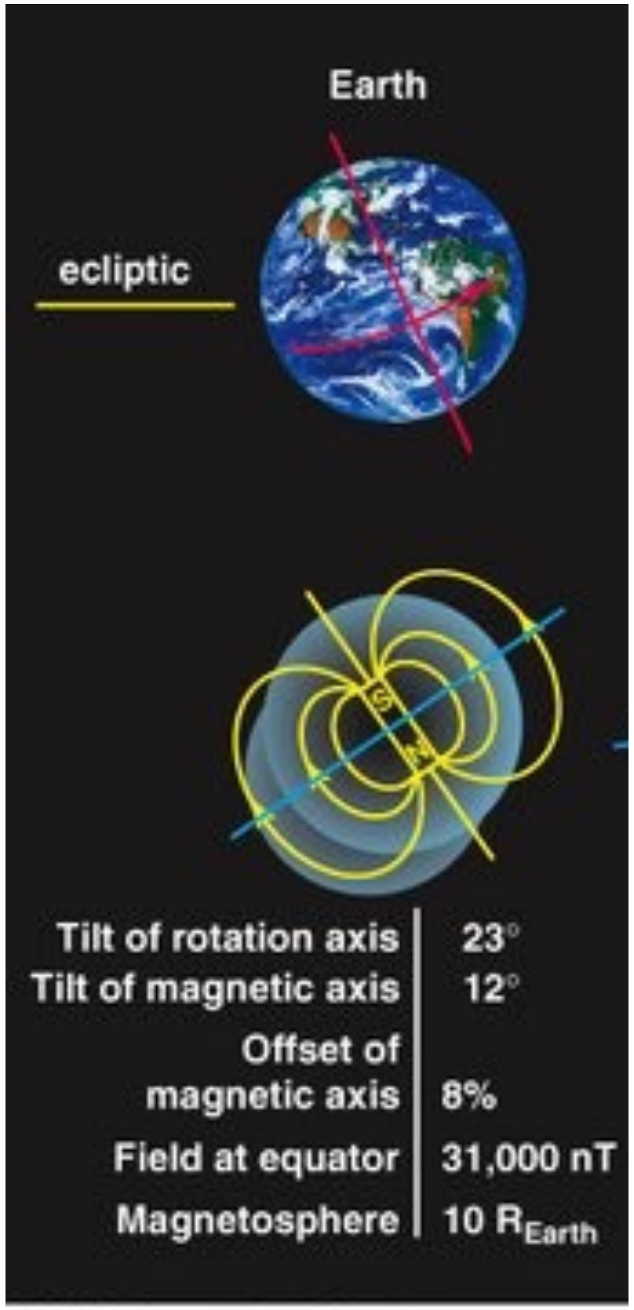


Magnetic Polarity
of Earth's Magnetic Field
Past 160 million years (left) &
Past 5 million years (above right)
Black is "normal" polarity,
White is reversed polarity



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Tilt of magnetic axis	12°
Offset of magnetic axis	8%
Field at equator	31,000 nT
Magnetosphere	10 R_{Earth}

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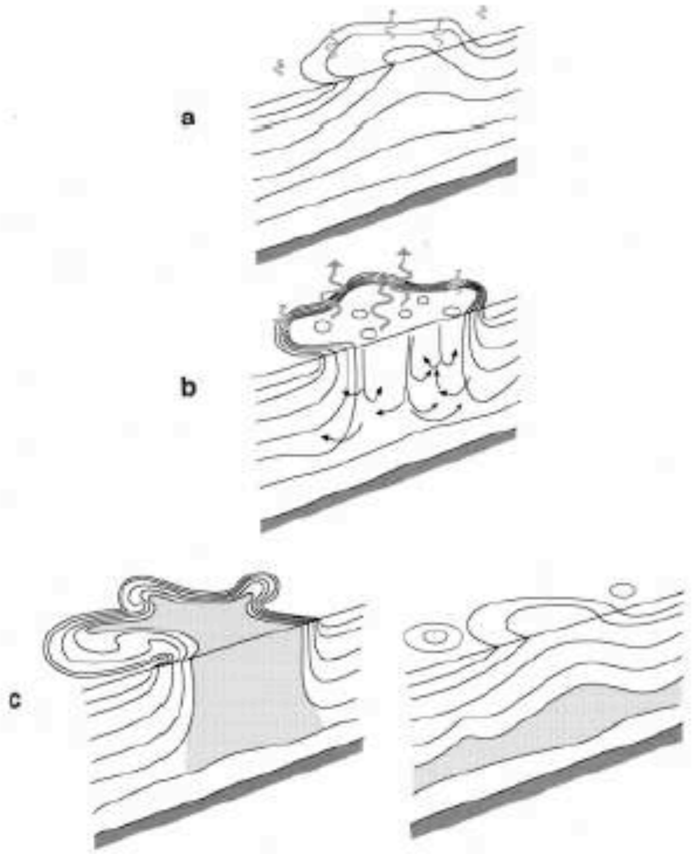
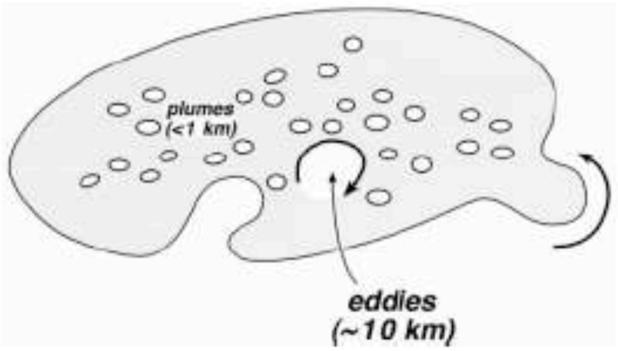
Outer liquid iron core - rotating convective motions sustain dynamo action, choatic reversals of 100K yr timescale

Phases of Open Ocean Deep Convection

(Marshall and Schott, JGR 1999)



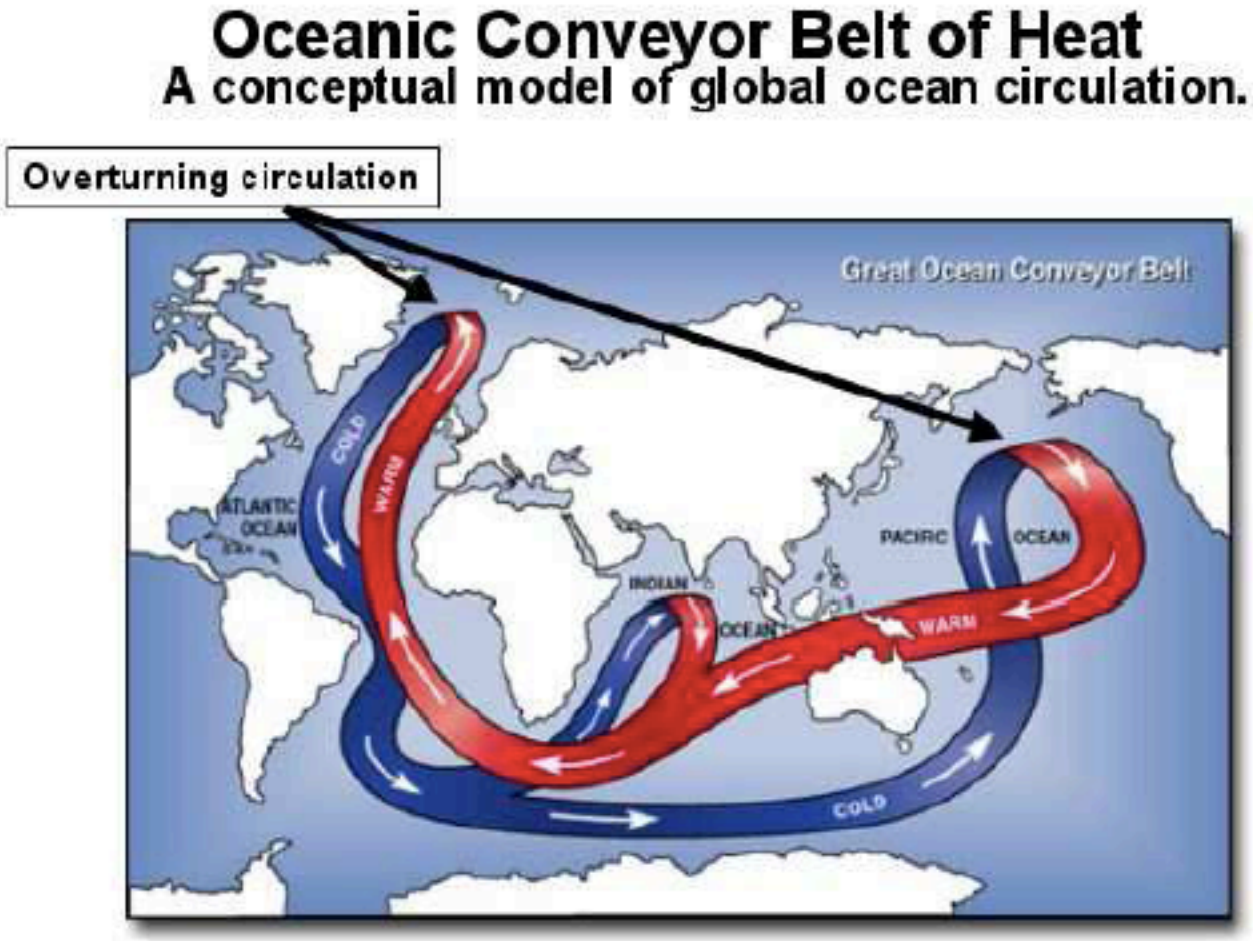


- preconditioning
  - cyclonic gyre domes isopycnals, $L \sim 100$km
- deep convection
  - cooling events trigger deep plumes, $L \lesssim 1$km $H \sim 2$km, $U \lesssim 10$ cm/s
- lateral exchange
  - geostrophic eddies, $L \sim 10$km
- influenced by rotation
  - natural Rossby number $Ro^* \sim 0.1 - 0.4$

$$Ro^* = \frac{L_{rot}}{H} = \left(\frac{B}{f^3 H^2}\right)^{1/2}$$

Meridional Overtuning Circulation

- density driven vertical overtuning
  - timescale: 1000yrs
- one contribution:
  - small scale deep convection
    - timescale: $\sim$ days
  - requires parameterization

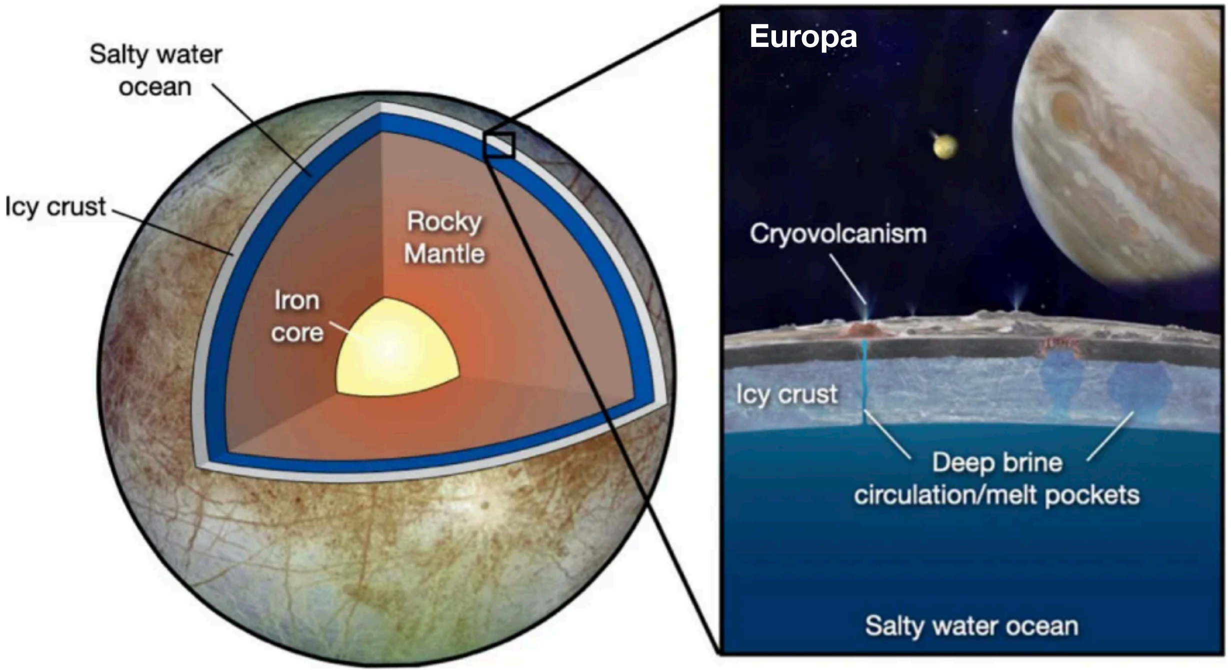


Oceanic Conveyor Belt of Heat

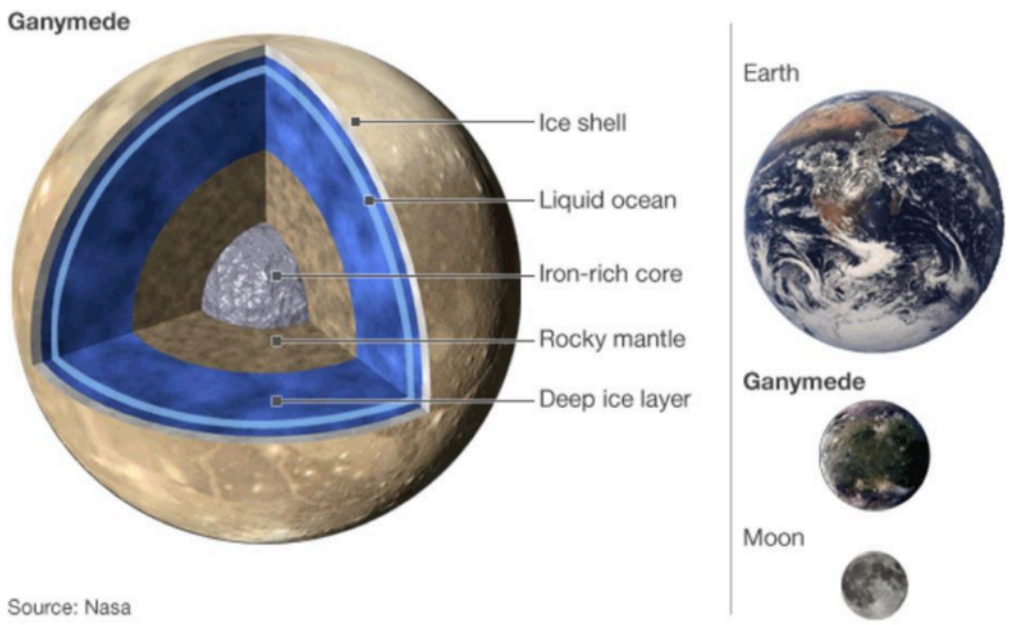
A conceptual model of global ocean circulation.

W.Broecker (Columbia U.); http://www.anl.gov/Media_Center/Frontiers/2003/

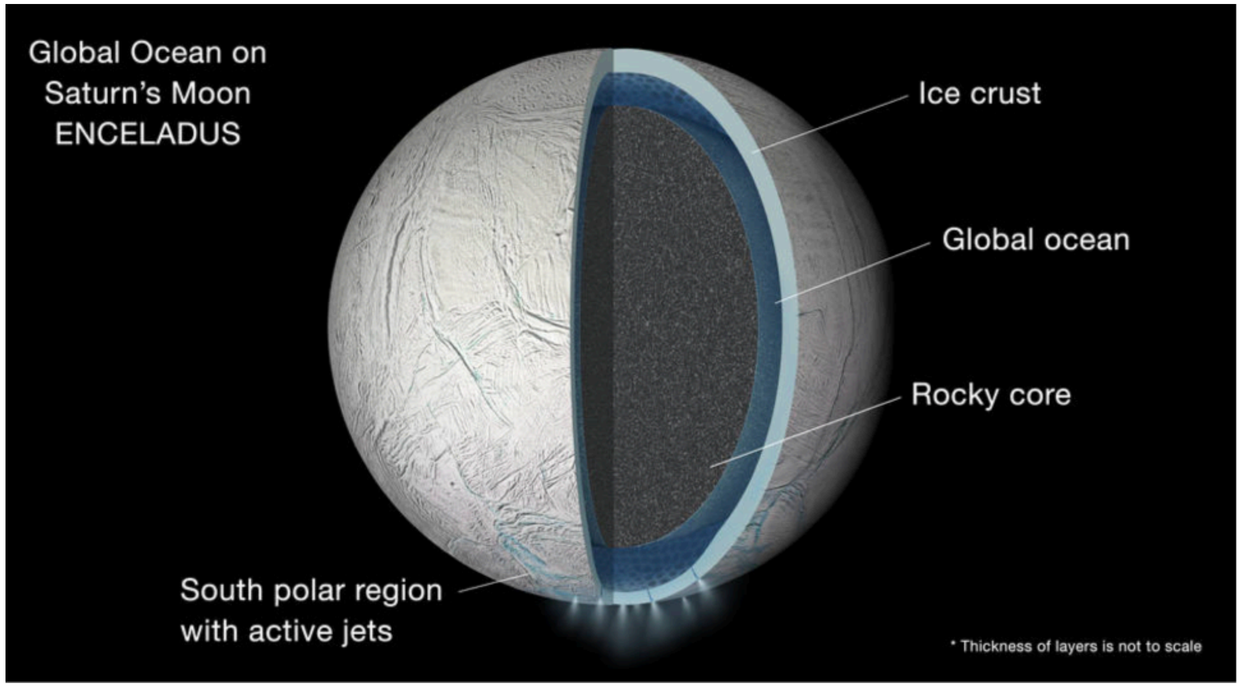
Subsurface off-world oceans



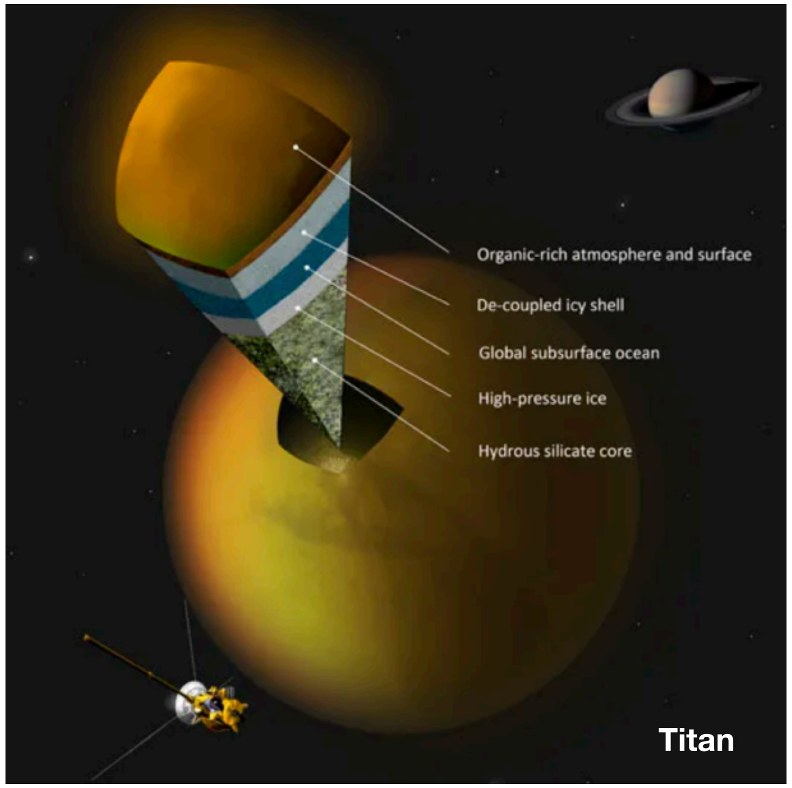
120 km/1561 km



300 km/2631 km



40 km/252 km



300 km/2578 km

Magnetic measurements - existence of subsurface oceans O(100km) deep on two moons of Jupiter/Saturn.

