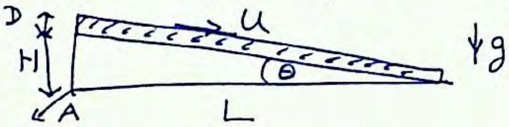
Introduction to Turbulence.

Motivation: How fast does a river flow?

This was the question that A.N. Kolmogorov apparently used to ask himself (according to G.I. Barenblatt). Let's try a variety of answers.

We could assume laminar flow. Then we would equate

$$\nu \frac{\partial^2 u}{\partial z^2} = g \sin\theta$$



$$\Rightarrow \quad \bar{u} \approx \frac{g \sin\theta\, D^2}{\nu}$$

The parameters for the Volga (I got from V. Lvov) are

$D \simeq 10\,\text{m} \quad L \simeq 3000\,\text{km} \quad H \simeq 300\,\text{m} \quad \theta \simeq 10^{-4}$

Giving: $\quad \bar{u} \sim 10^7\ \text{cm s}^{-1}!$ (a bit fast!)

We could try asking what speed we would get if we just converted potential energy to kinetic energy. This gives

$$\tfrac{1}{2}\rho u^2 = \rho g H \quad \Rightarrow \quad u \sim \sqrt{2gH} \sim 10^4\ \text{cm s}^{-1}! \quad \text{(still fast)}$$

The reason for these over-estimates is that we have neglected the nonlinearities in the Navier-Stokes equations. After we have understood some turbulence ideas, we'll see that the right way to make the estimate is the following argument.

Eddy turnover time $\tau = D/U$

Energy of eddy is $E \sim AD\rho U^2/2$

Energy dissipation rate in an eddy as it decays is

$$\epsilon = E/\tau = \frac{AL\rho U^2}{2}\frac{U}{L} = \frac{\rho A}{2} U^3$$

This dissipation occurs due to gravitational friction. The rate of working is $W =$ Force $\times$ velocity

$$= \rho A g L \sin\theta \times U.$$

Equating $\epsilon = W \Rightarrow \quad \rho \frac{A}{2} U^3 = \rho A g L \sin\theta . U$

$$\Rightarrow U \sim \sqrt{2gL\sin\theta} \sim 10\ \text{cms}^{-1}!$$

Turbulence.

NG. July 2001.

General reference: K. Sreenivasan, Rev. Mod. Phys. 71, S383–395 (Mar 1999).

Outline.

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- fully developed turbulence
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1 Turbulent phenomena.

(1) Why study turbulence?
Turbulence is widely considered to be the most important unsolved problem in classical physics. The Clay Mathematics Institute has included it in its list of 7 problems that if solved would substantially advance knowledge ($1 Million prize will be awarded for the solution). See

http://www.claymath.org

if you wish to claim the prize! Not only is it important because it is a fundamental process that

- enables internal combustion engines to work efficiently (turbulence-enhanced mixing)
- transports heat, dust, pollutants in the earth's atmosphere and oceans
- causes airplanes, ships and cars to waste energy (turbulent drag)

⋮

but it is also a fundamentally interesting process for all the same reasons that made solidification patterns interesting, i.e. it

- is a spatially-extended dynamical system with many degrees of freedom. Despite much hype and early claims to the contrary, low dimensional chaos has not been very helpful in understanding this problem, so far.
- is a multiscale phenomenon. Indeed, as we will see, it is the prime and perhaps first example of such, where the range of scales involved goes to infinity as the turbulence becomes more and more intense. As such it has much in common with the theory of critical phenomena, but so far, and despite claims to the contrary, this analogy has also not been as helpful as one would like. See G. Eyink + N.G. Phys. Rev. E 50, 4679–4683 (1994) for a discussion.

(ii) Drag.

