THE QUANTUM HALL EFFECTS

Peculiar Phenomena that take place in 2-Dimensional Electron Systems in Strong Magnetic Fields, at Low Temperatures

→ electrons trapped at the surface of a semiconductor or the interface between 2 semiconductors

** electron-doped GaAs
* hole-doped GaAs
* Si MOSFET's
+ some other systems.

*** Single-layer systems
** Double-layer systems
* Multi-layer systems

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* Samples with different electron concentration &/or different amounts of impurity scattering

Magnetic fields from ≈1 to 30 T

The Quantum Hall Effects

INTEGER QUANTIZED HALL EFFECT

FRACTIONAL QUANTIZED HALL EFFECT

UNQUANTIZED QUANTUM HALL EFFECT

- What happens at even-denominator filling fractions, such as $\nu = 1/2$, where quantized Hall plateaus do not occur.

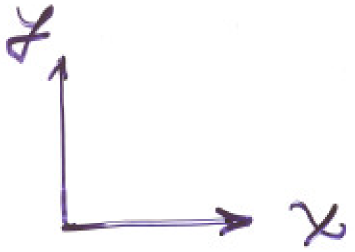
All of these occur in many varieties & flavors.

- Single layer systems; bilayers; multilayers
- Systems where spin-degree-of-freedom is important.
- Systems of different mobilities; carrier densities
- Magnetic fields ranging from ~1 to 30 T

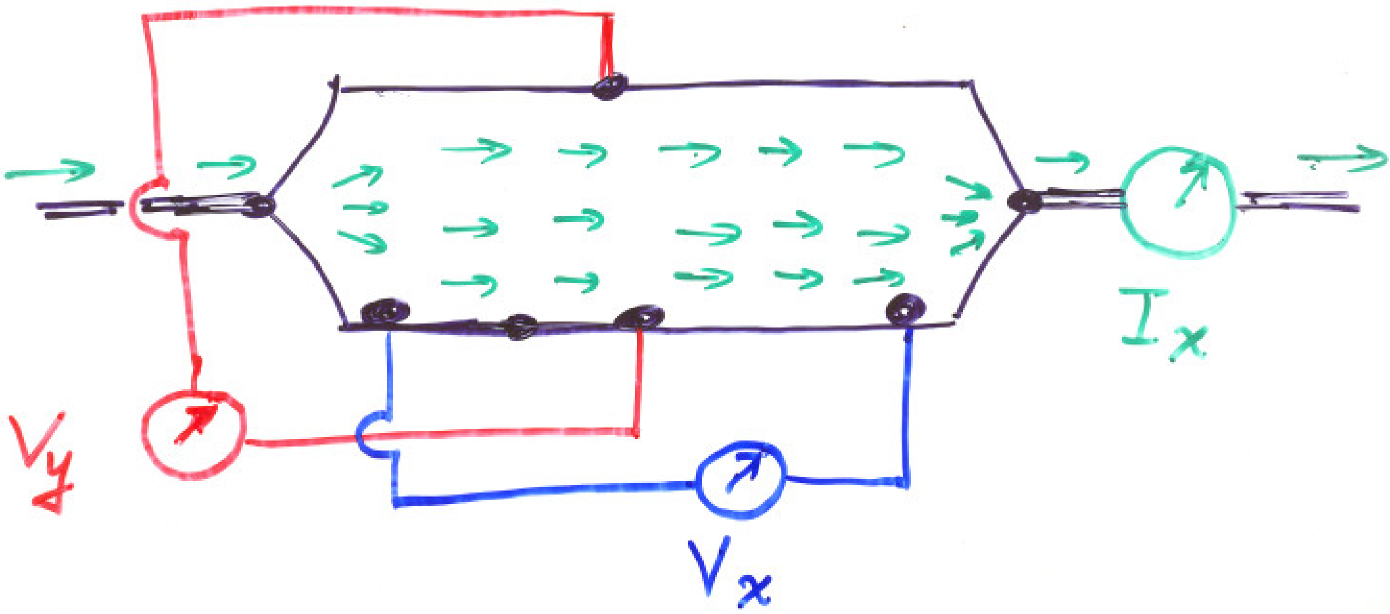
Many different measuring techniques.