Fluids motions are buoyantly forced - rotation important to dynamics.

Soderlund et al. GRL2019, Nature Geo. 2013

Navier-Stokes Equation

inertia *pressure* *viscous* *body force*

Cons. Mtm.

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{\mathcal{F}}$$

Cons. Mass

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = -\rho \nabla \cdot \boldsymbol{u}$$

Navier-Stokes Equation

inertia *pressure* *viscous* *body force*

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Incompressible fluid motions

Navier-Stokes Equation

inertia *pressure* *viscous* *body force*

$$St\,\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -Eu\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{u} + \boldsymbol{\mathcal{F}}$$

$\frac{L}{U\mathcal{T}}$ *Strouhal*

$\frac{p}{\rho_0 U^2}$ *Euler*

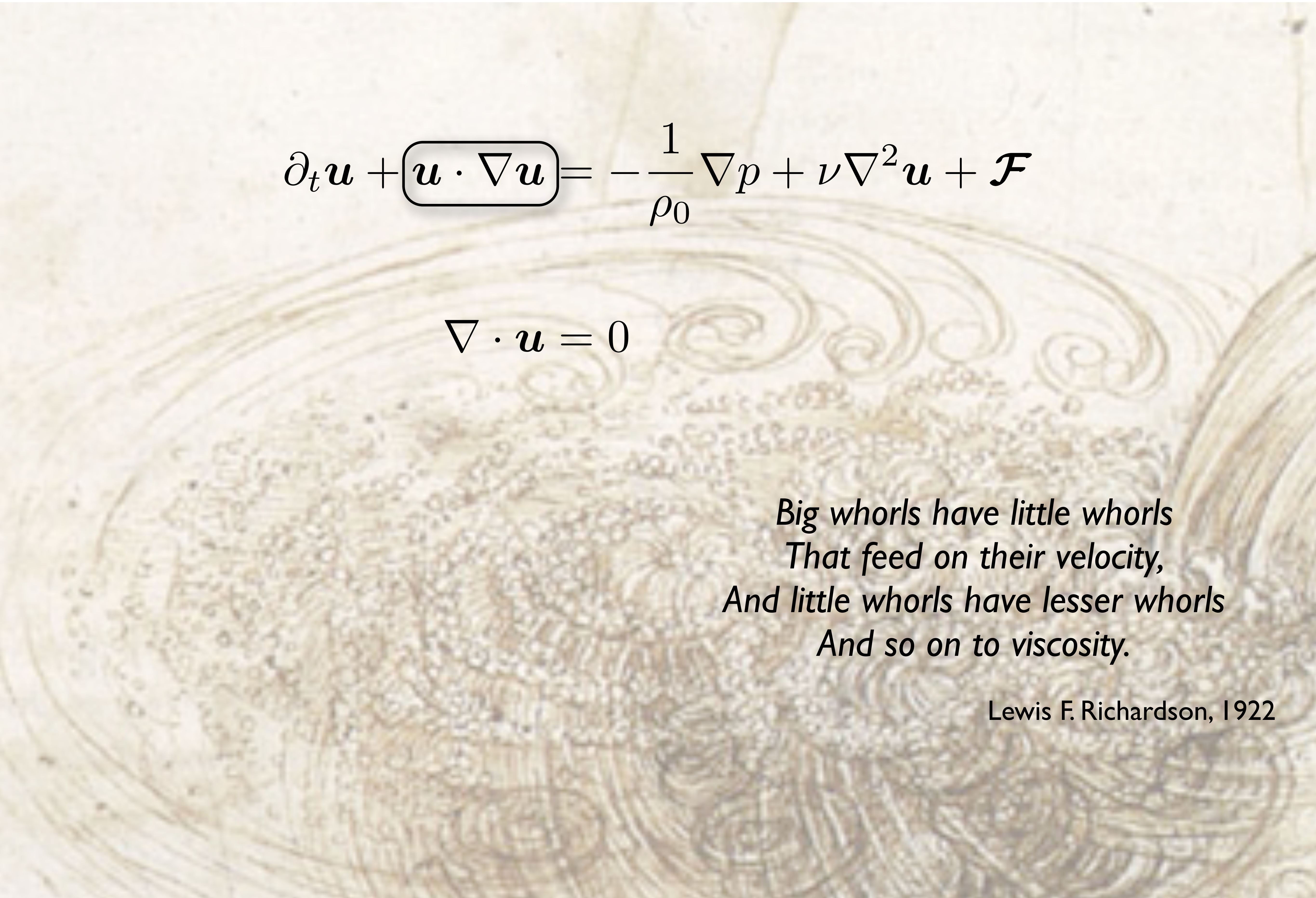
$\frac{UL}{\nu}$ *Reynolds*

$$\nabla\cdot\boldsymbol{u} = 0$$

incompressibility

Generic nondimensionalization $U,\ L,\ \mathcal{T},\ P$

Nonlinear Energy Cascade



$$\partial_t \boldsymbol{u} + \boxed{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{\mathcal{F}}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.

Lewis F. Richardson, 1922

Nonlinear Energy Cascade

$$\partial_t \boldsymbol{u} + \boxed{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{\mathcal{F}}$$

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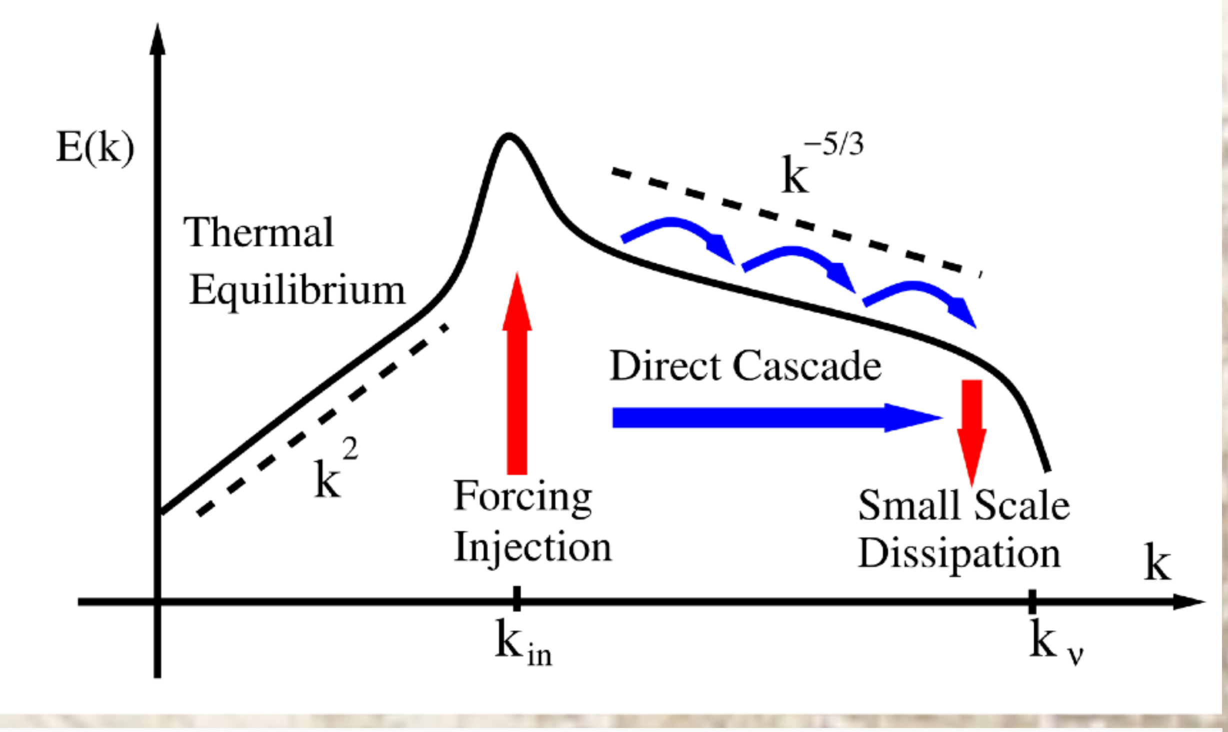

$$\langle E \rangle = \frac{1}{V} \iiint \frac{1}{2} (\mathbf{u} \cdot \mathbf{u}) \, dV = \int_0^\infty E(k) dk.$$

Big whorls have little whorls
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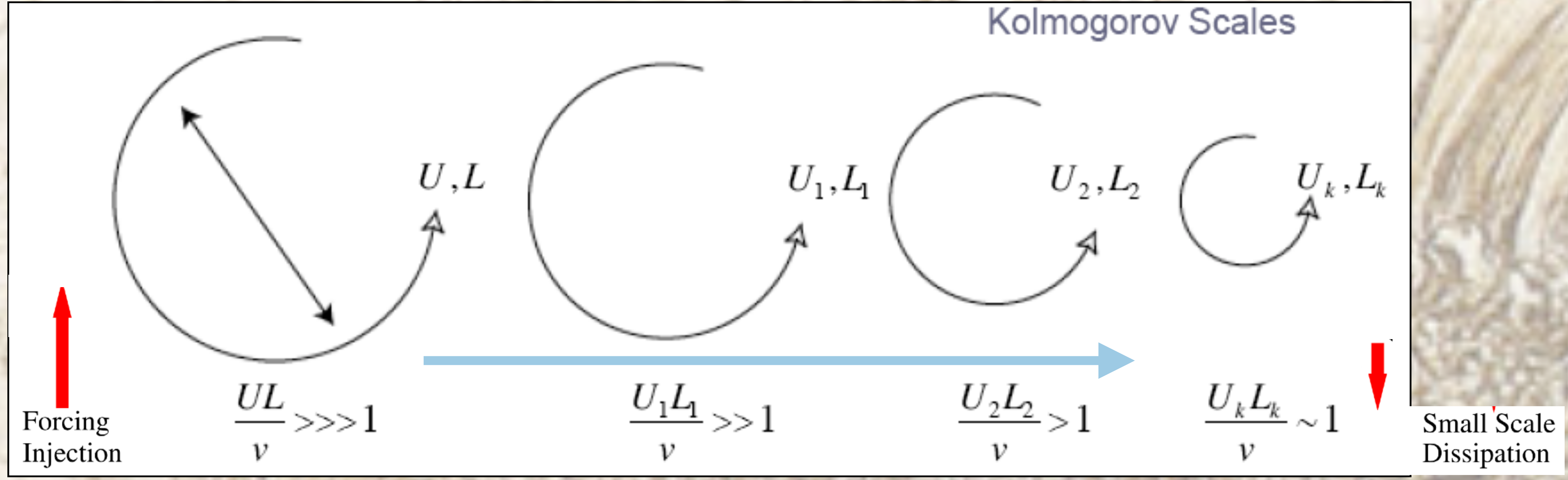
Viscosity is unimportant
Energy flux ε is conserved



$$\varepsilon_I \propto \frac{U^2}{\mathcal{T}} \sim \frac{U^3}{L}$$

Energy injection rate

$$\varepsilon \sim \varepsilon_I$$

$$\varepsilon \sim \varepsilon_d \propto \frac{U^3}{L}$$

Energy dissipation rate

Nonlinear Energy Cascade

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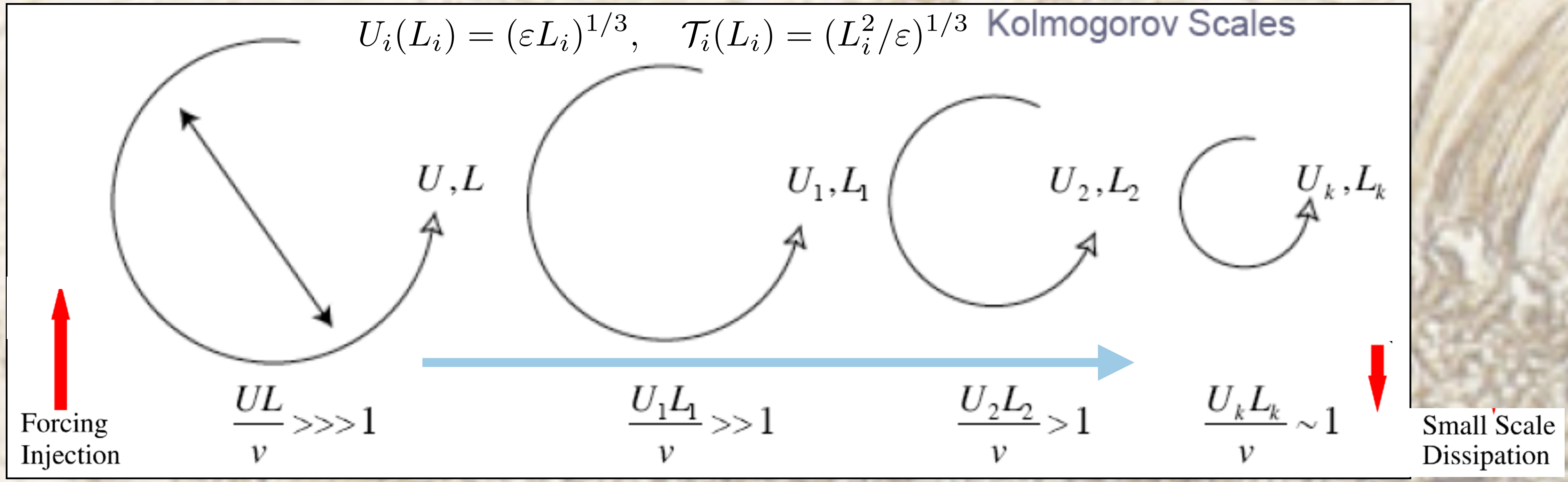

$U_i(L_i) = (\varepsilon L_i)^{1/3}, \quad \mathcal{T}_i(L_i) = (L_i^2/\varepsilon)^{1/3}$ Kolmogorov Scales

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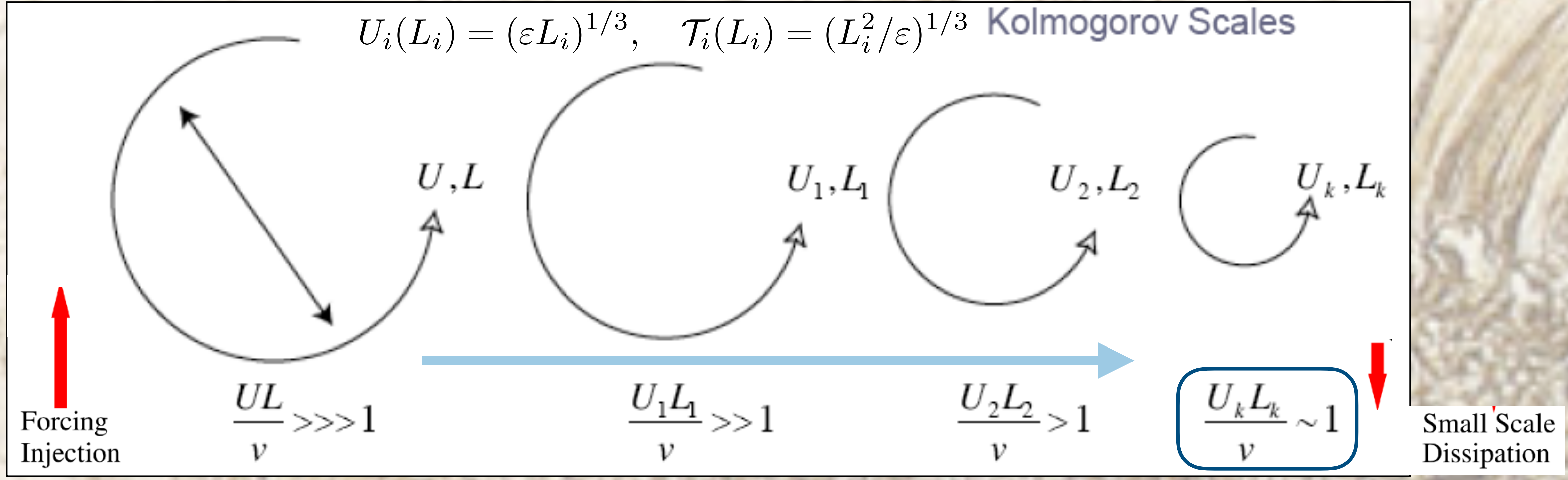

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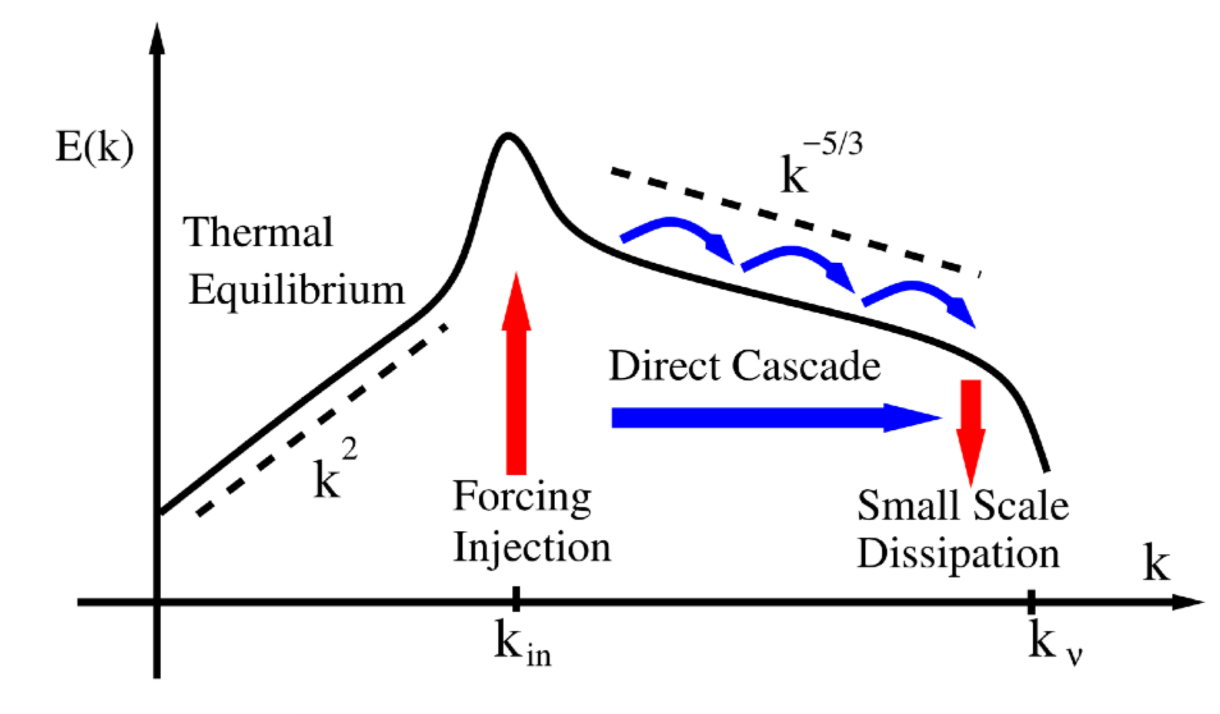

