Hydrodynamics

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Thermodynamics / statistical mechanics :

N>>1; large number of interacting particles !

Hydrodynamics :

L >> l; T >> t; long wavelength, slow time - average over (some) microscopic length and time scales ... continuum field theories !

microscopic length : l ? (particle size, mean-free path, pore size, ...)

microscopic time : t ? (particle relaxation times, hopping times, ...)

Continuum theory ?

1. Microscopic picture + systematic averaging ... "rigorous" !

2. Equations of state, constitutive equations ... "empirical" !

3. Symmetry, invariance ... "intuitive" !

Balance laws: mass, momentum (linear, angular), energy, entropy ...

Variables - independent ? $(\mathbf{r}, t) = (x, y, z, t)$ space-time

- dependent ? $\rho(\mathbf{r}, t), \mathbf{u}(\mathbf{r}, t), \sigma(\mathbf{r}, t), ...$

density (scalar), velocity (vector), stress (tensor), ...

order parameter

displacement ... elasticity; orientation ... liquid crystals; polarity ... magnetic field etc.

Frames:

1. Lagrangian - material frame : follow parcels of material ... finite bodies.

2. Eulerian - spatial (inertial) frame : lab-based ... infinite bodies.

Laws + boundary conditions:

Qualitative behavior of solutions:

Lecture 1. Balance laws, constitutive equations and boundary conditions. Symmetry, invariance and how to derive equations for simple and complex fluids. Compressibility, Viscosity, Inertia, Capillarity. Dimensional analysis and scaling laws. Analogies to other field theories (electrostatics, elasticity, transport).

Lecture 2. Stability and instability. Rayleigh instabilities. Pattern formation. Turbulence. Kolmogorov scaling. Low dimensional flows. Landau-Levich problem. Washburn problem.

Lecture 3. Porous media - convection, soft hydraulics. Complex fluids - suspensions, polymers. Locomotion in fluids.

Laws: differential form (integral form - useful in the presence of discontinuities - e.g. shocks)

Conservation of mass

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

Eulerian

$$\frac{D\rho(\mathbf{r},t)}{Dt} = (\partial_t + \mathbf{u} \cdot \nabla)\rho = -\rho \nabla \cdot \mathbf{u}$$

Material derivative

where $\frac{d\mathbf{r}}{dt} = \mathbf{u}$

velocity

Incompressible fluid $\frac{D\rho}{Dt} = 0$ or $\nabla \cdot \mathbf{u} = 0$

Conservation of linear momentum

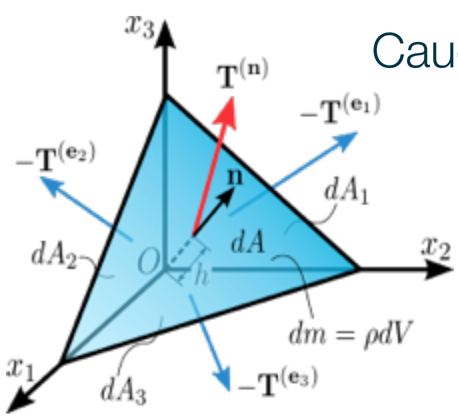
$$\rho \frac{D\mathbf{u}}{Dt} = \mathbf{f} + \nabla \cdot \sigma$$

accln force stress

~ force/area (interactions !)

Newton II for deformable continua (per unit volume)

ratio of two vectors ??



Cauchy stress theorem

$$\mathbf{T}^{(\mathbf{n})} dA = \Sigma_i T^{(\mathbf{e_i})} dA_i$$

$$dA_i = (\mathbf{n} \cdot \mathbf{e_i}) dA$$

$$\mathbf{T}^{(\mathbf{n})} = \sigma^T \mathbf{n}$$

Stress tensor completely defines force/area locally !

all images from wikipedia

Stress tensor (Newtonian fluid) - isotropic, homogeneous

$$\sigma = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + [-p + (\zeta - \frac{2}{3}\mu)(\nabla \cdot \mathbf{u})]\mathrm{I} \longleftarrow$$ 2 constant fluid !