MANY SURPRISING PHENOMENA!

Hall Geometry



$$\vec{B} = B_z$$



Sample: 2D electron system :

Semiconductor inversion layer
in Strong Magnetic Field
at Low-Temperature

HALL RESISTANCE : LOW TEMPERATURES

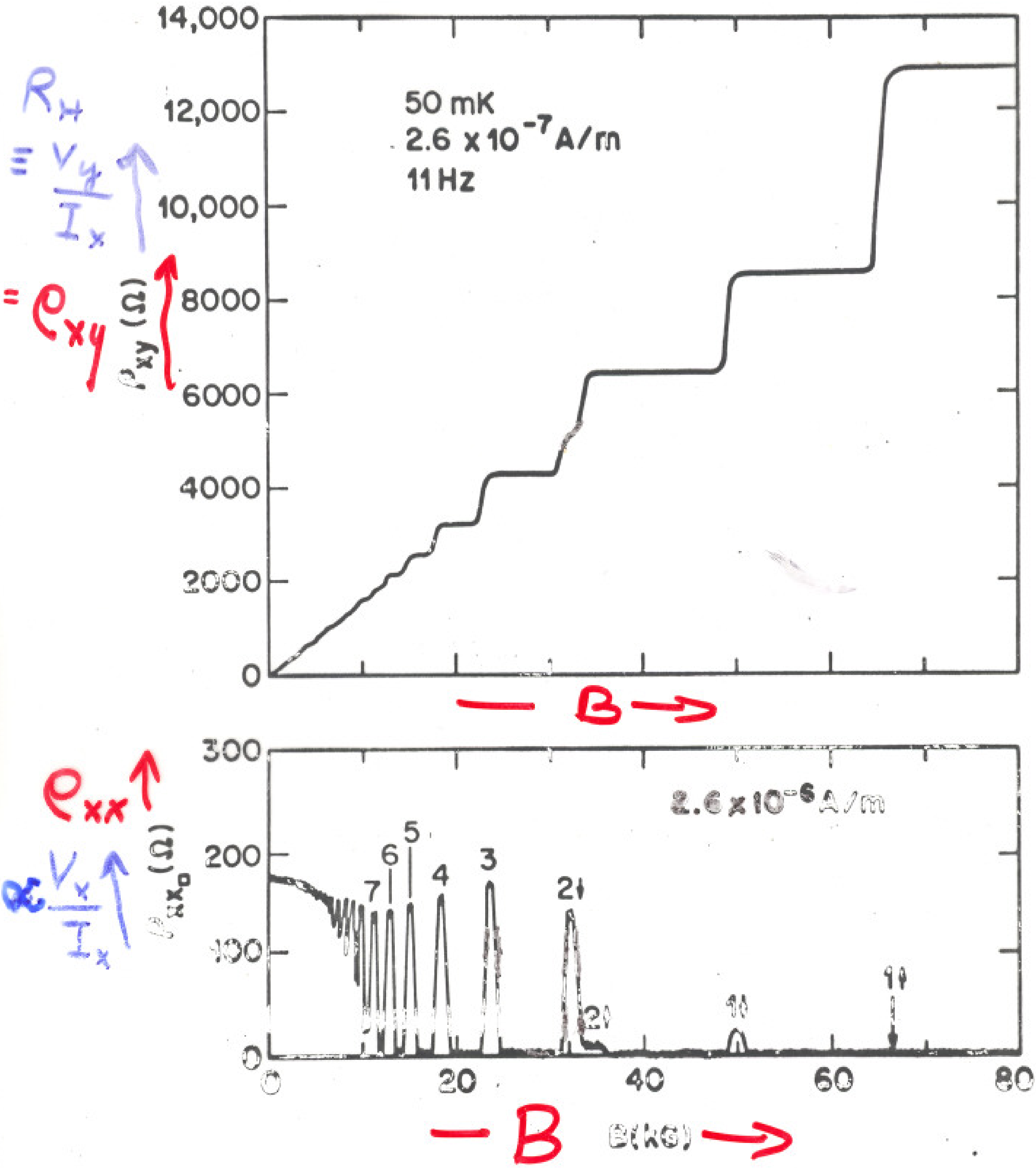


Figure 1

Paalanen, Tsui, & Gossard

GaAs - GaAlAs

Integer Quantized Hall Effect:

(von Klitzing · 1980)

On plateaus: In Limit $T \to 0$:

- $$\frac{1}{R_H} \equiv \frac{I_x}{V_y} = n\,\sigma_0$$

where n is an integer and

$$\sigma_0 \equiv \frac{e^2}{h} = \frac{\alpha c}{2\pi} \quad (\text{cgs})$$

$$\left[\frac{1}{\sigma_0} = \frac{2\pi \times 10^{-7} c}{\alpha} = 25\,812.82 \pm .02\,\Omega \quad (\text{MKS}) \right]$$

- Also: $V_x = 0$ (No dissipation)

All this is independent of precise shape or other details of the sample.

Note: When there is no dissipation: $(V_x = 0)$

$$\frac{1}{R_H} = \sigma_{xy} \quad : \text{"Hall conductivity"}$$

FRACTIONAL Q.H.E.

(Tsui, Stormer, Gossard, 1982)

In samples of very high mobility:

(Modulation-doped GaAs - GaAlAs heterojunctions)

$\exists$ additional plateaus

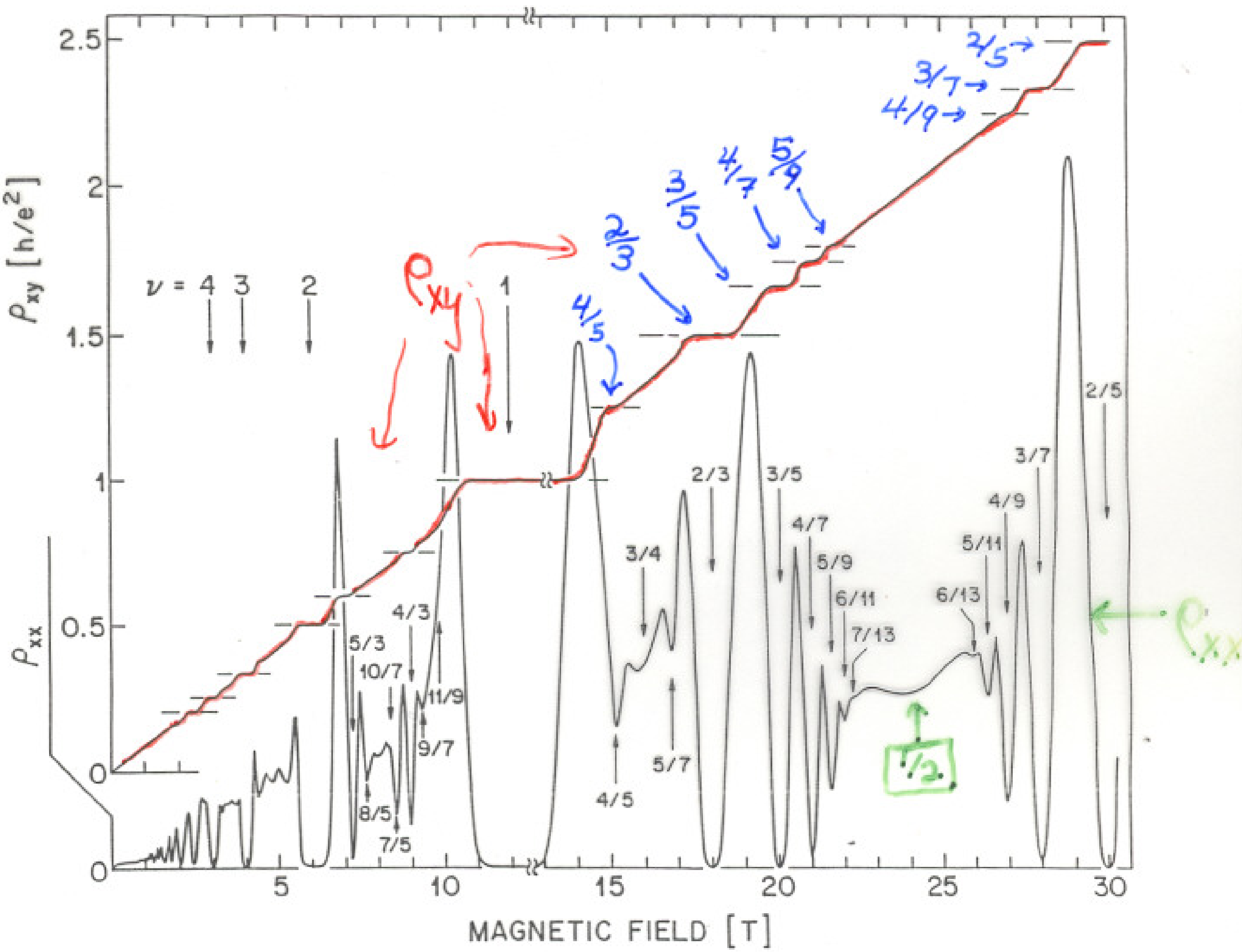
with $\sigma_{xy} = \nu \frac{e^2}{h}$

where ν is a simple rational fraction

Quantized Hall Effect in a very-high mobility 2-Dimensional Electron System