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Kolmogorov Scales

$$l_k = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$ Length

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Degrees of Freedom

$$L/l_k \sim Re^{3/4}$$

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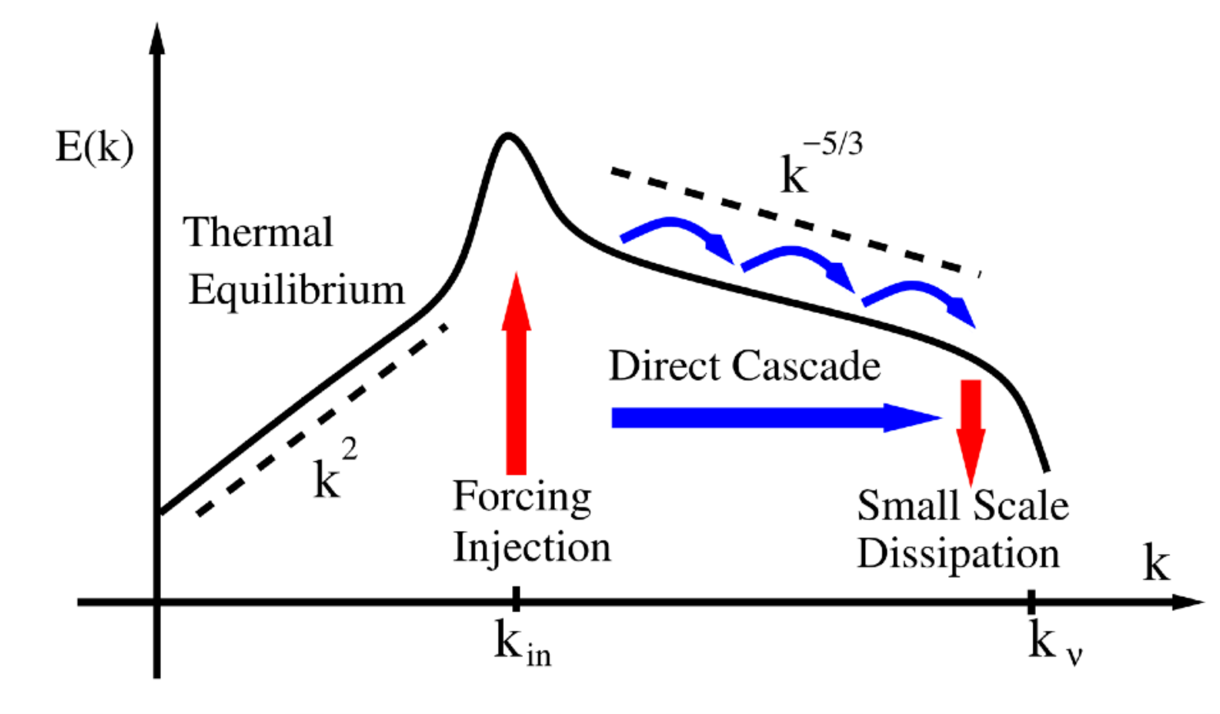
Nonlinear Energy Cascade



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From Re_I / Re_k

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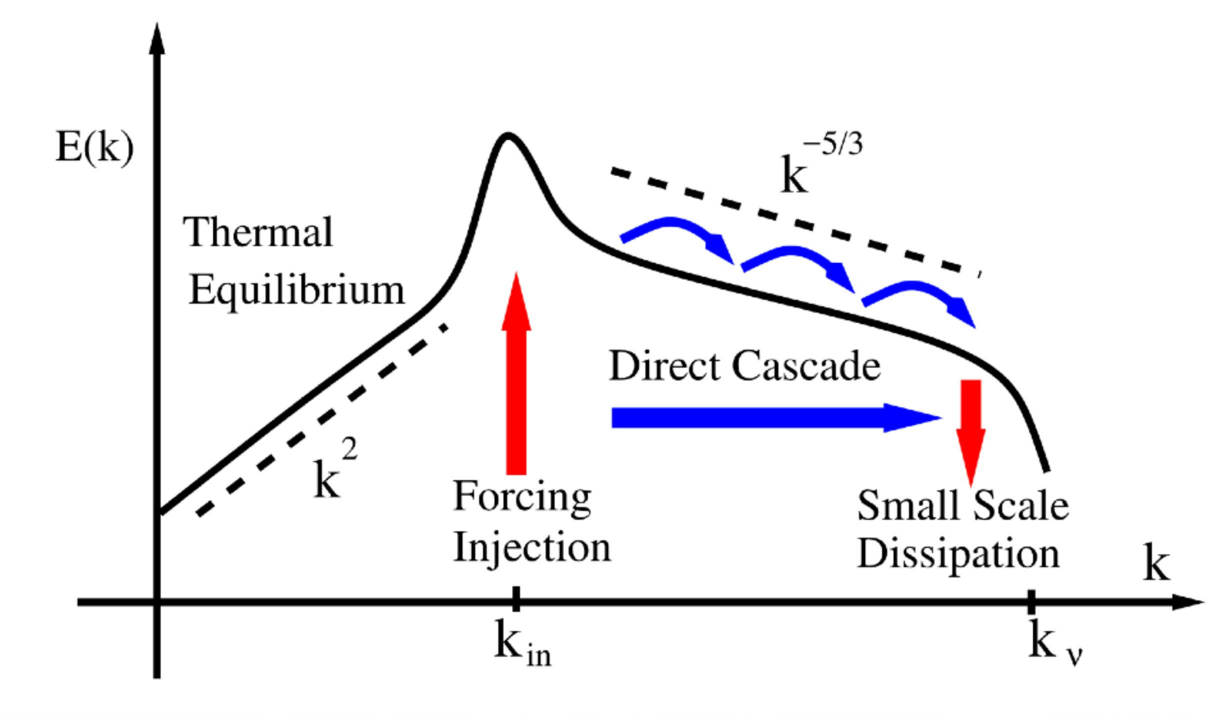

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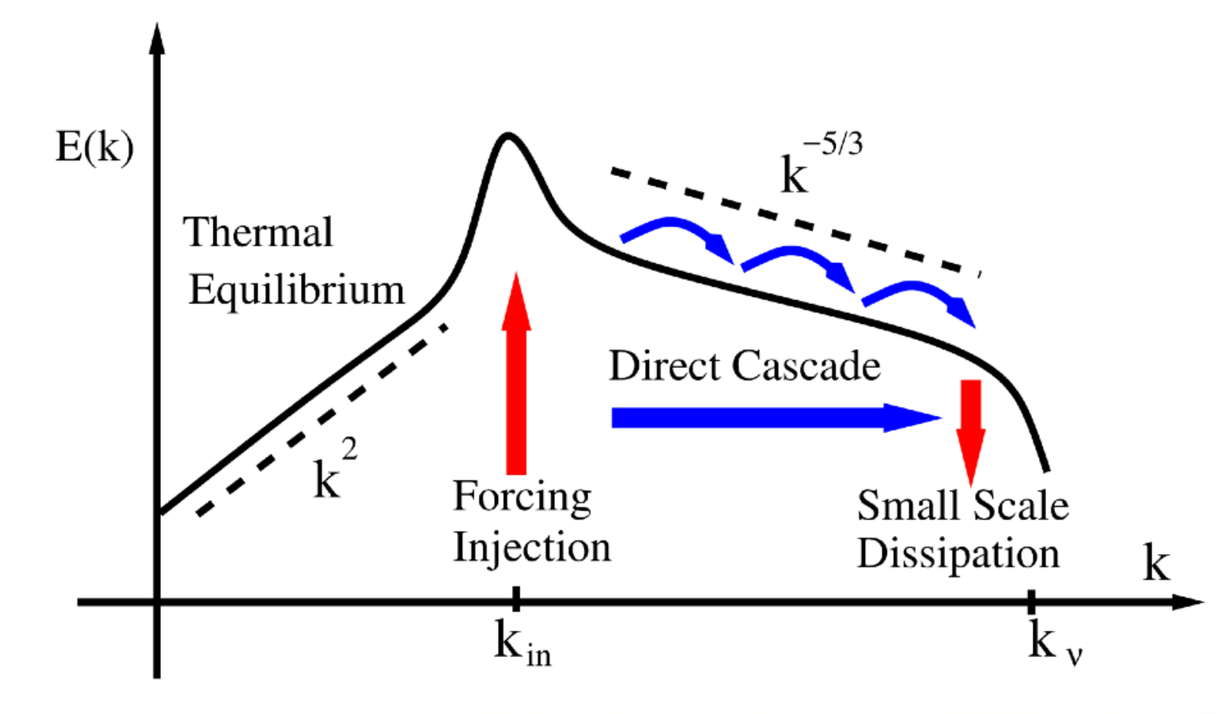
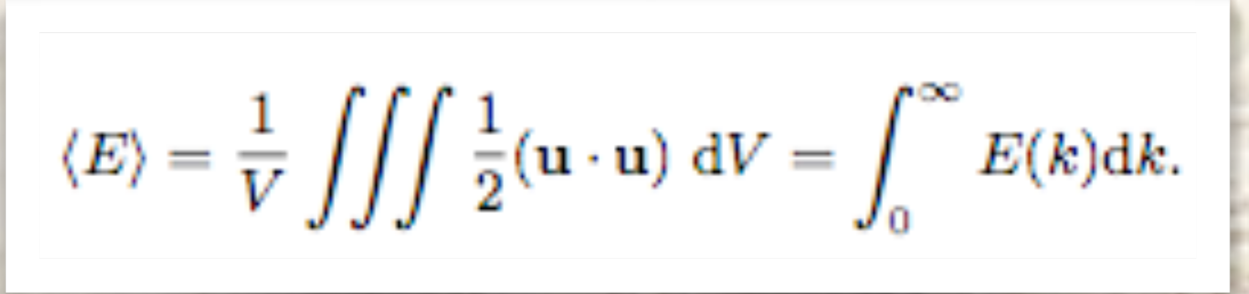
Kolmogorov scaling law:

Dimensional analysis of energy per unit mass per unit k

$$\frac{U^2}{k} \sim \frac{L^3}{\mathcal{T}^2} \sim \varepsilon^\alpha \left(\frac{1}{L}\right)^\beta = \left(\frac{L^3}{\mathcal{T}^3}\right)^{\overset{2/3}{\nearrow}\alpha} \left(\frac{1}{L}\right)^{\overset{-5/3}{\nearrow}\beta}$$

Nonlinear Energy Cascade





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Turbulence challenge $\Rightarrow N^3 \sim Re^{\frac{9}{4}}$

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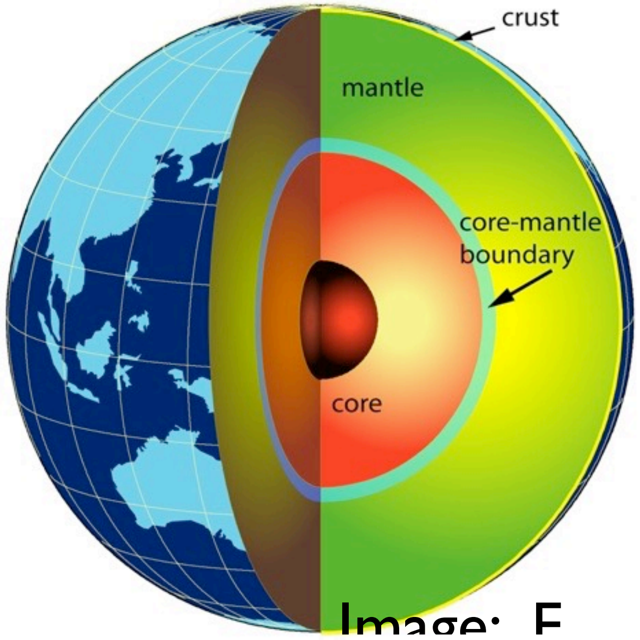
$\Omega \sim 7\times10^{-5}\ rad/s$

$L \sim 1{-}100\ km$



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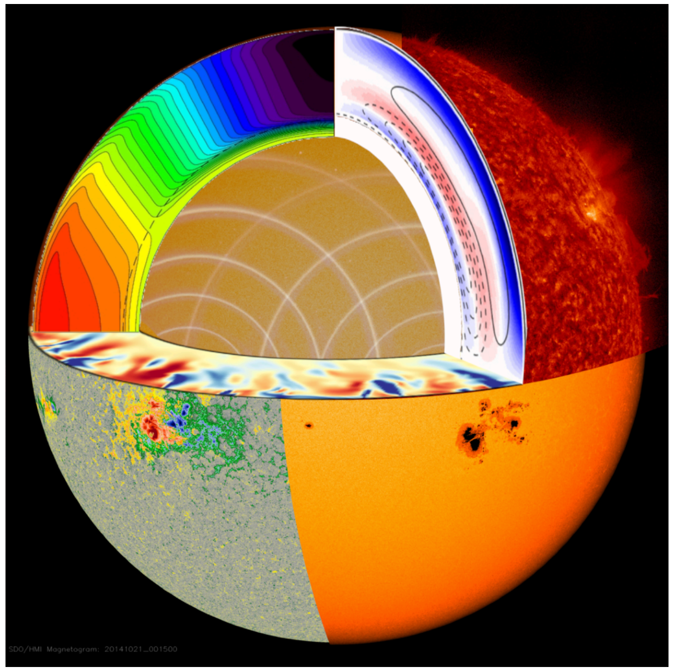

$U \sim 10 - 100 m/s$

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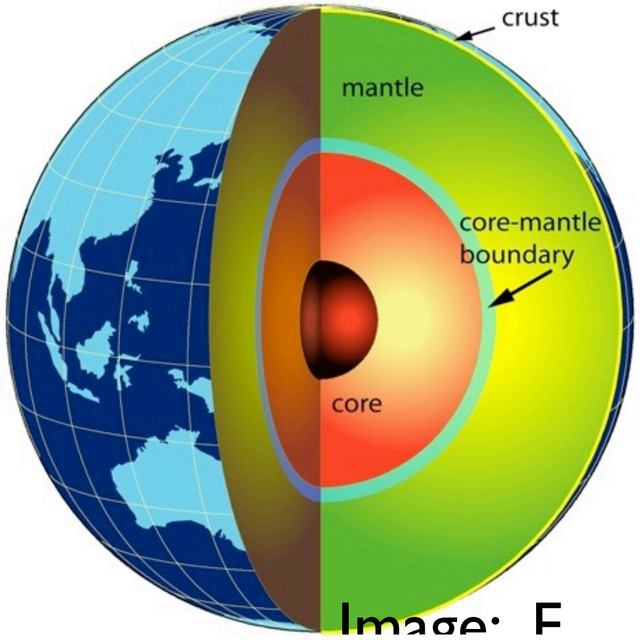

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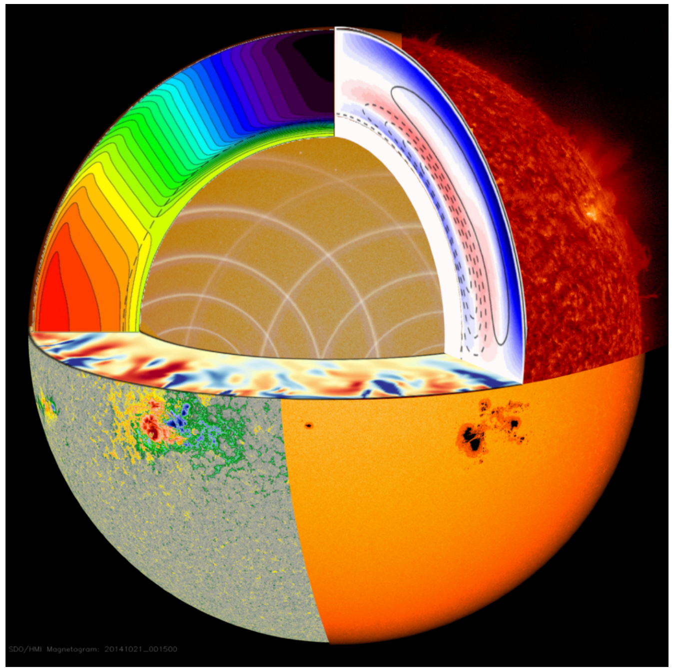

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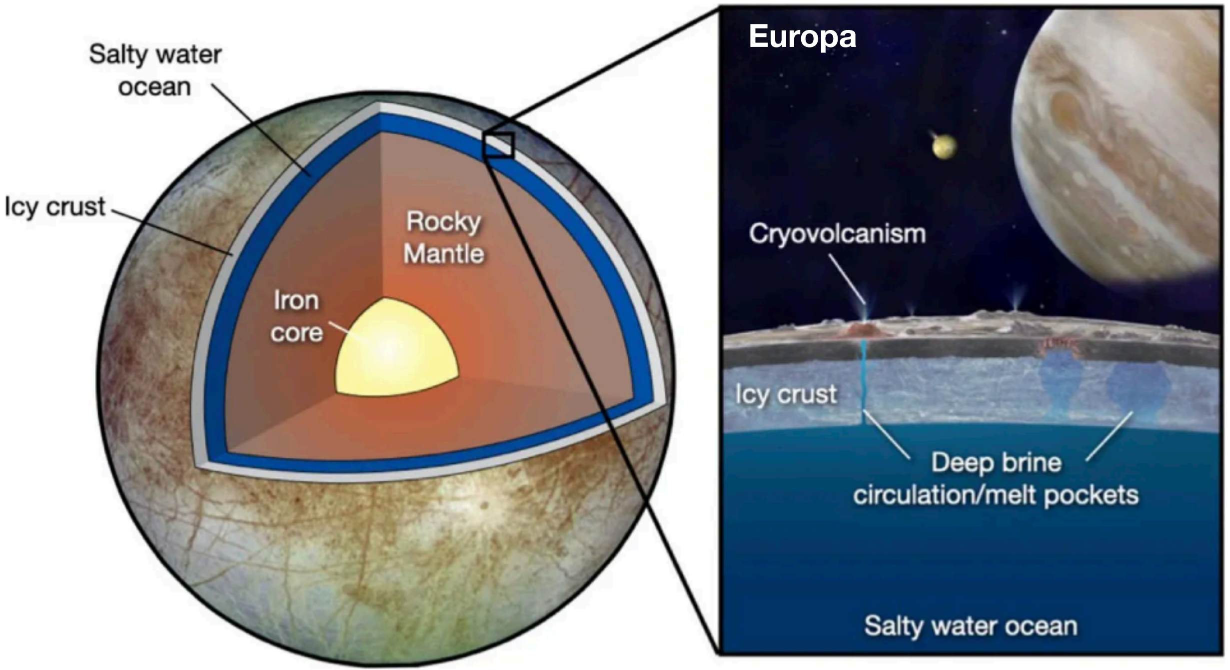
Subsurface off-world oceans

Soderlund et al. GRL2019, Nature Geo. 2013



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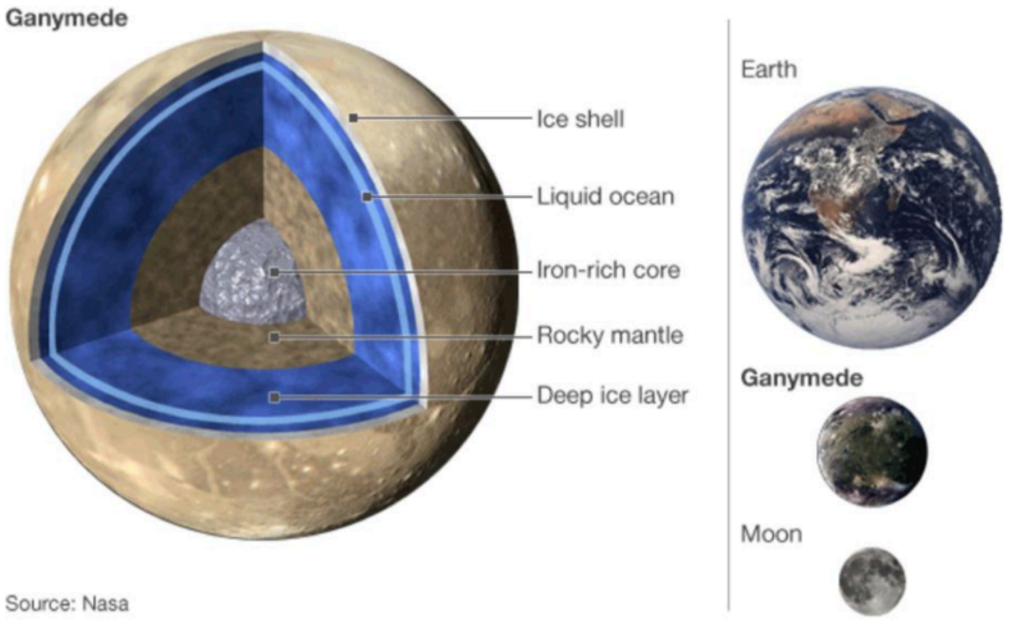