(anisotropic) shear (isotropic) "pressure" = thermodynamic + dynamic

pressure dilatation viscosity

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} + (\zeta + \mu/3)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f}$$

acceleration shear viscosity body force

+ equation of state

$$p = p(\rho, T)$$

Conservation
of angular momentum ?

$$\sigma = \sigma^T$$

heat conduction

Conservation
of entropy ?

$$\rho T(\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s) = \kappa \nabla^2 T + \frac{1}{2}\mu[\nabla \mathbf{u} + \nabla \mathbf{u}^T - \frac{2}{3}\nabla \mathbf{u} \mathrm{I}]^2 + \zeta(\nabla \cdot \mathbf{u})^2 + \mathbf{f} \cdot \mathbf{u}$$

entropy production viscous dissipation external work

Initial conditions: $\mathbf{u}(\mathbf{r}, 0)$ $T(\mathbf{r}, 0)$

unknowns $\mathbf{u}, p, T$

Boundary conditions

$\mathbf{u}|_{\partial\Omega} = \mathbf{U}_b$ or $p|_{\partial\Omega} = p_b$

$s|_{\partial\Omega} = s_b$ or $-\rho T \nabla s|_{\partial\Omega} = \mathbf{q}_b$

open boundaries

Solid (rigid) boundaries

$$\mathbf{u}|_{\partial\Omega} = 0$$

no slip

A very slippery condition !

Fails at

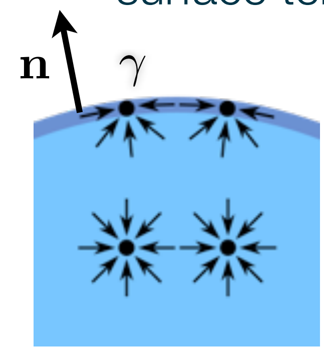
- low density
- porous interface
- hydrophobic surface
- contact line (triple phase boundary)

Free boundaries

traction is continuous

$$\sigma\mathbf{n} = -\gamma\kappa\mathbf{n}$$

surface tension



generalizing Laplace's law

$$\delta p = \gamma\kappa$$

$$\kappa = \frac{1}{R_1} + \frac{1}{R_2}$$

2 X mean curvature

Physical parameters:

μ

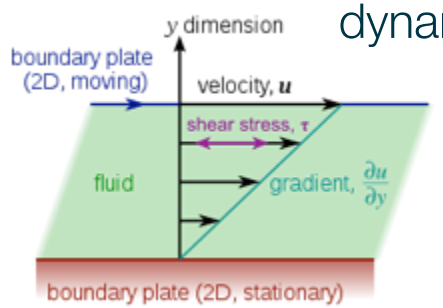
ρ fluid (intrinsic)

γ

L

U extrinsic

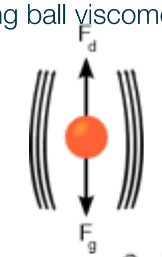
dynamic viscosity ?



$$\tau = \mu\frac{\partial u}{\partial y}.$$

cone-plate rheometer

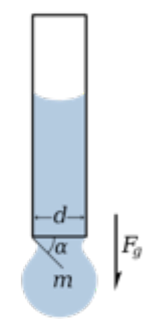
falling ball viscometer



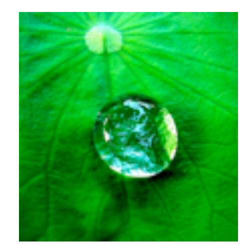
air 0.00001 Pa.s
water 0.001 Pa.s
honey 1.0 Pa.s

$$V_s = \frac{2}{9}\frac{r^2 g(\rho_p - \rho_f)}{\mu}$$

surface tension ?



pendant drop



sessile drop

$$mg = \pi d\gamma \sin\alpha$$

water 0.07 Pa.m
alcohol 0.02 Pa.m

Incompressible viscous fluid

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T) = D \nabla^2 T + \frac{\mu}{2\rho C_p}[\nabla \mathbf{u} + \nabla \mathbf{u}^T]^2 + \frac{\mathbf{f} \cdot \mathbf{u}}{\rho C_p}$$

Navier-Stokes eqns.