GaAs / GaAlAs Heterostructure $n = 3.0 \times 10^{11}\ cm^{-2}$

$\mu = 1.3 \times 10^{6}\ cm^2/V sec$



R Willett, J.P. Eisenstein, H.L. Störmer, D.C. Tsui, A.C. Gossard, & J H English, Phys Rev. Lett. 59 1776 (1987)

Data taken at Bitter Magnet Lab (MIT)

EXPERIMENTAL RESULTS

for Magnetic fields near $\nu = 1/2$

in high mobility single-layer systems.

- No quantized Hall plateau is observed
- ρ_{xx} is finite (250 - 2000 $\Omega/\square$)

$$\rho_{xx} \ll \rho_{xy} \approx 50{,}000\ \Omega$$

Both ρ_{xx} and ρ_{xy} are smooth functions of B. But . . .

- ∃ Strong Anomaly in propagation of Surface Sound Waves near $\nu = 1/2$

(Willett, Paalanen, Ruel, West, Pfeiffer & Bishop, 1990 + 1993)

Example: Integer Quantized Hall Effect:

Electrons in a uniform positive background - No impurities. Ignore electron-electron interaction

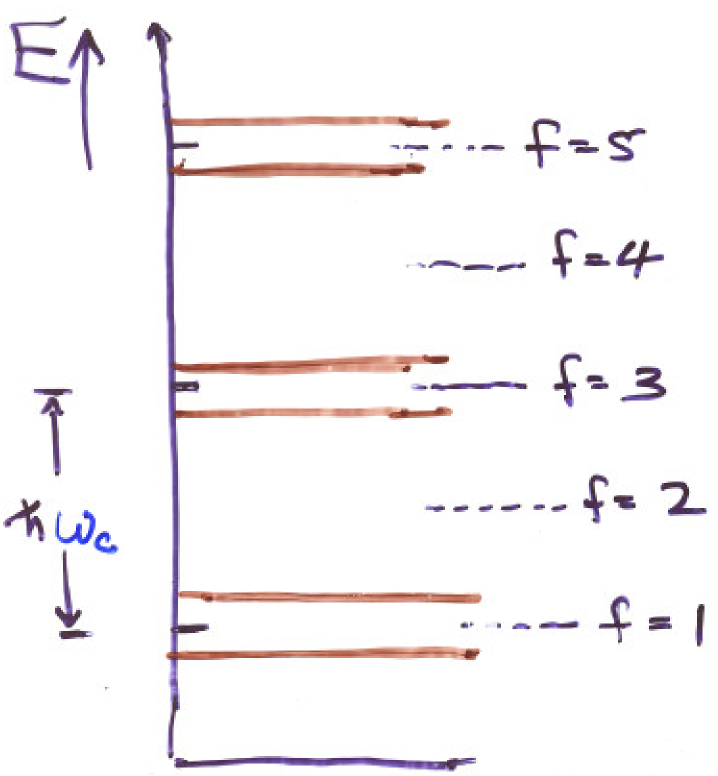
Electron Energy Levels: "Spin-split Landau Levels"