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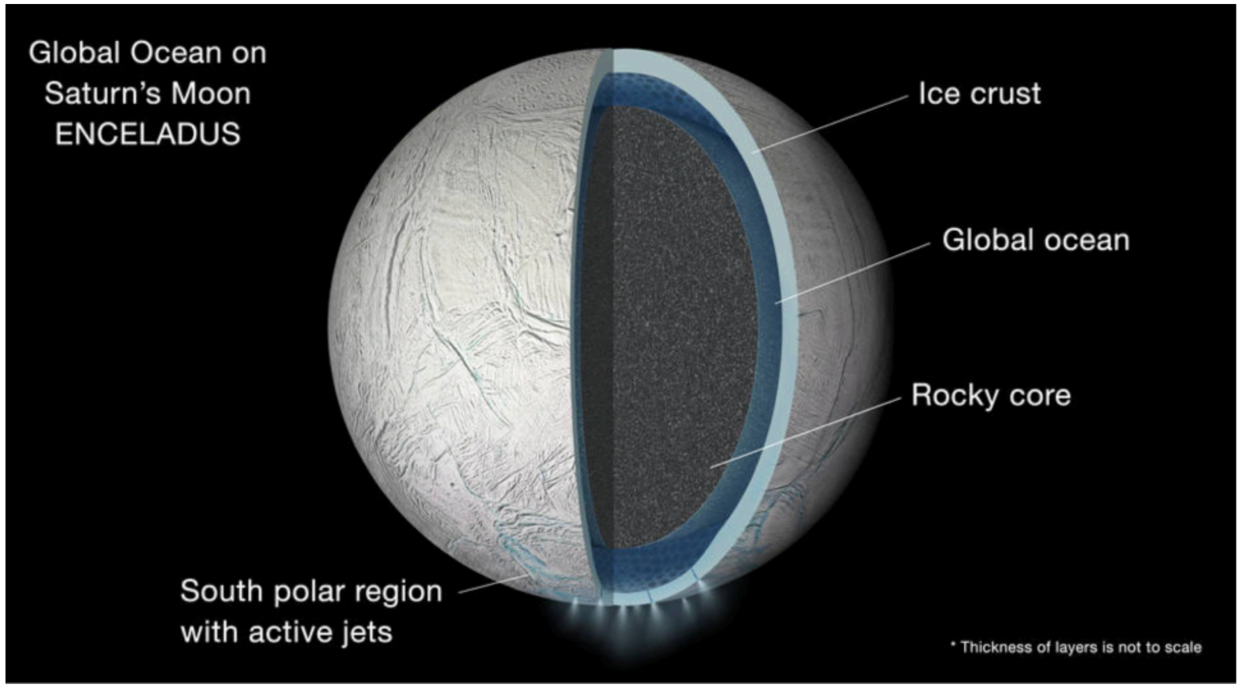

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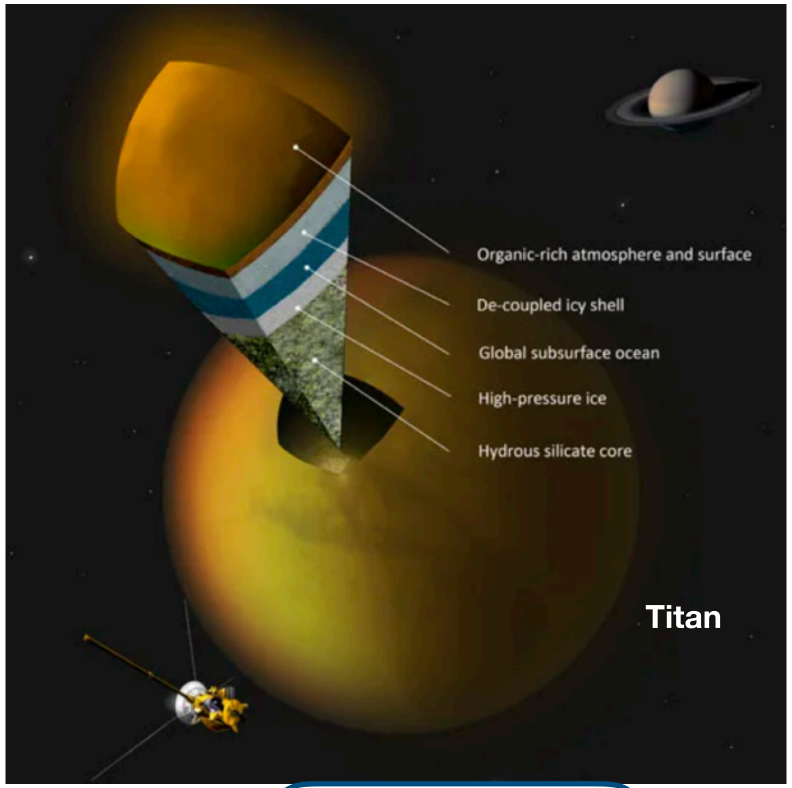

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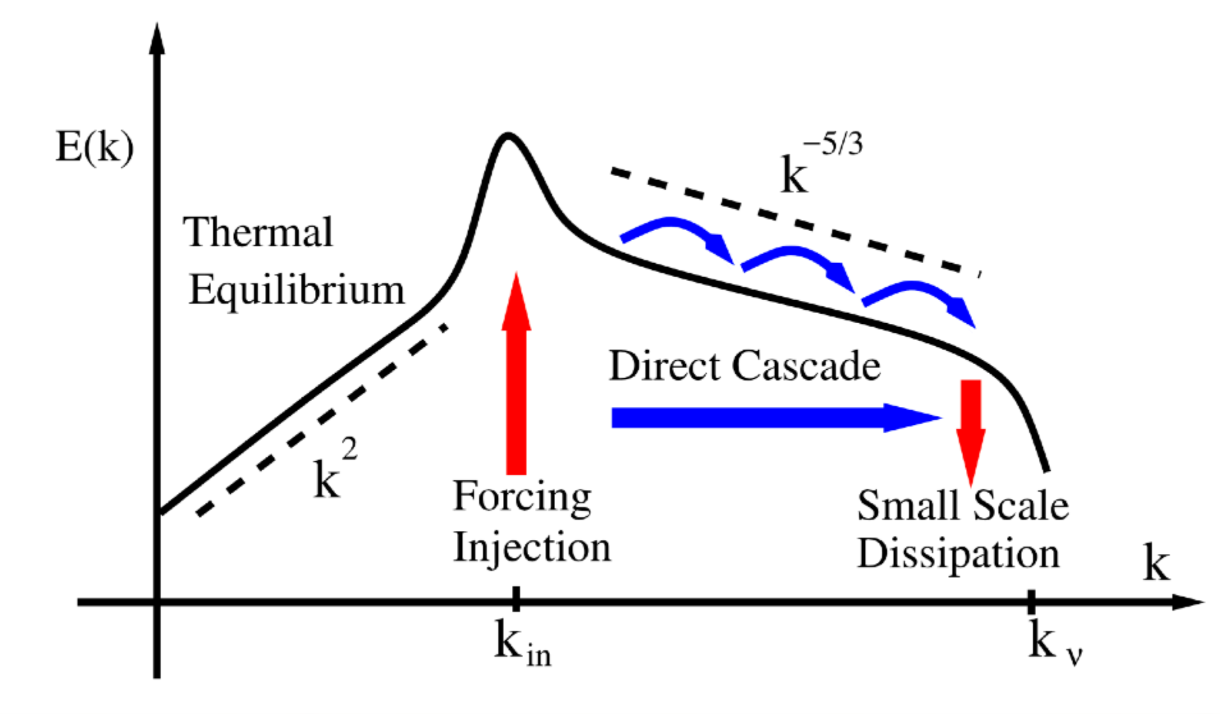

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Turbulence challenge $\Rightarrow N^3 \sim Re^{\frac{9}{4}}$

$(10^{6+})^3 \sim (10^{8+})^{\frac{9}{4}} \Rightarrow$ GAFD

$(10^3)^3 \sim (10^4)^{\frac{9}{4}} \Rightarrow$ Num Sim.

$$F_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}$$

Rotation/Coriolis Force

$$F_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}$$

Coriolis Force: pseudo-force that appears in a rotating frame of reference w/ fixed rotation

acts to deflect fluid parcels perpendicular to their direction of motion

$$\left[\frac{D}{Dt}\right]_i = \left[\frac{D}{Dt}\right]_r + \boldsymbol{\Omega}\times$$

$$\boldsymbol{r} : \quad \boldsymbol{u}_i = \boldsymbol{u}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\boldsymbol{u}_r : \quad [D_t \boldsymbol{u}_r]_i = [D_t \boldsymbol{u}_r]_r + \boldsymbol{\Omega} \times \boldsymbol{u}_r$$

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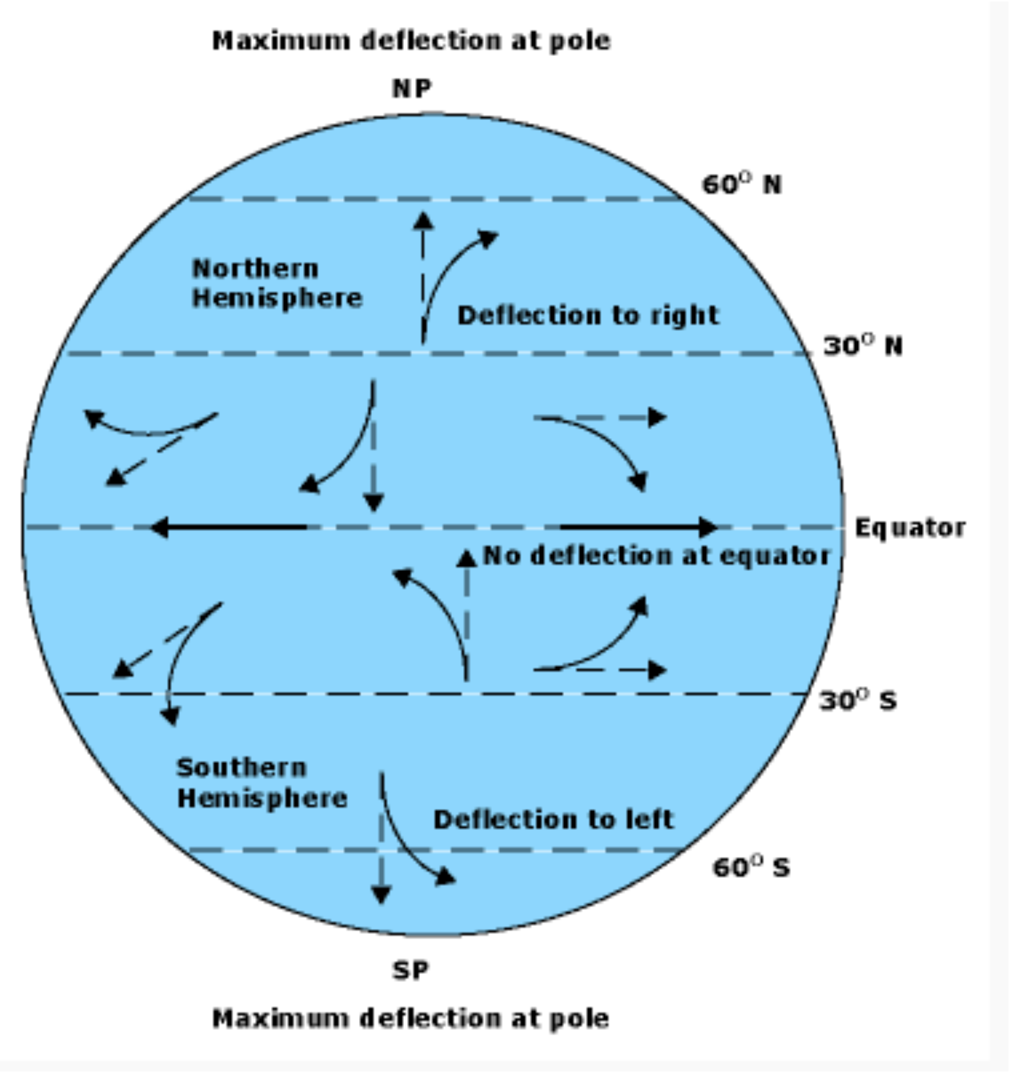

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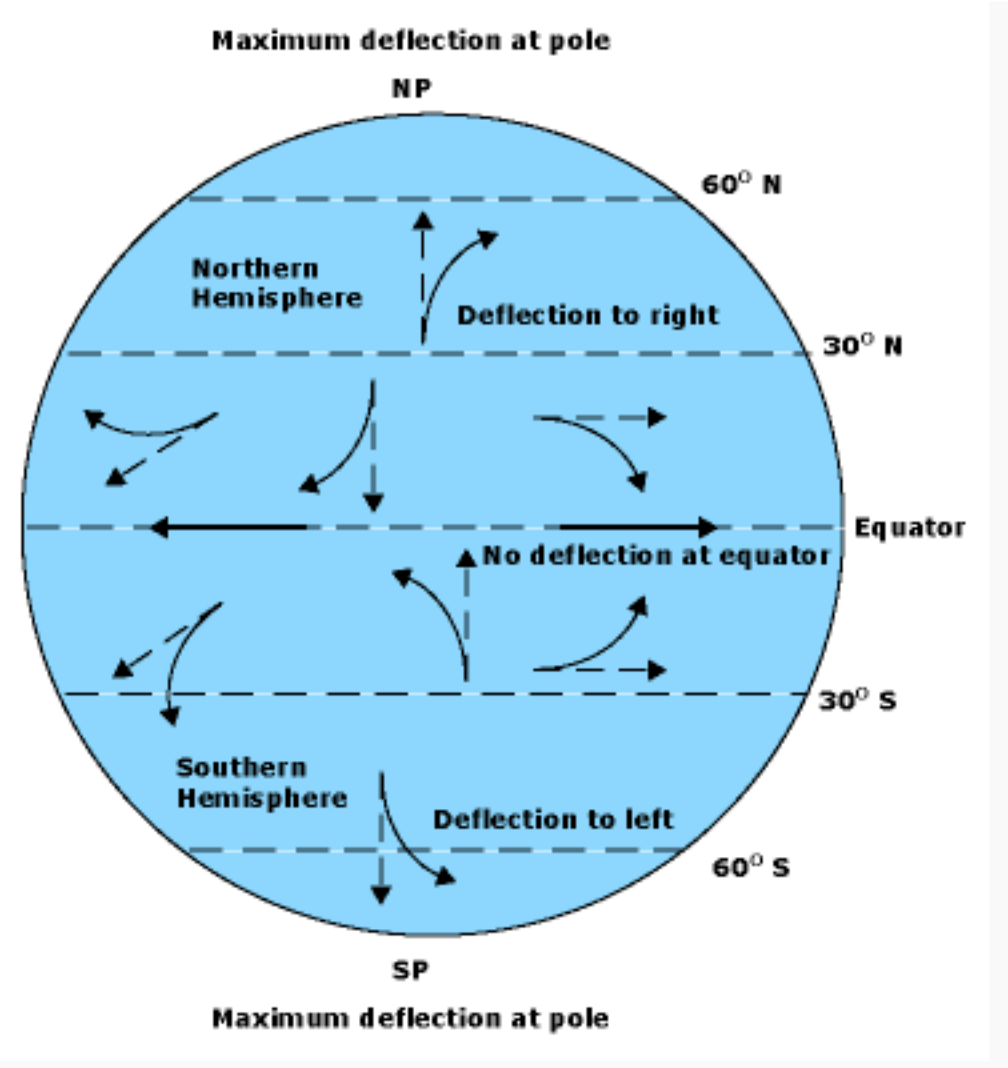

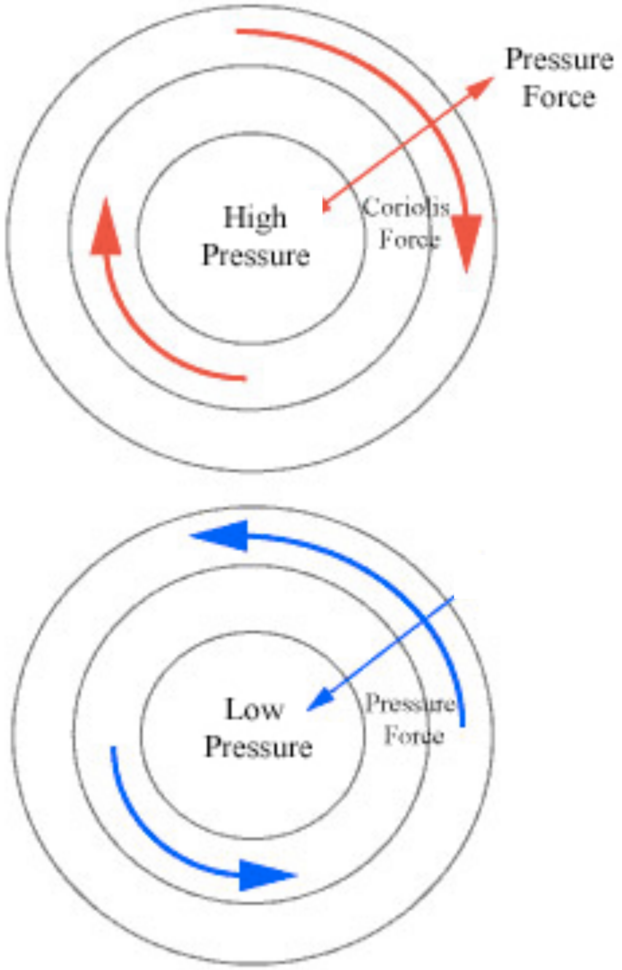

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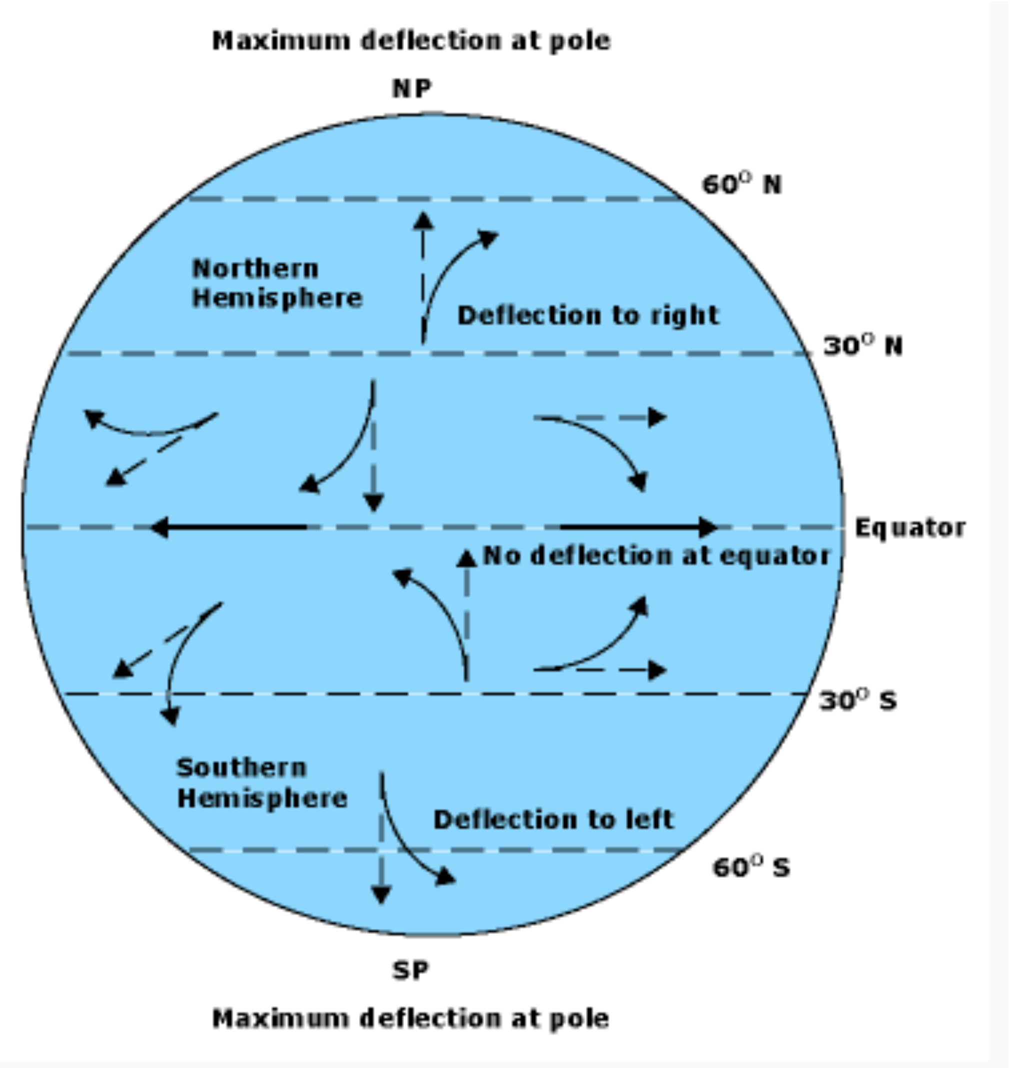