Incompressible, inviscid fluid (and f = 0)

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = \frac{Ds}{Dt} = 0$$

Euler eqns.

Steady incompressible, inviscid flow

$$\partial \mathbf{u}/\partial t = 0; \quad \rho \mathbf{u}^2/2 + p = \mathrm{const}$$

Bernoulli eqn. (energy cons.)

Incompressible, inertia-less flow

$$\nabla p = \mu \nabla^2 \mathbf{u} + \mathbf{f}$$
$$\nabla \cdot \mathbf{u} = 0$$

Stokes eqns.

Linear !

2d flow ? axisymmetric flow ?

Laws: symmetry + invariance ?

a hydrodynamic theory of self-propelled objects ?

birds, bees, fishes, pedestrians, ...

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \lambda_1 \mathbf{u} \cdot \nabla \mathbf{u} + \lambda_2 (\nabla \cdot \mathbf{u})\mathbf{u} + \lambda_3 \nabla(|\mathbf{u}|^2) =$$
$$\alpha \mathbf{u} - \beta |\mathbf{u}|^2 \mathbf{u} - \nabla p + D_1 \nabla^2 \mathbf{u} + D_2 \nabla(\nabla \cdot \mathbf{u}) + D_3 (\mathbf{u} \cdot \nabla)^2 \mathbf{u} + \mathbf{f}$$

not Galilean invariant

1. Truncation ? 2nd order in space + cubic in velocity ...

2. Coefficients ? empirical / experimental ...

- Unusual phases/ instabilities
- Long range fluctuation effects

Compare with balance laws : symmetry/invariance used in constitutive equation.

What next ?

- Nonlinear PDE ... all terms are not equally relevant in all situations.

- Approximations: analysis (perturbation/ asymptotics), computation, SCALING

Dimensional analysis and the Pi theorem

In any equation, all terms must have the same dimensions !

$$f(q_1, q_2, \ldots, q_n) = 0 \qquad F(\pi_1, \pi_2, \ldots, \pi_p) = 0 \qquad \pi_i = q_1^{a1}\, q_2^{a2} \cdots q_n^{a_n}$$

dimensionless numbers

n number of physical variables (dimensional)

k number of independent physical units

$$p = n - k$$

E. Buckingham (1914)

examples: $$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u}$$ force/volume

μ, ρ fluid parameters

L, U extrinsic

$$n = 4;\ k = 3;\ p = 1$$

dimensionless variables $\hat{t} = tU/L;\ \hat{\mathbf{u}} = \mathbf{u}/U;\ \hat{p} = pL/\mu U$

dimensionless equation: $$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\nabla} \hat{\mathbf{u}} = \frac{\mu}{\rho U L}(-\hat{\nabla}\hat{p} + \hat{\nabla}^2 \hat{\mathbf{u}})$$

$\nu = \mu/\rho$ kinematic viscosity (momentum diffusivity)

$$Re = \frac{UL}{\nu} = \frac{\rho U^2}{\mu U/L} = \frac{\text{inertial pressure}}{\text{viscous stress}}$$

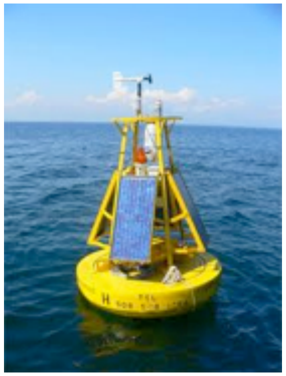
Reynolds #

$Re = 0$?; $Re = \infty$?