$$E_{n,\sigma} = \hbar\omega_c\left(n+\tfrac{1}{2}\right) + \frac{g^*}{2}\hbar\omega_c\sigma$$

$$\left(n = 0, 1, 2, 3, \ldots; \quad \sigma = \pm 1\right)$$

$$\omega_c \equiv \frac{e}{m^* c} B \qquad g^* = \frac{g\, m^*}{m}$$

(For GaAs: $g^*/2 \approx \frac{1}{60}$)



Degeneracy: 1 electron / flux quantum in each level.

Contribution to σ_{xy}: $= e^2/h$ for each filled level, according to classical Hall formula*: $\sigma_{xy} = \frac{Nec}{B}$

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*: Note: This formula follows from Lorentz Invariance: $\vec{E}$ and $\vec{J}$ must vanish in a frame moving with "drift velocity" $\vec{v}_D = \vec{E}\times\vec{B}c/|B|^2$

Remark: Model without impurities gives quantized Hall plateaus only if chemical potential is fixed, but not if density N is specified.

So, need a reservoir of states which do not contribute to Hall conduction to supply or remove conducting electrons as B is varied.

Impurities produce localized states (within layer) which can act as such a reservoir!

Remarkable Fact Impurities* do not affect the exactness of the quantization of the Hall conductance.

* Also: Sample-Boundaries, Electron-Electron Interaction, Bandstructure Effects

Explanations....

- Gauge Invariance Arguments
- Topological properties of phase of wavefunction in a toroidal geometry...
- etc

But must first find a stable state with no impurities

EXPLANATIONS of the FRACTIONAL Quantized Hall Effect:

- LAUGHLIN: 1983: Trial Wavefunctions for $\nu = 1/3, 2/3$ (also $1/5$, $4/5$)

  Exact for zero-range interactions
  (very good for Coulomb interactions)

- New kind of incompressible liquid.
- Energy gap for excitations
- Elementary charged excitations have

  fractional charge

  & fractional statistics*

  (Halperin 1984; Arovas, Schrieffer, Wilczek 1984)

* Particles intermediate between fermions & bosons:
Realization in nature of concept originally due to Leinaas & Myrheim 1977.

Composite Fermion Picture

- Introduced by Jain (1989) in the form of TRIAL WAVEFUNCTIONS for the groundstate, and quasiparticle excitations at the "principal" quantized Hall fractions (FQHE) of form $\nu = \frac{p}{2p+1}$: $\left(\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \ldots\right)$

- **F**ERMION **C**HERN · **S**IMONS Theory

used by Moore & Read; Lopez & Fradkin, Greiter, Wen, Wilczek } FQHE

Halperin, Lee, Read (1992), Kalmeyer & Zhang } Unquantized fractions such as $\nu = 1/2$

- also useful for Response Functions; Transport
- Similar approach used earlier in theory of "Anyon Superconductivity" by Laughlin, Kalmeyer, Fetter, Hanna, ..

2D Electrons in a Magnetic Field $\vec{B}$

$$\hbar = \frac{e}{c} = 1$$

$$H_e = \sum_i \frac{|\vec{p}_i + \vec{A}(r_i)|^2}{2m} + \frac{1}{2}\sum_{i\neq j} V(\vec{r}_i - \vec{r}_j)$$

$\vec{A}$ = external vector potential, $\ni$

$$\vec{\nabla} \times \vec{A}(\vec{r}) = \vec{B} \qquad \left(\begin{array}{l} \vec{B} \parallel \hat{z} \\ \vec{A} \text{ in } x\text{-}y \text{ plane} \end{array} \right)$$

$$V(\vec{r}_i - \vec{r}_j) = \frac{e^2}{\varepsilon\, |\vec{r}_i - \vec{r}_j|}$$

<u>Wavefunction</u> $\Psi_e(\vec{r}_1, \ldots \vec{r}_N)$ obeys <u>Fermi statistics</u>: <u>changes</u> <u>sign</u> under interchange

$$\vec{r}_i \leftrightarrows \vec{r}_j$$

<u>Schrödinger Eqn</u>: $\displaystyle i\frac{\partial \Psi_e}{\partial t} = H_e \Psi_e$