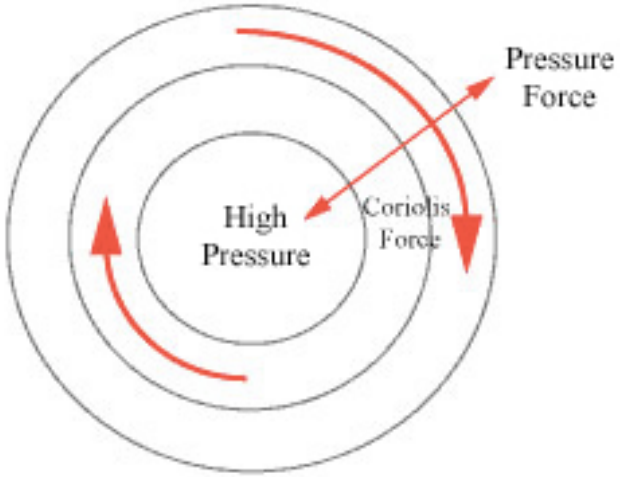

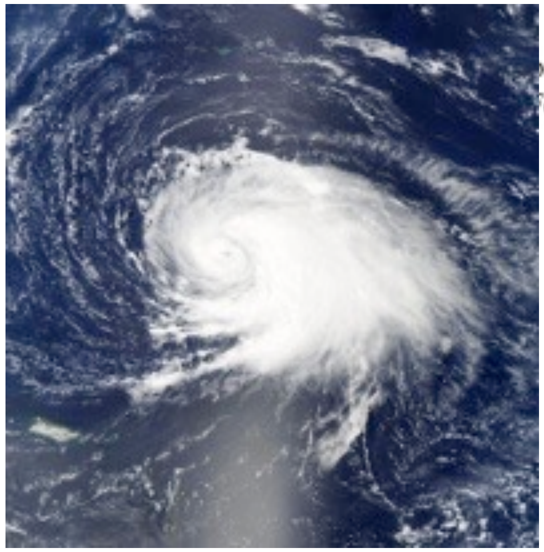

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Governing Equations: Effects of Rotation

inertia Coriolis pressure viscous body force

$$St\,\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\hat{\mathbf{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -Eu\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{u} + \boldsymbol{\mathcal{F}}$$

$\frac{L}{U\mathcal{T}}$	$\frac{U}{2\Omega L}$	$\frac{p}{\rho_0 U^2}$	$\frac{UL}{\nu}$
Strouhal	*Rossby*	*Euler*	*Reynolds*

$$\nabla\cdot\boldsymbol{u} = 0$$

incompressibility

$$Ro = \frac{\text{inertia}}{\text{Coriolis}} = \frac{U^2/L}{2\Omega U} = \frac{U/L}{2\Omega} = \frac{U}{2\Omega L}$$

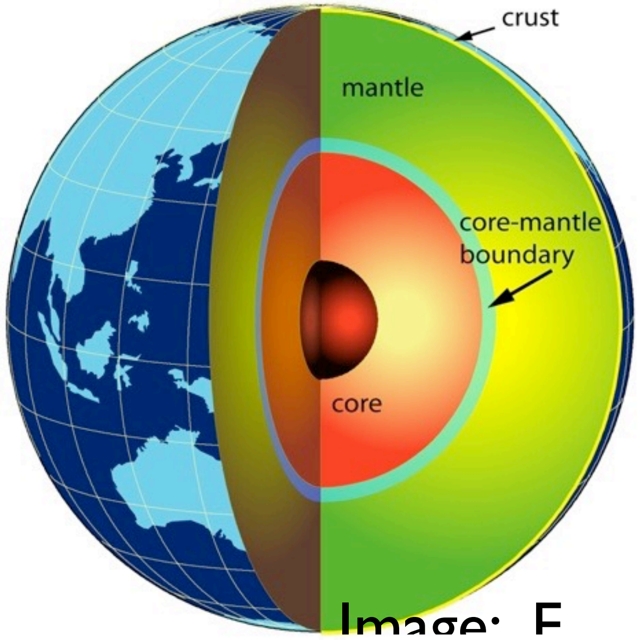
Types of motion affected by the Coriolis force:

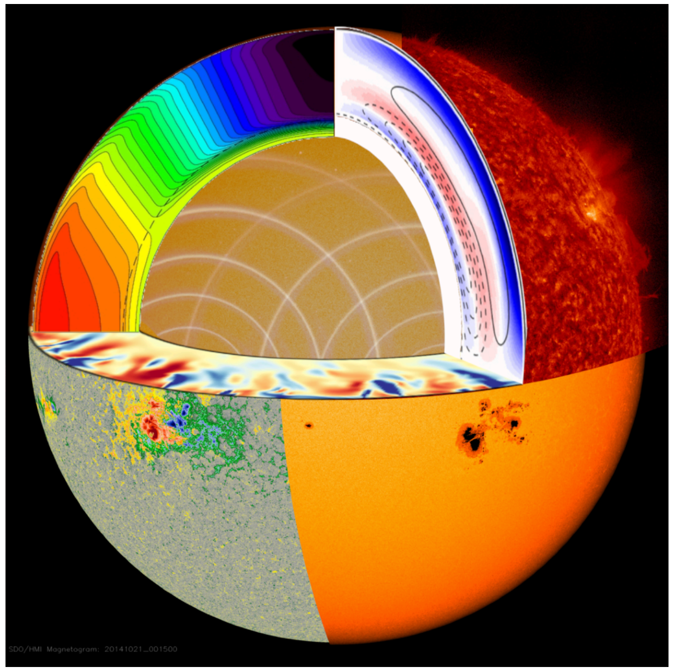
Relative vorticity is less than planetary vorticity $Ro \leq 1$ or $U/L \leq 2\Omega$

Vary on timescales greater than a planetary day $Str\,Ro \leq 1$ or $T \geq 1/2\Omega$

Nondimensional Parameters: Extreme

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{e}}_{\boldsymbol{\Omega}} \times \boldsymbol{u} = -Eu\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{u} + \boldsymbol{\mathcal{F}}$$





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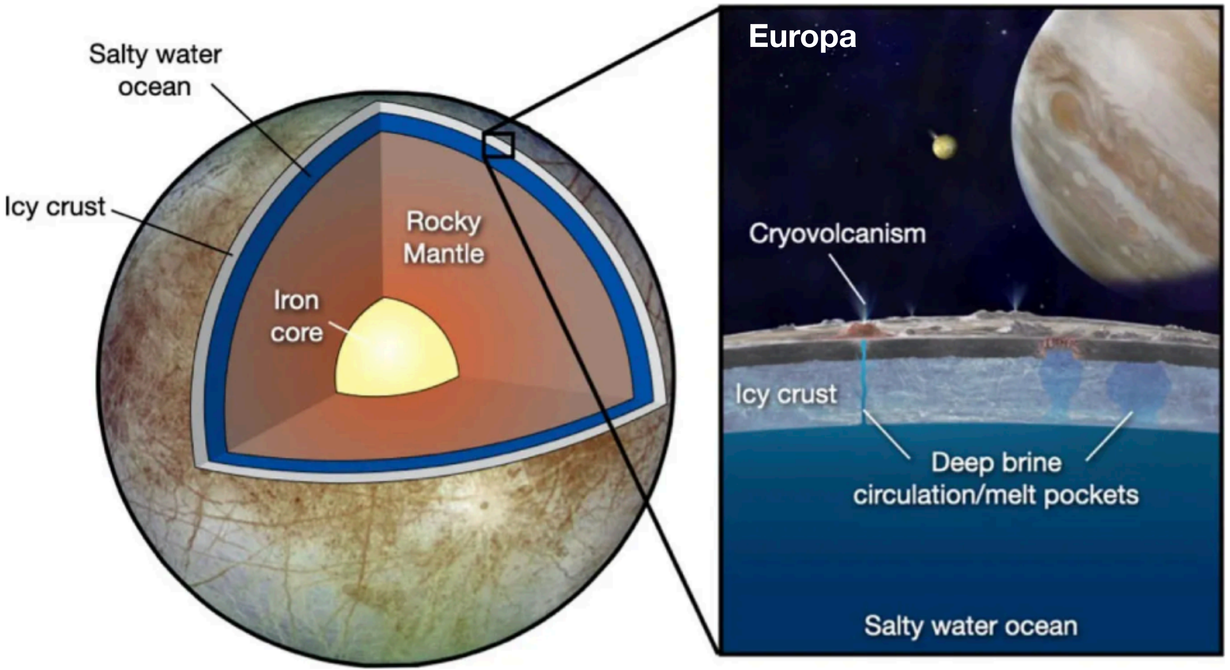
$\Omega \sim 7\times10^{-5}\ rad/s$	$\Omega \sim 7\times10^{-5}\ rad/s$	$\Omega \sim 2\times10^{-4}\ rad/s$	$\Omega \sim 2\times10^{-6}\ rad/s$
$L \sim 1\text{–}100\ km$	$L \sim 2260\ km$	$L \sim 15\ Mm$	$L \sim 200\ Mm$
$Re \sim 10^8$	$Re \sim 10^{8^+}$	$Re \sim 10^{12^+}$	$Re \sim 10^{12+}$
$Ro \sim 1\text{–}10^{-2}$	$Ro \sim 10^{-6}$	$Ro \sim 10^{-2}$	$Ro \sim 10^{-2}\text{–}1$

Subsurface off-world oceans

Bire & Marshall et al. GRL2022

Soderlund et al. GRL2019, Nature Geo. 2013



120 km/1561 km

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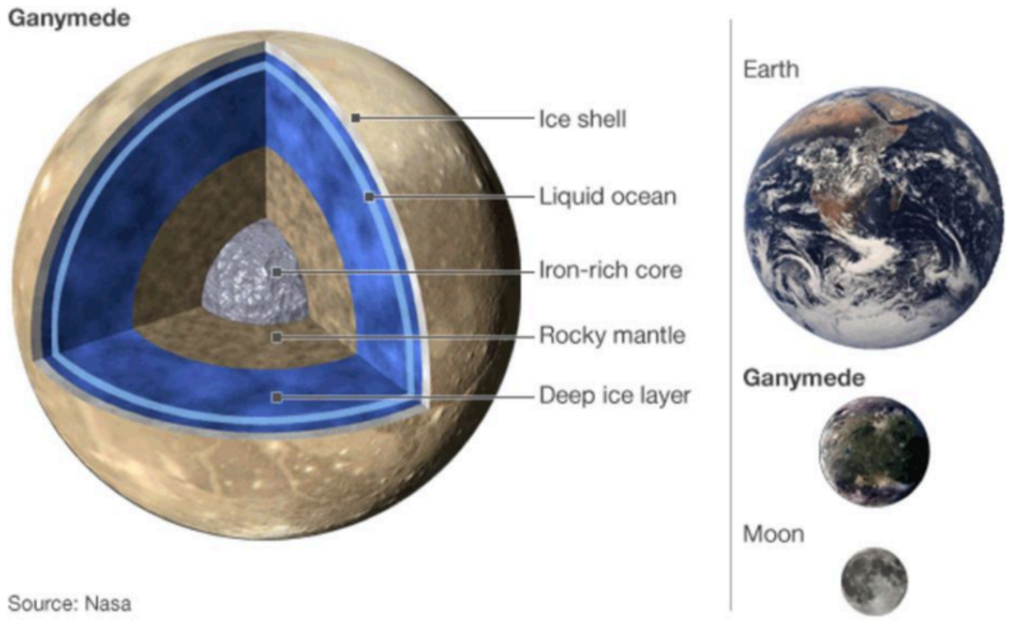

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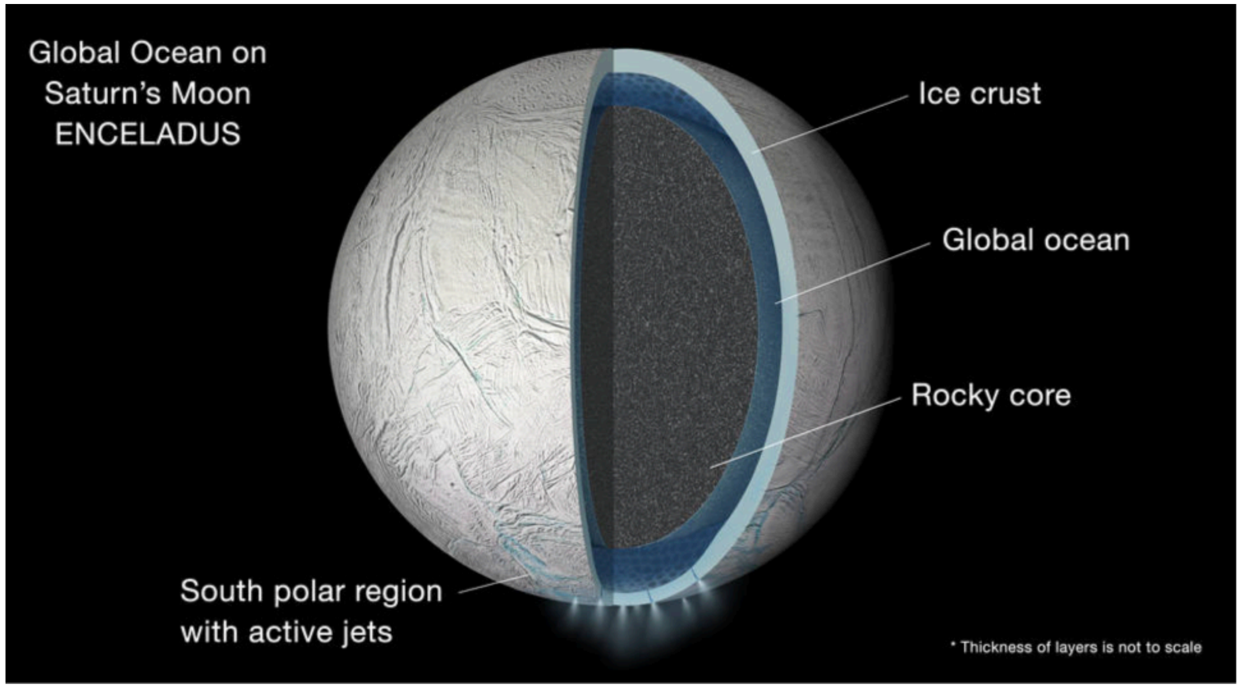

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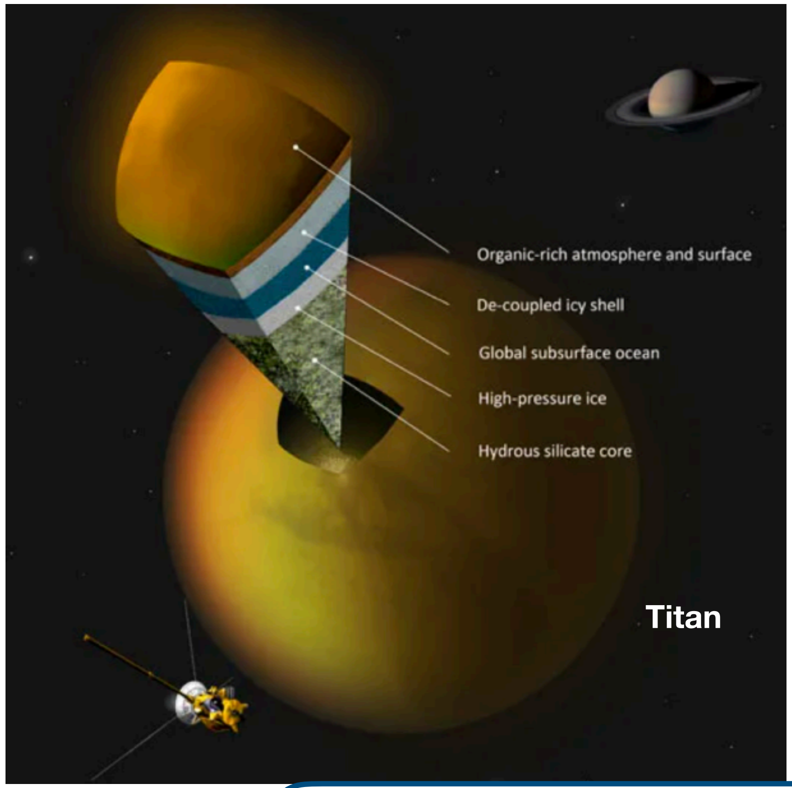

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Inertial Oscillations

Consider case *St Ro* ~ 1, *Eu* << 1/*Ro*, inviscid motions *Re >> 1*

$$\frac{\partial u}{\partial t} - \frac{1}{Ro} v = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{Ro} u = 0$$

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Inertial oscillation: velocities oscillate with period $\mathcal{T}_i = \pi/\Omega$

$$u = \sigma U \sin(\sigma Ro^{-1} t), \quad v = U \cos(\sigma Ro^{-1} t) \longrightarrow u_d = \sigma U_d \sin(\sigma 2\Omega t), \quad v_d = U_d \cos(\sigma 2\Omega t)$$