Stokes Euler

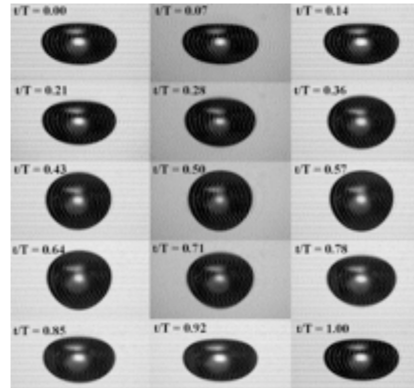
Fun with scaling !

1. Frequency of a bobbing buoy ?



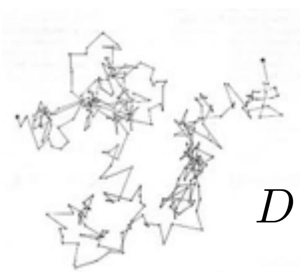
$$\omega = f(\Delta\rho, \rho_w, R, g)$$

2. Oscillations of a drop ?



$$\omega = f(\gamma, \rho, R)$$

3. Diffusion of a polymer ?



$$D = f(k_B T, \mu, R)$$

4. Tsunami warning time ?

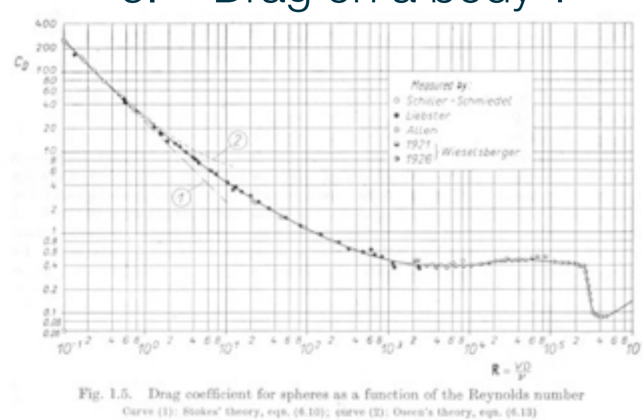
$$T = f(L, g, H, \lambda)$$

5. Yield of an atomic explosion ?

T + 0.006s

$$E = f(R, t, \rho)$$

6. Drag on a body ?



$$F = f(R, \rho, U, \mu)$$

Small does not mean negligible !

$$F = \rho U^2 g(\frac{UL}{\nu}) \qquad C_d = \frac{F}{\rho U^2} = g(Re)$$

$$Re \to \infty \ ?$$

Example $\mu\dot{x} + kx = f; \; x(0) = 0$ exact solution ... exponentials !

i.e. $\tau\dot{x} + x = f/k; \; x(0) = 0$ natural time scale $\tau = \mu/k$

2 regimes $t \ll \tau; \quad x(t) = ft/k\tau = ft/\mu$ $\qquad t \gg \tau; \quad x(t) = f/k$

initial layer ... memory of init. condn.

- Singular ODE
- Divide and conquer !

Home work ?

Langevin equation

$$\epsilon\ddot{x} + \dot{x} = \zeta(t); \; x(0) = 0, \; \dot{x}(0) = 0$$

$$< \zeta(t) >= 0, \quad < \zeta(t)\zeta(t') >= G\delta(t - t')$$

In space - **boundary layers** .... Prandtl (1905), but also Laplace, Stokes, Rayleigh, Lamb,

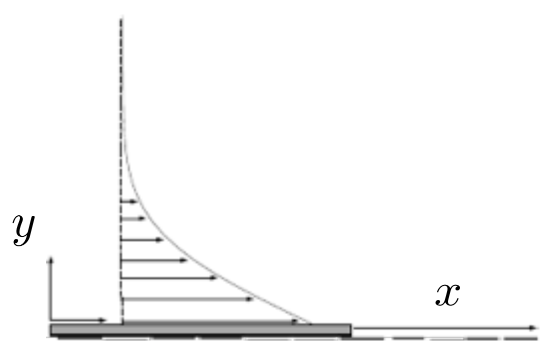
Stokes' problems: : fluid driven by transient wall motion

$$\mathbf{u} = (u(x, y, t), v(x, y, t))$$

boundary data:

I $u(x, y = 0, t) = UH(t);$

II $u(x, y = 0, t) = U \sin\omega t$



- Scaling approach
- Analytical approach

I Suddenly moving boundary ... no intrinsic/extrinsic length scale !

zone of influence: $y \sim (\nu t)^{1/2}$ momentum diffusion

II Oscillating boundary .. extrinsic frequency scale ω

zone of influence: $l_S \sim (\nu/\omega)^{1/2}$ Stokes length

$\nabla \cdot \mathbf{u} = 0$ ✓

N.S. $\partial_t u = \nu \partial_{yy} u$

I similarity solution (heat equation)

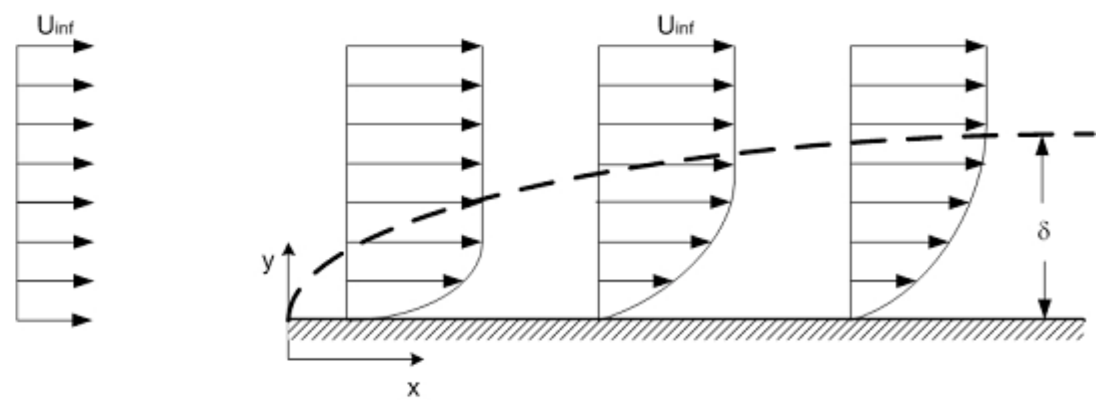
II separation of variables

Home work !

Prandtl boundary layer

Steady flow past

a semi-infinite flat plate



- effect of wall is limited to a (small) neighborhood of the wall ... at low viscosity (high Re !)

- zone of influence cannot depend on the length of the plate (infinite !) ... must be self-similar

$$\nabla \cdot \mathbf{u} = 0 \rightarrow \quad u/x \sim v/\delta \qquad\qquad \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u}$$

boundary layer assumption $\delta \ll x \rightarrow \quad v \ll u$ $\quad \rho u^2/x \sim \mu u/\delta^2 (\sim p/x) \longrightarrow \delta \sim (\frac{\nu x}{U})^{1/2}$

$\frac{\delta}{x} \sim Re_x^{-1/2}$ where $Re_x = \frac{Ux}{\nu}$ so that $Re_x \gg 1 \rightarrow \delta \ll x$

Wall shear stress $\sigma \sim \mu \partial_y u \sim \mu U/\delta \sim (\frac{\rho \mu U^3}{x})^{1/2}$ decreases with distance along plate !

Total drag force / width $\int_0^l \sigma dx \sim (\rho \mu U^3 l)^{1/2}$ skin drag

Comparison $F_p = C_d \rho U^2 l$ $\qquad F_s = C \mu U$

pressure drag / width $\qquad$ Stokes drag / length

Careful analysis: Stream function approach (see Wikipedia or Batchelor, Landau/Lifshitz)

Q. Are these steady state solutions stable ?

Q. Instability and transition (to turbulence) ?

NEXT TIME: Hydrodynamic Instability, Turbulence (briefly) + Free surface flows