<u>Assume</u> all spins polarized $\parallel \vec{B}$; $\therefore$ <u>ignore</u> spin degree of freedom

Unitary Transformation (Generalized Gauge Transformation)

$$\Psi_{tr}\{\vec{r}_i\} \equiv \Psi_e\{\vec{r}_i\} \cdot \prod_{i<j} \left[\frac{z_i - z_j}{|z_i - z_j|}\right]^{\tilde{\phi}}$$

$(z_j \equiv x_j + i y_j)$

$\tilde{\phi}$ = even integer → preserves Fermi statistics

(We shall use $\tilde{\phi} = 2$.)

Then:

$$i \frac{\partial \Psi_{tr}}{\partial t} = H_{tr} \Psi_{tr} \, , \quad \text{where}$$

$$H_{tr} = \sum_i \frac{|\vec{p}_i + \vec{A}_i - \vec{a}_i|^2}{2m} + \frac{1}{2} \sum_{i<j} v(\vec{r}_i - \vec{r}_j)$$

$$\vec{a}_i \equiv \tilde{\phi} \sum_{j \neq i} \frac{\hat{z} \times (\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2} \equiv \vec{a}_i(\vec{r}_i)$$

$\vec{a}_i(\vec{r}) \equiv$ "Chern-Simons vector potential"

"Magnetic field" $\vec{b}_i(\vec{r}) \equiv \vec{\nabla} \times \vec{a}_i(\vec{r})$

$$= 2\pi \tilde{\phi} \sum_{j \neq i} \delta(\vec{r} - \vec{r}_j) = 2\pi \tilde{\phi} \, \rho(\vec{r})$$

where $\rho(\vec{r})$ = electron density at point $\vec{r}$, excluding electron

Mean-Field Approximation

(ie: Hartree Approximation)

Replace true Chern-Simons "magnetic field" b_i by its average value:

$$b_i \to \langle b \rangle = 2\pi\tilde{\phi}\, n_e$$

Ignore $v(\vec{r}_i - \vec{r}_j)$.

$\Rightarrow$ **Free fermions in an effective magnetic field** $\Delta B = B - 2\pi\tilde{\phi}\, n_e$

Define "Effective filling factor" $p \equiv \dfrac{2\pi n_e}{\Delta B}$

Recall: true filling factor $f \equiv \dfrac{2\pi n_e}{B}$

$$\boxed{\therefore \quad \frac{1}{p} = \frac{1}{f} - \tilde{\phi}}$$

For $\tilde{\phi} = 2 \implies f = \dfrac{p}{2p+1}$

If "effective filling factor" p is an integer:

Mean field theory ⇒ stable state!

⇒ $|p|$ filled Landau levels

Energy gap = cyclotron frequency in effective field ΔB:

$$E_g = \frac{|\Delta B|}{m} = \frac{4\pi n_e}{m\, p}$$

⇒ Original electron system has a stable state at filling factor $f = \frac{p}{2p+1}$

These are in fact the largest fractional QHE states that are observed! (Jain)

If the true filling factor is $f = \frac{1}{2}$:

(2 flux quanta per electron): $\Delta B \equiv B - 4\pi n_e = 0$

Mean-field approximation $\Rightarrow$ FREE FERMIONS in ZERO MAGNETIC FIELD

Ground state = filled Fermi sea

$$k_F = (4\pi n_e)^{1/2}$$

If this is correct, then:

$\Rightarrow$ No energy gap

$\Rightarrow$ No QHE

$\Rightarrow$ Should be able to calculate all properties of $f = \frac{1}{2}$ state using perturbation theory starting from mean-field state.