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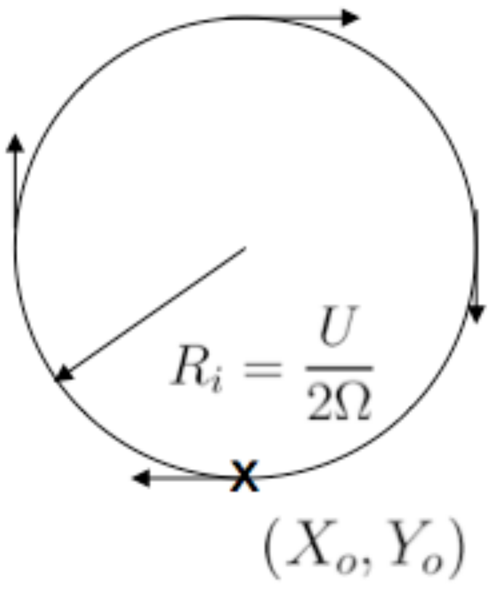
$$\frac{D\boldsymbol{X}_\perp}{Dt} = \boldsymbol{u}_\perp, \quad \boldsymbol{X}_\perp = \boldsymbol{X}_{0\perp}$$

$$X = \frac{U_d}{2\Omega}(1 - \cos(\sigma 2\Omega t) + X_0$$

$$Y = \sigma \frac{U_d}{2\Omega} \sin(\sigma 2\Omega t) + Y_0$$



Inertial Oscillations

Consider case *St Ro* ~ 1, *Eu* << 1/*Ro*, inviscid motions *Re >> 1*

$$\frac{\partial u}{\partial t} - \frac{1}{Ro} v = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{Ro} u = 0$$

$$\Rightarrow \left[\partial_{tt} + \frac{1}{Ro^2}\right] u = 0 \quad \Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i\sigma \\ 1 \end{pmatrix} U e^{-(i\frac{\sigma}{Ro}t)} + c.c., \quad \omega = \frac{\sigma}{Ro}, \sigma = \pm 1$$

Inertial oscillation: velocities oscillate with period $\mathcal{T}_i = \pi/\Omega$

$$u = \sigma U \sin(\sigma Ro^{-1} t), \quad v = U \cos(\sigma Ro^{-1} t) \longrightarrow u_d = \sigma U_d \sin(\sigma 2\Omega t), \quad v_d = U_d \cos(\sigma 2\Omega t)$$

Inertial circles: fluid parcels trace a circle of radius R_i in one inertial period

$$\frac{D\boldsymbol{X}_\perp}{Dt} = \boldsymbol{u}_\perp, \quad \boldsymbol{X}_\perp = \boldsymbol{X}_{0\perp}$$

$$X = \frac{U_d}{2\Omega}(1 - \cos(\sigma 2\Omega t) + X_0$$

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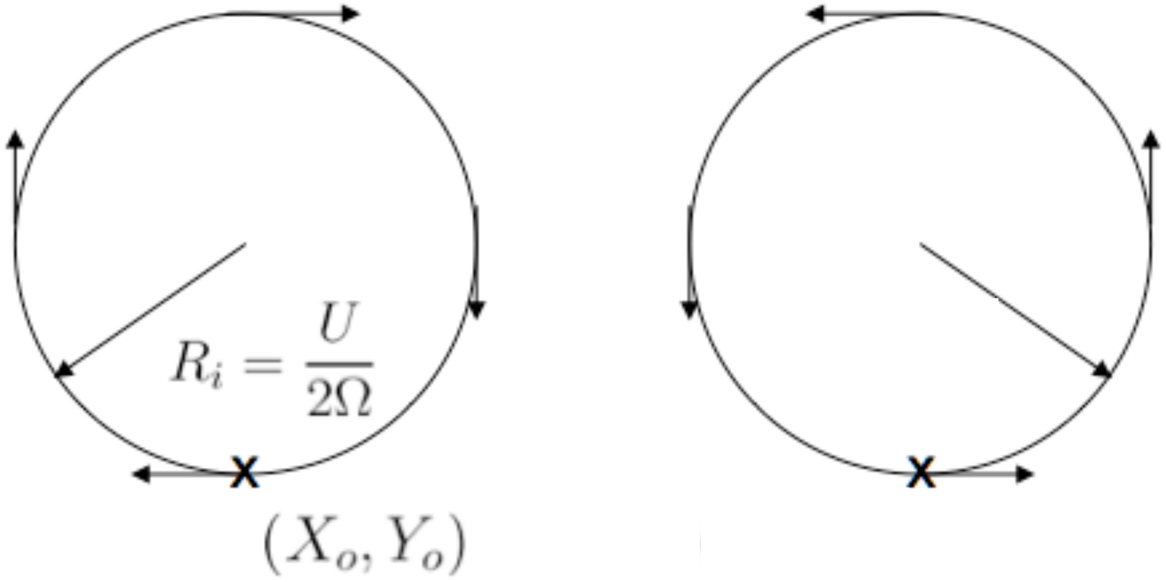
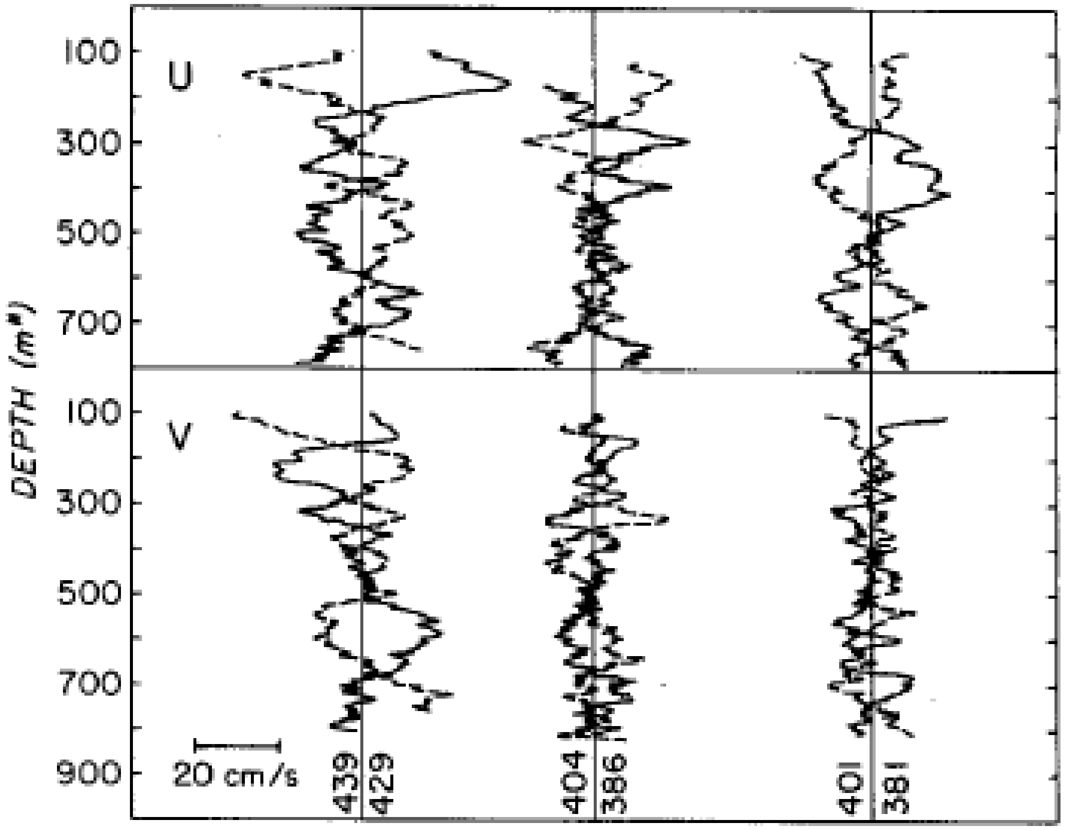
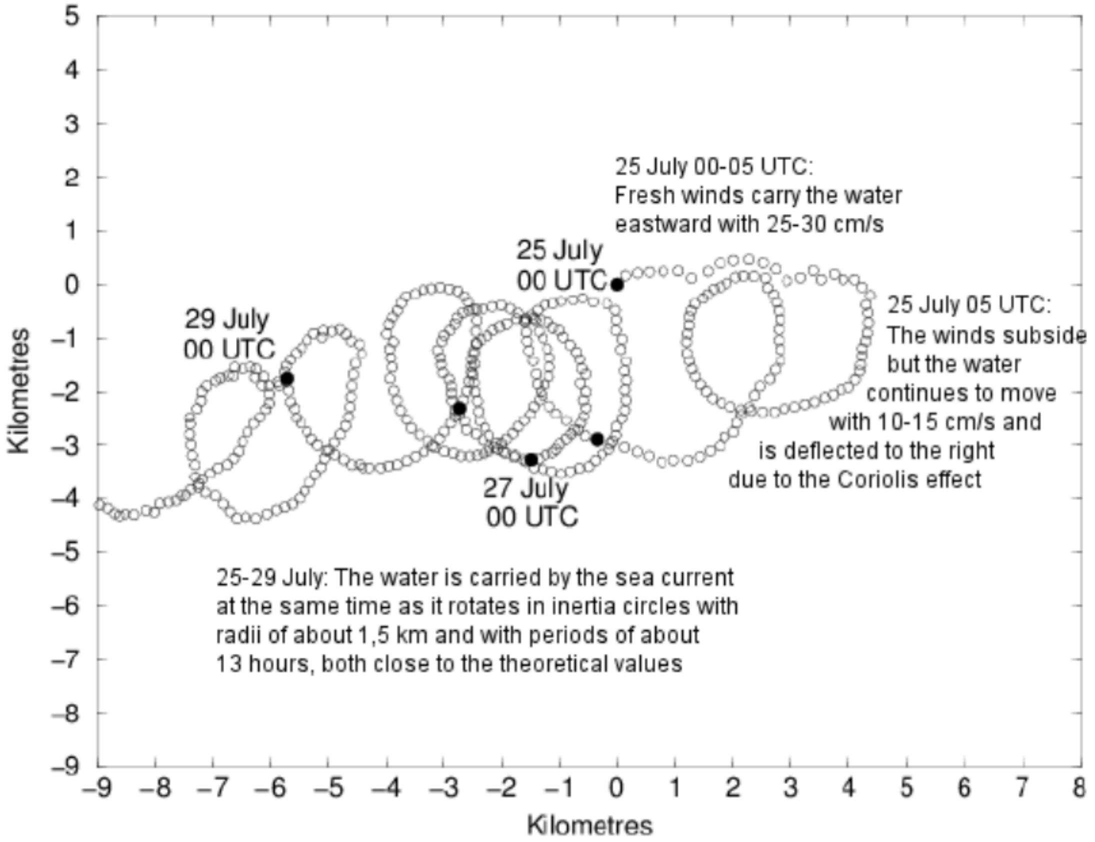


Inertial Oscillations; Oceanic Example





25 July 00-05 UTC:
Fresh winds carry the water
eastward with 25-30 cm/s

25 July 05 UTC:
The winds subside
but the water
continues to move
with 10-15 cm/s and
is deflected to the right
due to the Coriolis effect

25-29 July: The water is carried by the sea current
at the same time as it rotates in inertia circles with
radii of about 1,5 km and with periods of about
13 hours, both close to the theoretical values

Image copyright: Anders Persson.

Horizontal velocity profiles taken a half inertial period apart are near mirror-images of each other suggesting flows are dominated by inertial motions

Half the energy variance in the IW band is explained by inertial motions in the upper ocean (Ferrari & Wunsch. Ann. Rev. Fluid Mech 2008).

Inertial Waves

Consider case Eu ~ 1/Ro (incorporate mass cons) & inviscid motions (Re >>1), $\hat{\boldsymbol{\Omega}} \sim \hat{\boldsymbol{z}}$

$$\partial_t \boldsymbol{u} + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Inertial Waves

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$$\nabla \cdot \boldsymbol{u} = 0$$

Consider plane waves thru the normal mode assumption: wavenumber $\boldsymbol{k}$, frequency $\boldsymbol{\omega}$

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\sigma\omega t)}, \quad \boldsymbol{k} \cdot \boldsymbol{u} = 0, \quad \boldsymbol{k} = (k_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial Waves

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Eliminate pressure term: $\nabla \times$, $\nabla \times \nabla \times$ Vorticity $\boldsymbol{\zeta} = \nabla \times \boldsymbol{u}$

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Eliminate pressure term: $\nabla \times$, $\nabla \times \nabla \times$ Vorticity: $\boldsymbol{\zeta} = \nabla \times \boldsymbol{u}$

$$\partial_t \boldsymbol{\zeta} - \frac{1}{Ro} (\widehat{\boldsymbol{\Omega}} \cdot \nabla) \boldsymbol{u} \approx 0$$

$$\partial_t \nabla^2 \boldsymbol{u} + \frac{1}{Ro} (\widehat{\boldsymbol{\Omega}} \cdot \nabla) \boldsymbol{\zeta} \approx 0$$

Inertial Waves

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$$\Longrightarrow \quad \left[\partial_{tt} \nabla^2 + \frac{1}{Ro^2} (\hat{\boldsymbol{\Omega}} \cdot \nabla)^2 \right] \boldsymbol{u} = 0$$

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Dispersion relation

Inertial Waves

Dispersion relation

$$\omega = \frac{\sigma}{Ro}\frac{(\hat{\boldsymbol{\Omega}}\cdot\boldsymbol{k})}{|\boldsymbol{k}|} = \frac{\sigma}{Ro}\sin\phi$$

Inertial Waves

Dispersion relation

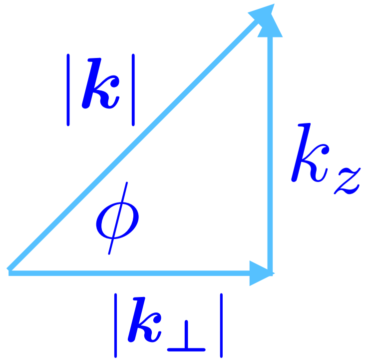
$$\boxed{\omega = \frac{\sigma}{Ro}\frac{(\hat{\boldsymbol{\Omega}}\cdot\boldsymbol{k})}{|\boldsymbol{k}|} = \frac{\sigma}{Ro}\sin\phi}$$

$$\Longrightarrow$$

dependent solely on orientation

$\phi \to 0$ freq. of oscill'n $\omega \to 0$

$\phi \to \frac{\pi}{2}$ freq. of oscill'n $\omega \to \frac{1}{Ro}(\sim 2\Omega)$



Inertial Waves

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Phase velocity of inertial plane waves:

$$\boldsymbol{v_p} = \frac{\omega}{|\boldsymbol{k}|}\frac{\boldsymbol{k}}{|\boldsymbol{k}|} = \frac{\omega}{|\boldsymbol{k}|^2}(\boldsymbol{k}_\perp, k_z) = \frac{1}{Ro}\frac{\sin\phi}{|\boldsymbol{k}|}(\cos\phi, \sin\phi)$$

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$\phi \to 0$ $\boldsymbol{v_p} \sim \frac{1}{Ro|\boldsymbol{k}|}\delta\phi\,(1,0)$ “Slow” horiz. propagation

$\phi \to \frac{\pi}{2}$ $\boldsymbol{v_p} \sim \frac{1}{Ro|\boldsymbol{k}|}(0,1)$ “Fast” horiz. propagation

Inertial Waves

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Group velocity of inertial plane waves:

$$\boldsymbol{v_g} = \nabla_k\omega = \frac{\omega}{|\boldsymbol{k}|^2}\left(-|\boldsymbol{k}_\perp|, \frac{|\boldsymbol{k}_\perp|^2}{k_z}\right) = \frac{\sigma}{Ro}\frac{\sin\phi\cos\phi}{|\boldsymbol{k}|}(-1, \cot\phi)$$

Deduction: wave packets propagate $\perp$ar to phase velocity $\boldsymbol{v_p}\cdot\boldsymbol{v_g} = 0$

Inertial Waves

Dispersion relation

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$\phi \to 0 \quad \boldsymbol{v_p} \sim \dfrac{1}{Ro|\boldsymbol{k}|}\delta\phi\,(1,0) \quad \boldsymbol{v_g} \sim \dfrac{1}{Ro|\boldsymbol{k}|}(0,1)$ Slow horiz. propag'n: axial wave energy transport

$\phi \to \dfrac{\pi}{2} \quad \boldsymbol{v_p} \sim \dfrac{1}{Ro|\boldsymbol{k}|}(0,1) \quad \boldsymbol{v_g} \sim \dfrac{1}{Ro|\boldsymbol{k}|}\delta\phi(-1,0)$ Fast vert. propag'n: horiz wave energy transport

Inertial Waves

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wave energy propag'n concentrates along rotation axis, associated with slowly propagating waves

Vallls 2017

Inertial Waves

Dispersion relation

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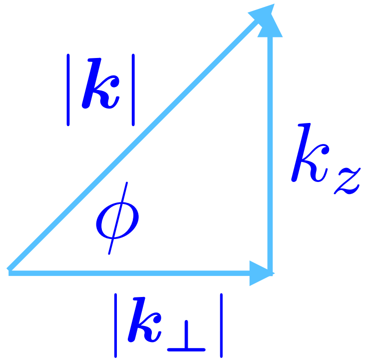
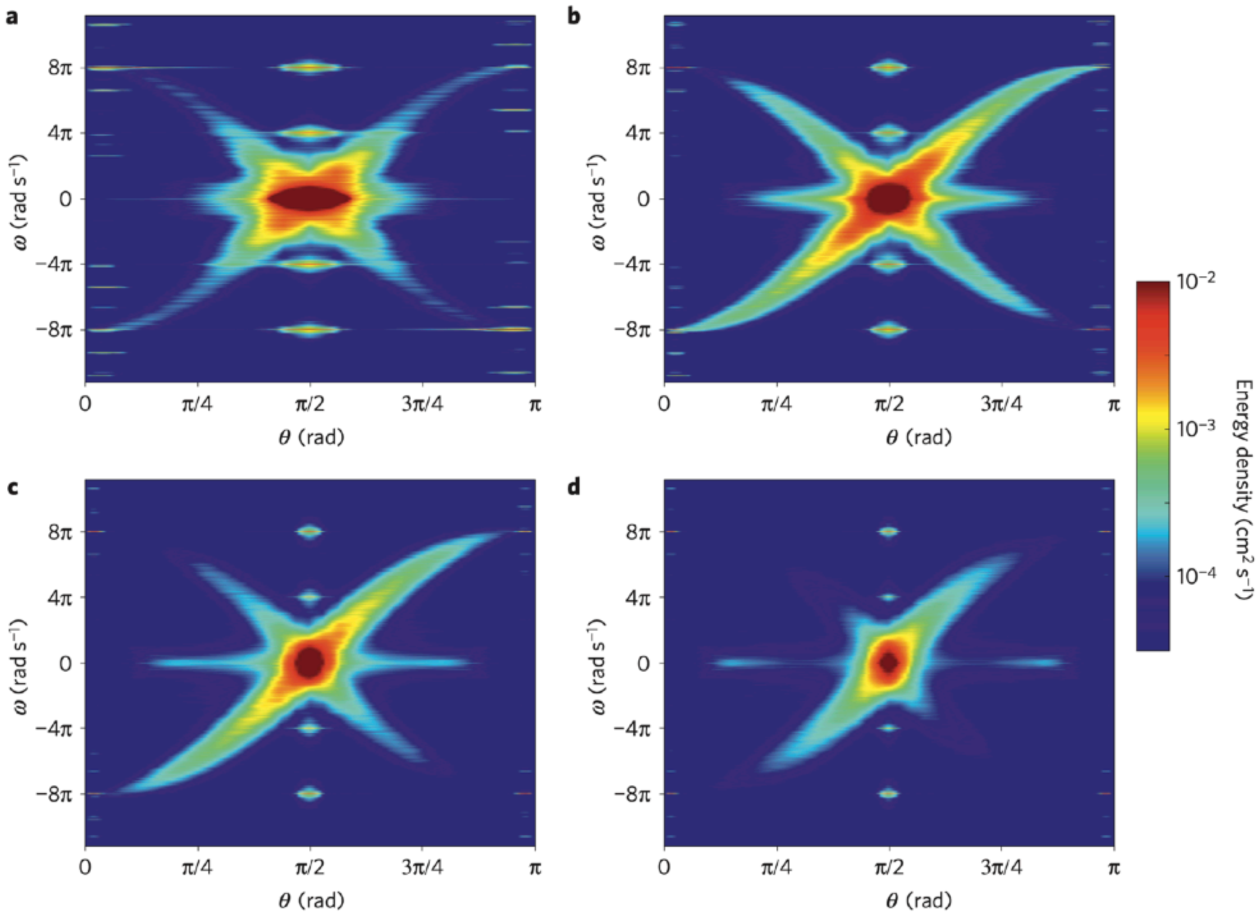


Inertial weak wave turbulence



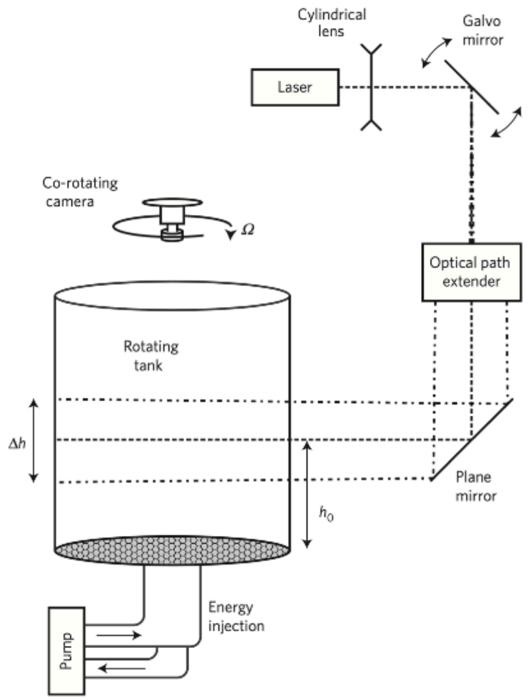


Figure 3 | Direct measurements of the inertial wave spectrum. The energy spectrum is localized along curves that correspond to the dispersion relation (equation (1)) of inertial waves. Integration across different wavenumber ranges (0.78–1.39 (**a**); 1.42–2.03 (**b**); 2.06–2.64 (**c**); 2.67–3.28 (**d**) rad cm^{-1}) shows that the shape of the curve is independent of $|k|$. In the high wavenumber range (**c**,**d**), the upward propagating waves carry more energy than the downward propagating waves. The data were taken at $h_0 = 68.5$ cm, with $\Delta h = 25.9$ cm and $\Omega = 4\pi$ rad s^{-1}. The spikes at $\theta = \pi/2$ and $\omega = \pm 4\pi, \pm 8\pi$ rad s^{-1} are measurement noise corresponding to the rotation rate and its harmonics.