Perturbation includes: effects of $V(\vec{r}_i - \vec{r}_j)$ & fluctuations in the Chern-Simons field:

$$\Delta b_i \equiv b_i - \langle b \rangle$$

Linear Response Functions

We wish to calculate the change in charge-density $\langle \delta \rho \rangle$ and current $\langle \delta \vec{j} \rangle$ produced by an external electromagnetic field at wavevector $\vec{k}$, frequency ω.

In RPA ($\equiv$ time-dependent Hartree approximation) we treat particles as free-fermions, responding to self-consistent field, consisting of external electromagnetic field + coulomb potential produced by $\langle \delta \rho \rangle$ + self consistent Chern-Simons field:

$$\langle \delta b \rangle = 2\pi \tilde{\phi} \, \langle \delta \rho \rangle$$

$$\langle \delta \vec{e} \rangle = -2\pi \tilde{\phi} \, \hat{z} \times \langle \delta \vec{j} \rangle$$

Better approximation: include corrections from m^* and Landau fermi liquid interaction parameters.

Halperin, Lee, Read analysis:

Important effect of fluctuations in gauge field $\Rightarrow$ Large renormalization of effective mass m^*

Limited by the electron-electron interaction
(ie - the Coulomb repulsion)

If interaction $\to 0$, $m^* \to \infty$, particles do not move, picture breaks down.

Also: fluctuations in gauge field must be taken into account (via the Random Phase Approximation) to calculate transport properties, linear response to an external electric field, or to Surface Acoustic Wave.

Linear Response

EXAMPLE. HALL CONDUCTIVITY

at $f = \frac{p}{2p+1}$

"Composite Fermions" are in an integer QHE state with Hall conductivity

$$\sigma_{xy}^{f} = p \cdot \frac{1}{2\pi} \qquad \left(\frac{e^2}{h} = \frac{1}{2\pi}\right)$$

For $\vec{E}$ in y-direction:

$$\langle j_x \rangle = \sigma_{xy}^{f} \left[E_y + \langle \delta \vec{e} \rangle_y \right]$$

$$\langle \delta \vec{\varepsilon} \rangle_y = 2 \cdot 2\pi \cdot \langle j_x \rangle$$

$$\therefore \quad \langle j_x \rangle = \sigma_{xy} E_y$$

with $\sigma_{xy} = \frac{1}{2\pi} \cdot \frac{p}{2p+1}$

Hamiltonian Formulation: RPA

(Time-dependent Hartree Approximation)

$$H = \frac{1}{2m}\int d^2r\, \psi^\dagger(r)\left[\frac{1}{i}\vec\nabla + \vec A(r) - \vec a(r)\right]^2 \psi(\vec r)$$
$$+ \frac{1}{2}\int d^2r\, d^2r'\, v(\vec r - \vec r')\, \psi^\dagger(\vec r)\psi^\dagger(\vec r')\psi(\vec r')\psi(\vec r)$$

$$\vec a(\vec r) = \tilde\phi \int d^2r'\, \vec G(\vec r - \vec r')\, \psi^\dagger(\vec r')\psi(\vec r')$$

$$\vec G(\vec r - \vec r') = \frac{\hat z \times (\vec r - \vec r')}{|\vec r - \vec r'|^2}$$ (excluding infinitesimal region about $\vec r = \vec r'$)

(Note that $\vec\nabla \cdot \vec a = 0$; we are working in Coulomb gauge)

Approximation: Assume $\Psi(r_1, \dots r_N) = \frac{1}{\sqrt{N!}} \det[\phi_i(\vec r_j)]$

In Hartree-Fock Approx: $E_{HF}(\Psi) = \langle \Psi | H | \Psi \rangle$

In Hartree Approx: $E_{Hartree}(\Psi) = K_1 + K_2 + K_3 + V$

$$K_1 = \frac{1}{2m}\int d^2r \left\langle \psi^\dagger(\vec r)\left[\frac{1}{i}\vec\nabla + \vec A\right]^2 \psi(\vec r)\right\rangle = \frac{1}{2m}\sum_{j=1}^{N} \int \phi_j^* \left[\frac{\vec\nabla}{i} + A\right]^2 \phi_j\, d^2r$$

$$K_2 = -\frac{1}{m}\int d^2r\, \langle \vec a(\vec r)\rangle \cdot \langle \vec j_p(\vec r)\rangle$$

$$K_3 = \frac{1}{2m}\int d^2r\, |\langle \vec a(r)\rangle|^2 \langle \rho(\vec r)\rangle$$

$$V = \frac{1}{2}\int d^2\vec r\, d^2\vec r'\, v(\vec r - \vec r')\langle \rho(\vec r)\rangle\langle \rho(\vec r')\rangle$$

where

$$\langle j_p(\vec{r})\rangle = \mathrm{Im}\frac{1}{m}\langle \psi^\dagger(r)\vec{\nabla}\psi(r)\rangle = \frac{1}{m}\sum_{j=1}^{N}\mathrm{Im}\left(\phi_j^*\vec{\nabla}\phi_j\right)$$

$$\langle \rho(\vec{r})\rangle = \langle \psi^\dagger(\vec{r})\psi(\vec{r})\rangle = \sum_{j=1}^{N}\phi_j^*(\vec{r})\phi_j(\vec{r})$$

$$\langle \vec{a}(\vec{r})\rangle = \int d^2r'\, \vec{G}(\vec{r}-\vec{r}\,')\langle \rho(\vec{r}\,')\rangle$$

Time dependent Hartree approximation:

$$i\frac{\partial \phi_j(\vec{r})}{\partial t} = \hat{H}^{(j)}_{\mathrm{Hartree}}\phi_j = \frac{\delta E_{\mathrm{Hartree}}}{\delta \phi_j^*(\vec{r})}$$

$$\hat{H}^{(j)}_{\mathrm{Hartree}} = \int d^2r\, \psi^\dagger(r)\left[\frac{\vec{\nabla}}{i} + A(\vec{r}) - \langle \vec{a}(r)\rangle\right]^2\psi(r) + \int d^2r\, U_{\mathrm{eff}}(\vec{r})\,\psi^\dagger(\vec{r})\psi(\vec{r})$$

$$U_{\mathrm{eff}}(\vec{r}) = \int d^2r'\, v(\vec{r}-\vec{r}\,')\langle \rho(\vec{r}\,')\rangle + \langle a_0(\vec{r})\rangle$$

$$\langle a_0(\vec{r})\rangle \equiv -\tilde{\phi}\int d^2r'\, \vec{G}(\vec{r}-\vec{r}\,')\, j_{\mathrm{Hartree}}(r')$$

$$\vec{j}_{\mathrm{Hartree}}(\vec{r}) = \langle j_p(\vec{r})\rangle - \langle \vec{a}(\vec{r})\rangle\langle \rho(\vec{r})\rangle$$

Lagrangian Formulation:

$$Z = \int D\psi\, D\psi^*\, Da_i\, Da_0\, e^{-S}$$

$$S = \int_0^\beta d\tau\, d^2r \left[\mathcal{L} + \mathcal{L}_{coulomb}\right]$$

$$\mathcal{L} = \psi^*(\partial_\tau - ia_0)\psi - \frac{1}{2m}\psi^*\left[\partial_i - iA_i + ia_i\right]^2\psi$$

$$-\frac{i}{4\pi\tilde{\phi}}\,\varepsilon_{\mu\nu\lambda}\, a_\mu\, \partial_\nu\, a_\lambda \qquad -\mu\,\psi^*\psi$$

$i, j = 1, 2, \quad \mu, \nu, \lambda = 0, 1, 2.$

- Integration over a_0 gives constraint:

$$b \equiv \varepsilon_{ij}\,\partial_i\, a_j = 2\pi\tilde{\phi}\,\psi^*\psi$$

- Saddle point condition $\frac{\delta\mathcal{L}}{\delta a_i} = 0$ gives:

$$e_i \equiv i(\partial_i a_0 + \partial_\tau a_i) = \frac{1}{m}\varepsilon_{ij}\, \mathrm{Im}\left[\psi^*(\partial_j + iA_j - ia_j)\psi\right]$$