Yarom & Sharom. Nature Phys 2014

Taylor-Proudman Constraint

Consider case Eu ~ 1/Ro, inviscid motions (*Re* >>1), and *Str Ro* << 1

$$Str\partial_t \boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{\Omega}} \times \boldsymbol{u} + Eu\nabla p \approx 0$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$\Longrightarrow$

horiz. phase velocity of inertial plane waves:

slowly propagating i.w's

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$$\frac{1}{Ro}\hat{\boldsymbol{\Omega}} \times \boldsymbol{u} + Eu\nabla p \approx 0 \qquad \Longrightarrow$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Geostrophic balance & Taylor-Proudman Theorem.

Proudman-Taylor Theorem (1916, 1923): $\hat{\boldsymbol{\Omega}}\cdot$ and $\nabla\times$

$$\hat{\boldsymbol{\Omega}} \cdot \nabla(\boldsymbol{u}, p) \approx 0$$

fluid motions are inherently columnar, two-dimensional

J. Proudman 1888-1975

GI. Taylor 1886-1975

Taylor-Proudman Constraint

Consider case Eu ~ 1/Ro, inviscid motions (*Re* >>1), and *Str Ro* << 1

$$\begin{aligned} \frac{1}{Ro}\hat{\boldsymbol{\Omega}} \times \boldsymbol{u} + Eu\nabla p \approx 0 \\ \nabla \cdot \boldsymbol{u} = 0 \end{aligned} \quad \Longrightarrow \quad \hat{\boldsymbol{\Omega}} \cdot \nabla(\boldsymbol{u}, p) \approx 0$$

Taylor-Proudman Constraint

Consider case Eu ~ 1/Ro, inviscid motions (*Re* >>1), and *Str Ro* << 1

$$\frac{1}{Ro}\hat{\boldsymbol{\Omega}} \times \boldsymbol{u} + Eu\nabla p \approx 0 \qquad \Longrightarrow \qquad \hat{\boldsymbol{\Omega}} \cdot \nabla(\boldsymbol{u}, p) \approx \mathcal{O}(Ro)$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Axial variations can occur if geometrically allowed

Barotropic Vorticity Equation

Axial variations geometrically disallowed $L \gg H$

T-P $\hat{\boldsymbol{z}} \cdot \nabla(\boldsymbol{u}_\perp^G, p) \approx 0$ then $\boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \boldsymbol{u}_\perp^{AG}$ $\boldsymbol{u}_\perp^G = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$

$$\partial_t \boldsymbol{u}_\perp + \boldsymbol{u}_\perp \cdot \nabla_\perp \boldsymbol{u}_\perp + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -Eu \nabla_\perp p^{AG} + F + D$$

$$\nabla_\perp \cdot \left(\boldsymbol{u}_\perp^{AG} \right) = 0$$

Barotropic Vorticity Equation

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$$\partial_t \zeta + \boldsymbol{u}_{\perp} \cdot \nabla_{\perp} \zeta = \mathcal{F} + \mathcal{D}$$

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$$\partial_t \nabla_\perp^2 \psi + J[\psi, \nabla_\perp^2 \psi] = \mathcal{F} + \mathcal{D},$$

$\mathcal{D} = -\alpha\zeta + \nu \nabla_\perp^2 \zeta$

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Two conserved quantities: Energy $E = \left\langle |\nabla_\perp \psi|^2 \right\rangle$, Enstrophy $\mathcal{Z} = \left\langle |\zeta|^2 \right\rangle$

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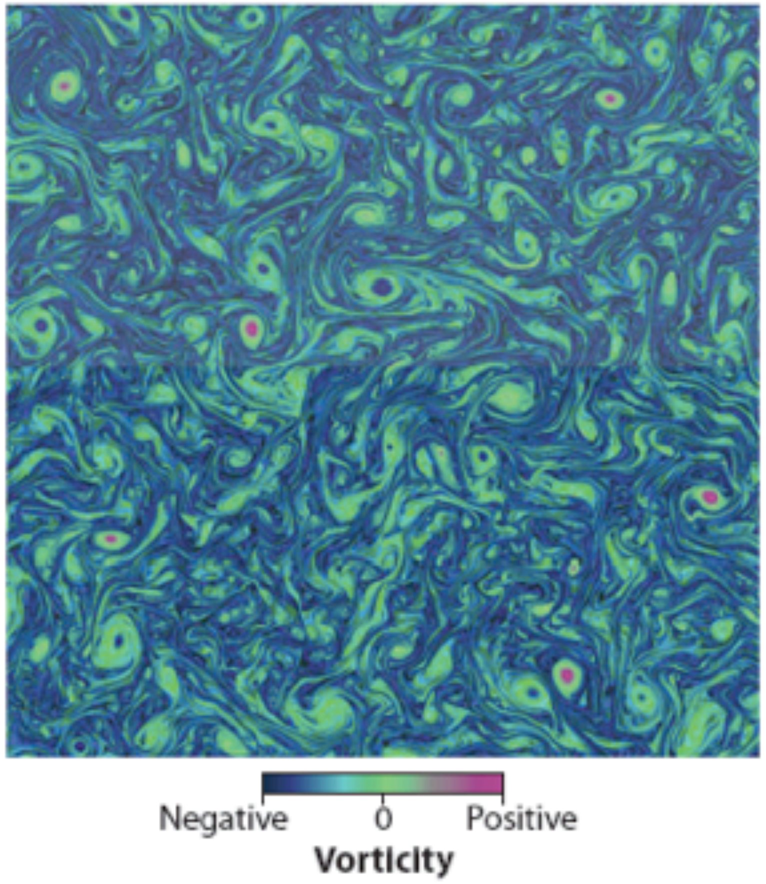

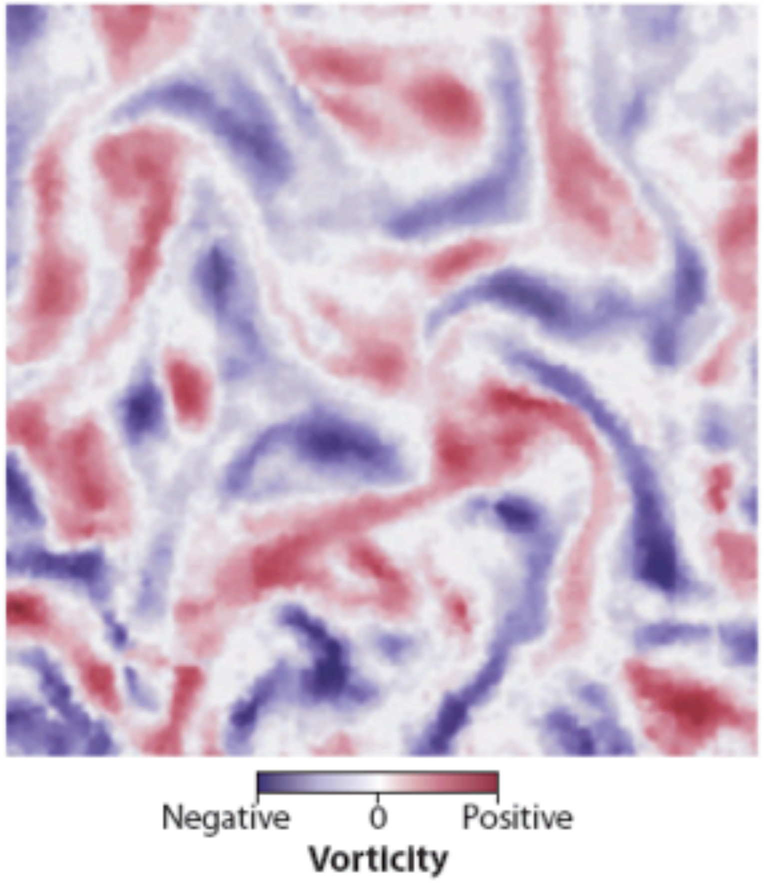

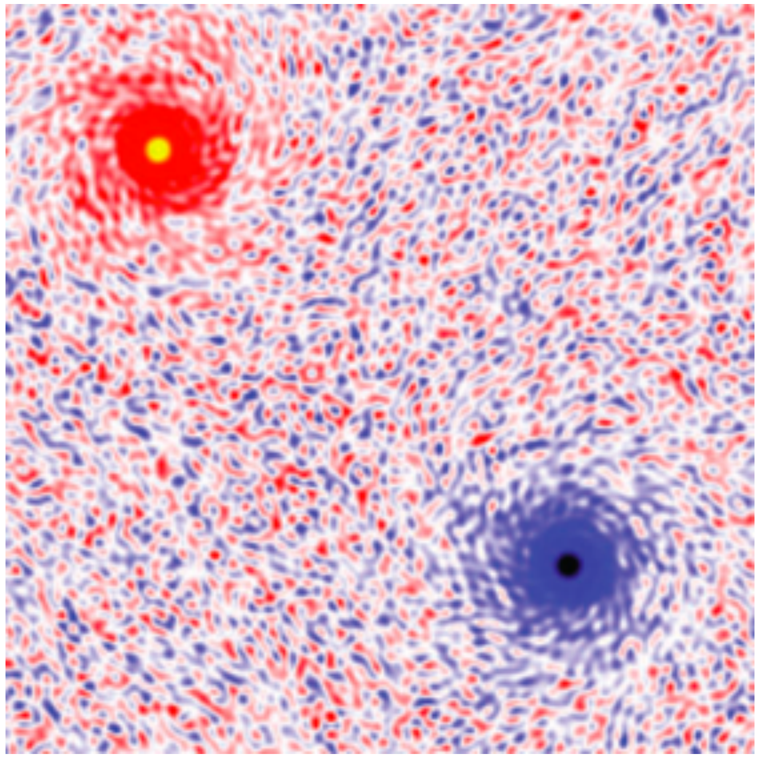

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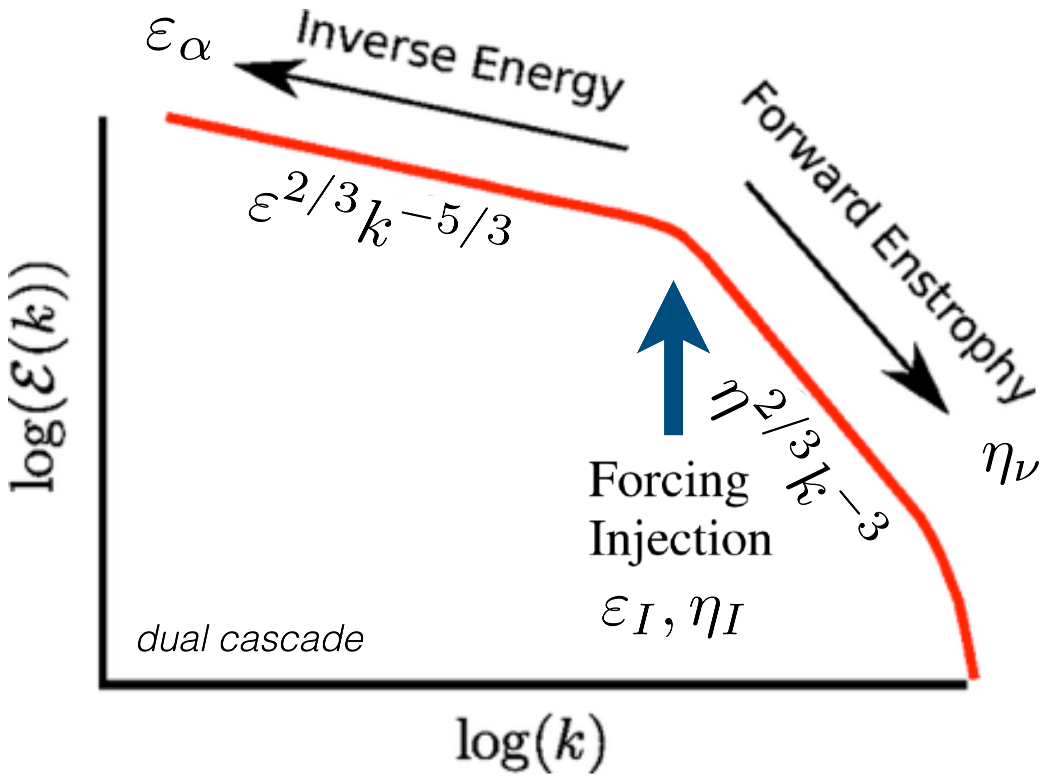
$$\partial_t \nabla_{\perp}^{2} \psi + J[\psi, \nabla_{\perp}^{2} \psi] = \mathcal{F} + \mathcal{D},$$

$$\mathcal{D} = -\alpha\zeta + \nu \nabla_{\perp}^{2} \zeta$$

Two conserved quantities: Energy $E = \langle |\nabla_{\perp} \psi|^2 \rangle$, Enstrophy $\mathcal{Z} = \langle |\zeta|^2 \rangle$

$$\varepsilon_I = \varepsilon_\alpha + \varepsilon_\nu \qquad \ell_\alpha^2 = \frac{\varepsilon_\alpha}{\eta_\alpha}, \quad \ell_\nu^2 = \frac{\varepsilon_\nu}{\eta_\nu},$$
$$\eta_I = \eta_\alpha + \eta_\nu$$



Bofetta & Ecke, Ann. Rev. Fluid Mech. 2012

Barotropic Vorticity Equation

Axial variations geometrically disallowed $L \gg H$

T-P $\hat{\boldsymbol{z}} \cdot \nabla(\boldsymbol{u}_\perp^G, p) \approx 0$ then $\boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \boldsymbol{u}_\perp^{AG}$ $\boldsymbol{u}_\perp^G = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$

$$\partial_t \nabla_\perp^2 \psi + J[\psi, \nabla_\perp^2 \psi] = \mathcal{F} + \mathcal{D},$$

$$\mathcal{D} = -\alpha\zeta + \nu \nabla_\perp^2 \zeta$$

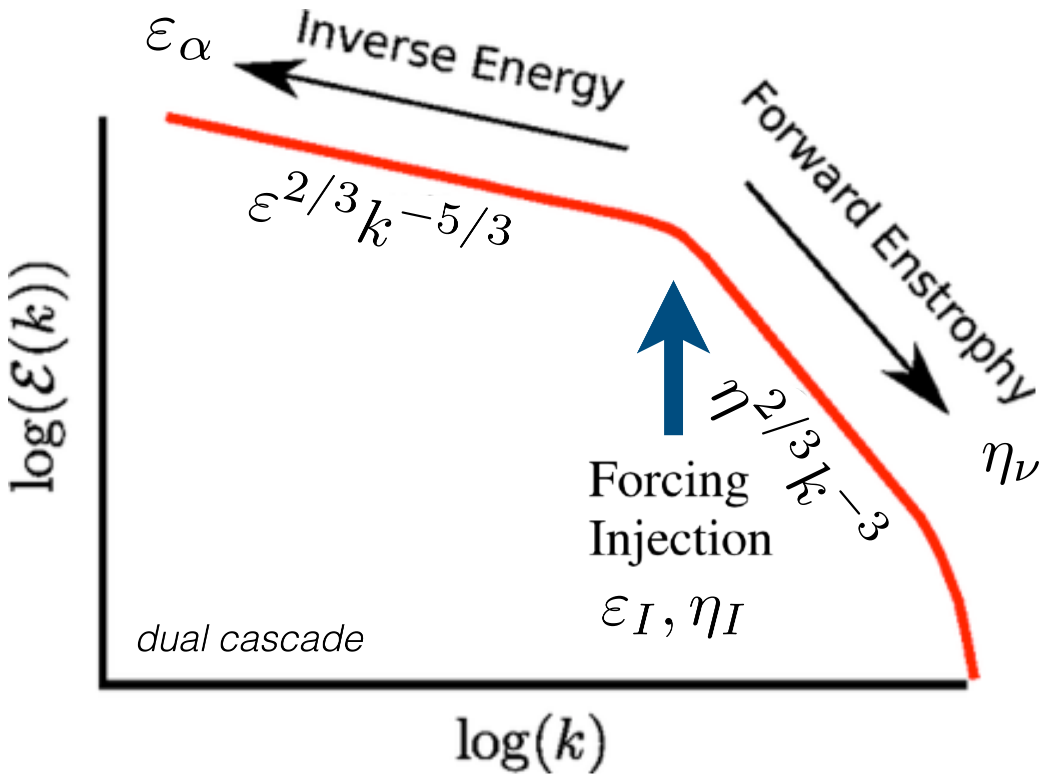
Two conserved quantities: Energy $E = \left\langle |\nabla_\perp \psi|^2 \right\rangle,$ Enstrophy $\mathcal{Z} = \left\langle |\zeta|^2 \right\rangle$

$$\varepsilon_I = \varepsilon_\alpha + \varepsilon_\nu \qquad \ell_\alpha^2 = \frac{\varepsilon_\alpha}{\eta_\alpha}, \quad \ell_\nu^2 = \frac{\varepsilon_\nu}{\eta_\nu},$$

$$\eta_I = \eta_\alpha + \eta_\nu$$

$$\frac{\varepsilon_\nu}{\varepsilon_\alpha} = \left(\frac{\ell_\nu}{\ell_f}\right)^2 \left(\frac{\ell_f}{\ell_\alpha}\right)^2 \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2},$$

$$\frac{\eta_\nu}{\eta_\alpha} = \frac{(\ell_\alpha/\ell_f)^2 - 1}{1 - (\ell_\nu/\ell_f)^2}.$$



Bofetta & Ecke, Ann. Rev. Fluid Mech. 2012

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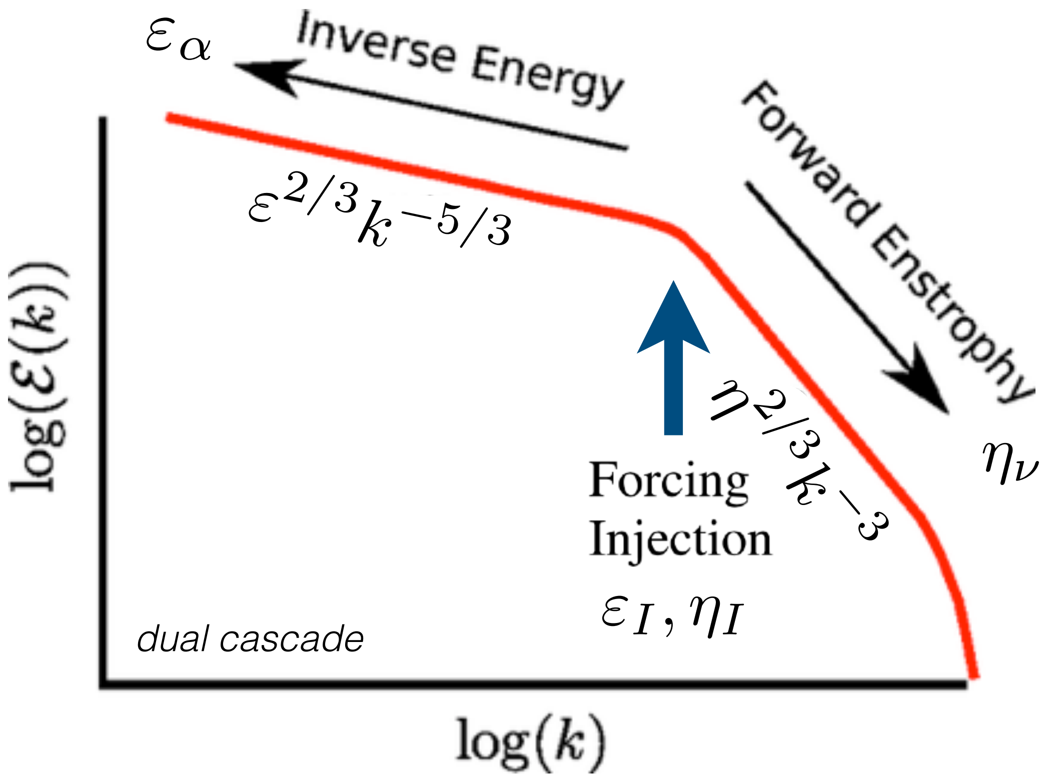

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$$\ell_\nu \ll \ell_f \implies \frac{\varepsilon_\nu}{\varepsilon_\alpha} \to 0$$
$$\ell_\alpha \gg \ell_f \implies \frac{\eta_\alpha}{\eta_\nu} \to 0$$

Barotropic Vorticity Equation: *β - plane*

$$f \approx 2\Omega \sin\vartheta + \beta y$$

$$\partial_t \zeta + \boldsymbol{u}_\perp \cdot \nabla_\perp \zeta + \beta v = \mathcal{F} + \mathcal{D}, \qquad \zeta_a = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} + Ro^{-1} + \beta y$$

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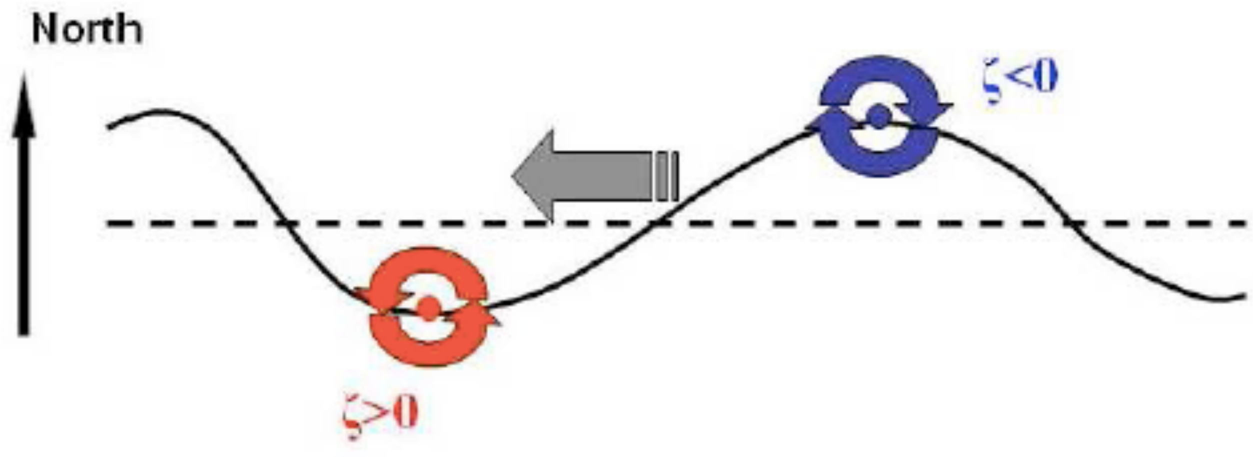
Dispersion Rel'n

$$\omega = \frac{\beta k_x}{k_x^2 + k_y^2}$$

westward propagation

Mechanism



Rossby wave

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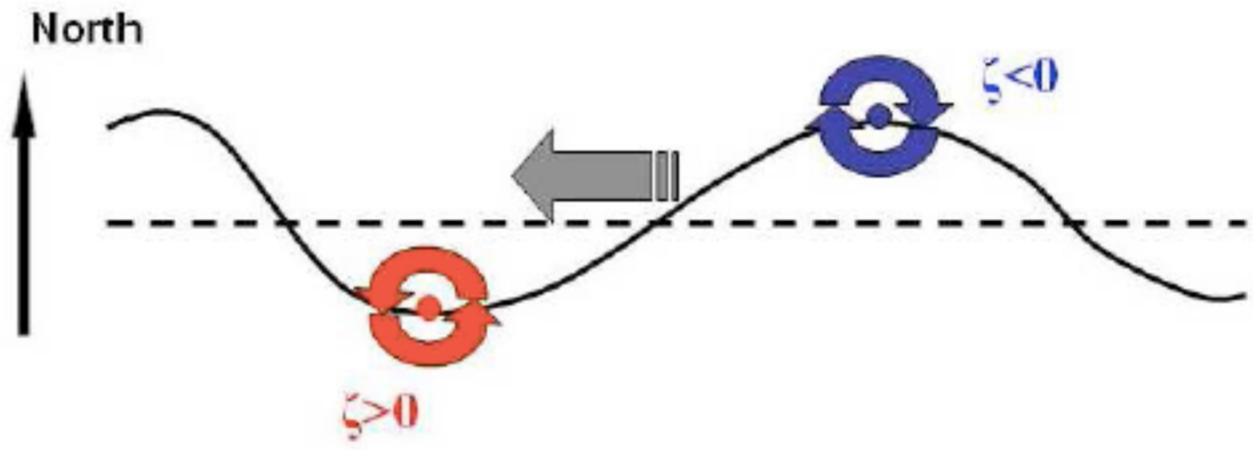


Rossby wave

Barotropic Vorticity Equation: *β - plane*

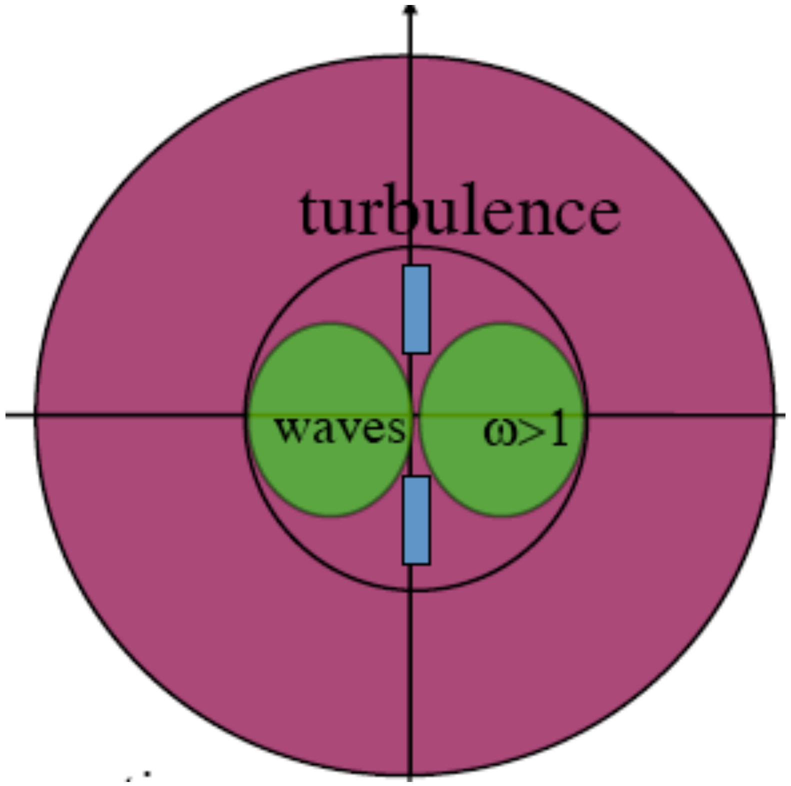
Inverse energy cascade is suppressed by low wavenumber Rossby waves (*Rhines JFM 1975*)

$$\omega_{turb} \approx \omega_\beta \quad \Rightarrow \quad \varepsilon^{1/3} k_\perp^{2/3} \approx \frac{\beta k_x}{k_\perp^2}$$

Inverse cascade barrier

$$k_x \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \cos^{8/5}\vartheta$$

$$k_y \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \sin\vartheta \cos^{3/5}\vartheta$$



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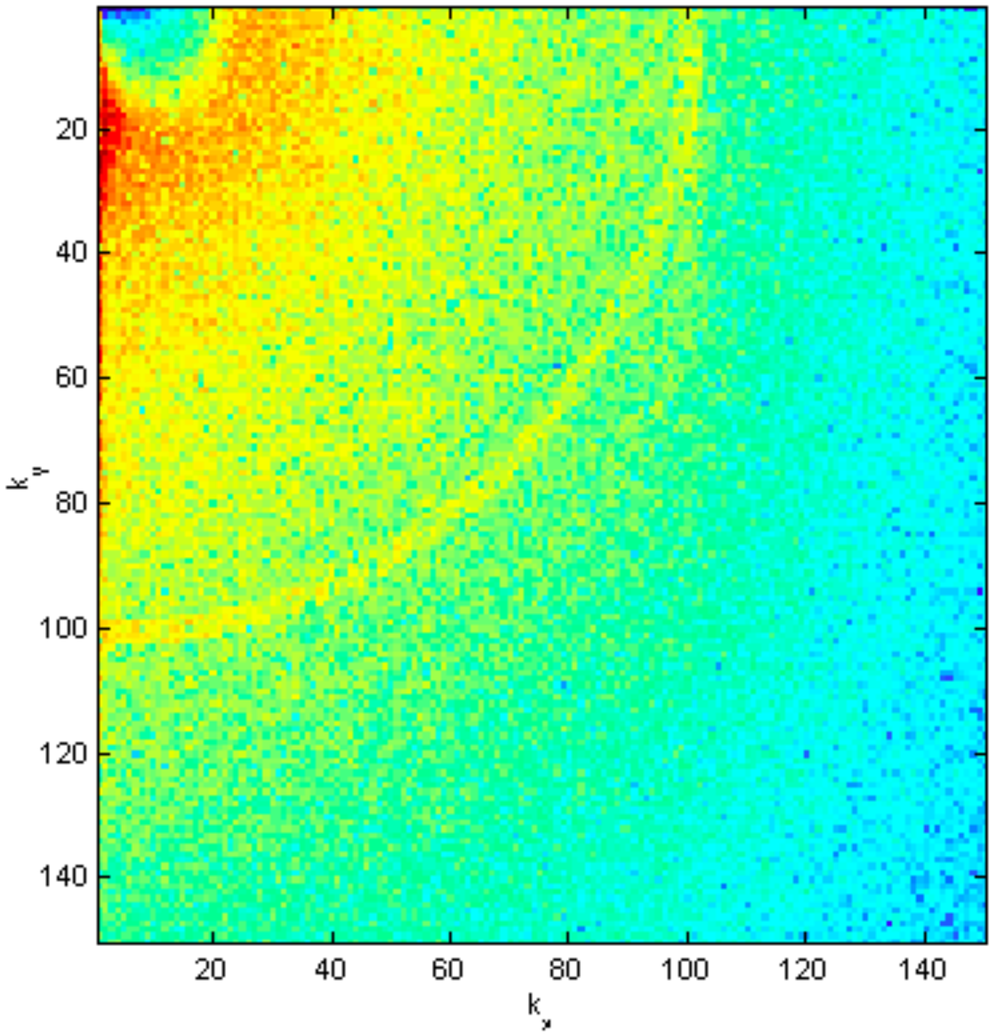
Beta-plane spectrum



Gurarie 2004

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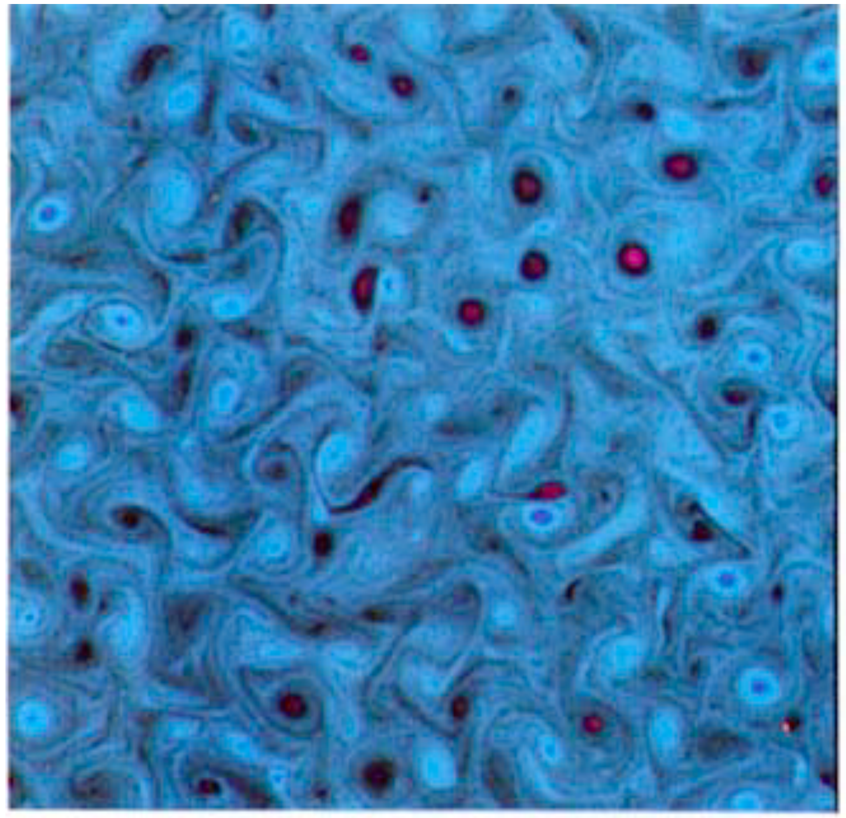
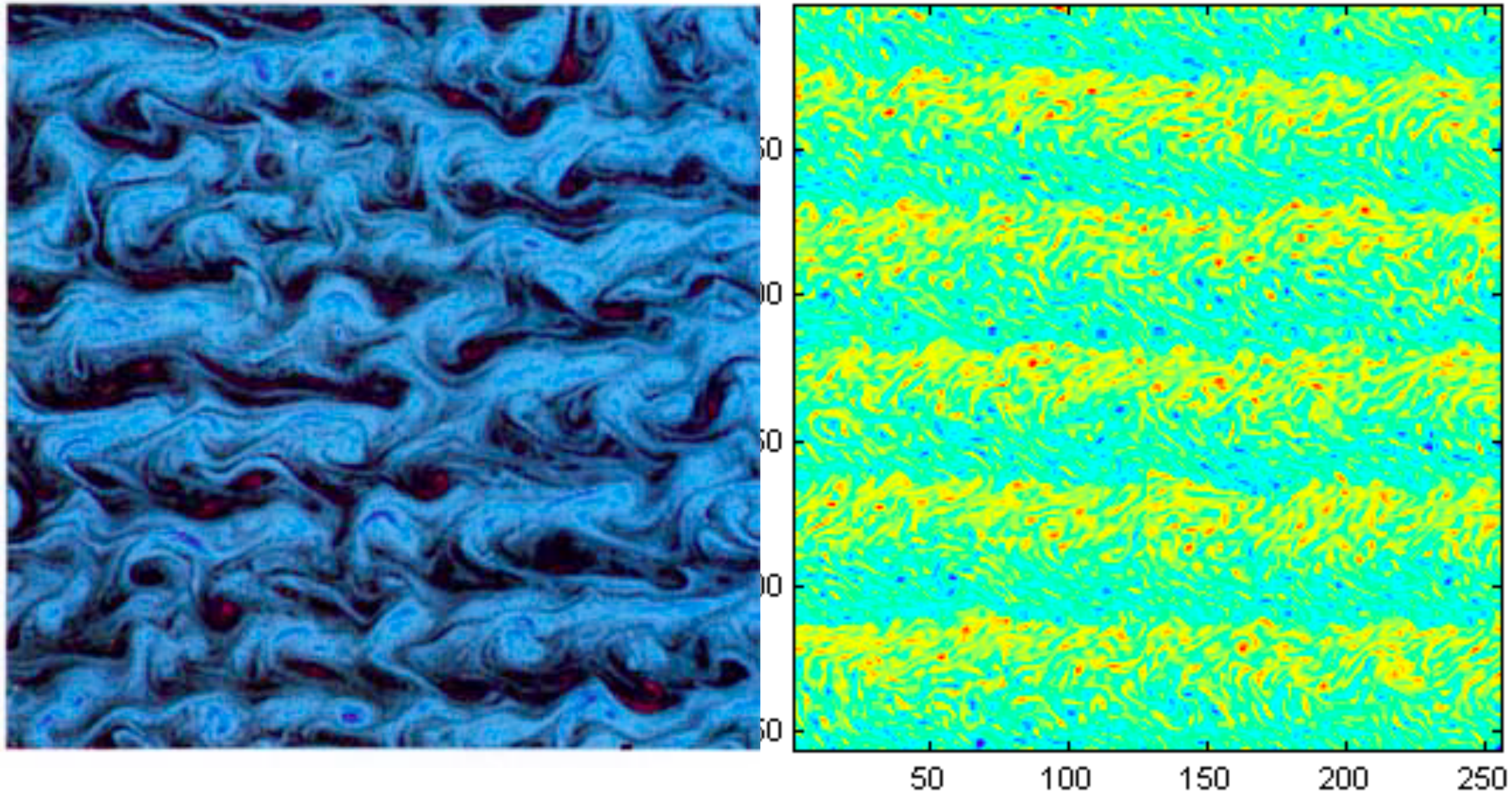
$\beta = 0$



$\beta \neq 0$



Maltrud & Vallis JFM 1992, Gurarie 2004