At low speeds, the drag on an object, such as a sphere is proportional to the velocity with which it moves through a fluid. However at higher speeds,

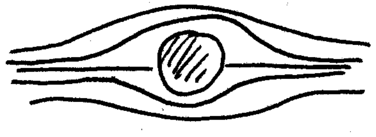
$$F_D = \frac{1}{2} C_D A \rho v^2$$

where ρ is the fluid density, A the cross-sectional area of the body, and C_D is the coefficient of drag. The reason is that at high speeds the fluid flow pattern becomes turbulent. We can make a dimensionless number that measures a priori how turbulent a flow is the Reynolds number

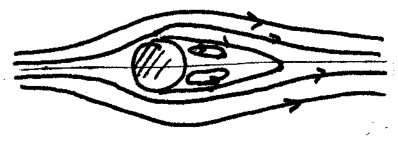
$$Re = \frac{UL}{\nu} \qquad \nu = \frac{\eta}{\rho}$$

where η is the viscosity, ρ is the density, ν is known as the kinematic viscosity, U is the characteristic velocity field, and L is a characteristic length scale of the body. For water $\nu = 10^{-2}\,\mathrm{cm^2\,s^{-1}}$. Notice that as one goes to larger and larger scales (L), the Reynolds number goes up. As the viscosity gets smaller, the Reynolds number goes up. And as the velocity goes up, the Reynolds number goes up.

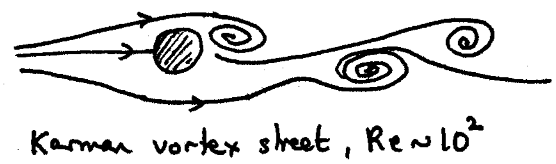
Here are some flow pictures:



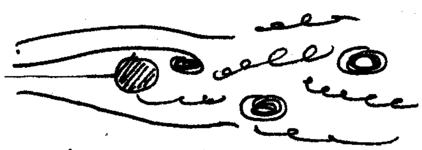
laminar, $Re \sim 10^{-2}$



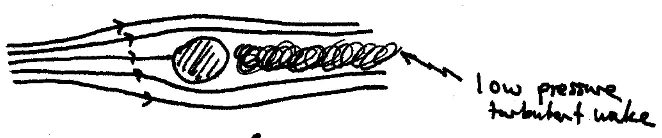
laminar, boundary layer, $Re \sim 20$



Karman vortex street, $Re \sim 10^2$

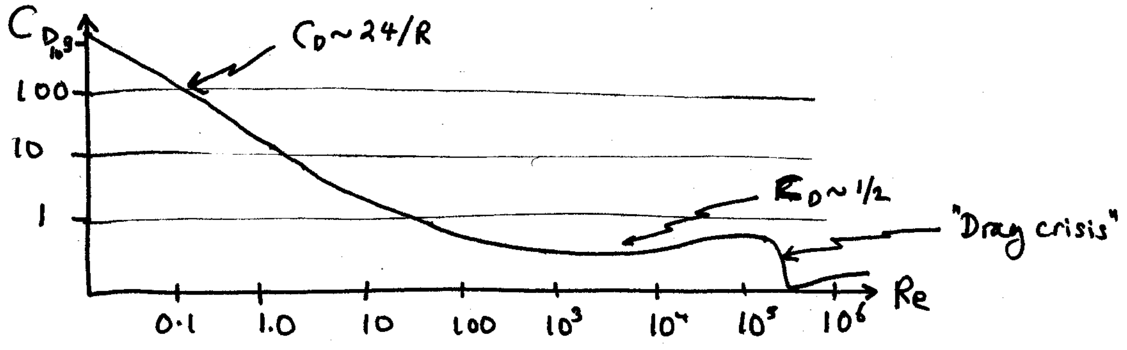


Turbulent vortex, $Re \sim 10^4$



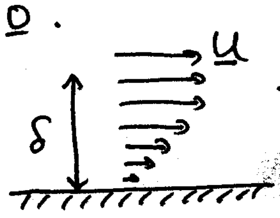
$Re \sim 10^5$ Turbulent wake

Now let's go back to the drag coefficient. The amount of drag should depend on the state of the fluid, whether or not it is turbulent, and how turbulent it actually is. This has been measured, and the result is



Q. Why doesn't C_D increase as Re increases? You'd think that the more turbulence there is, the greater would be the drag force.

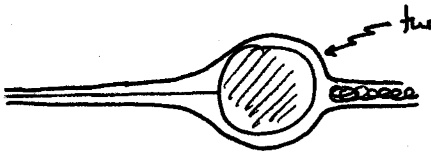
A. The answer has to do with boundary layers. Let's think about the boundary condition for the fluid velocity at the surface of the object: $\underline{v}_\perp = \underline{0}$, saying that the velocity transverse to the boundary must be zero, otherwise fluid would flow into the ball. And then there is a no-slip boundary condition: the fluid molecules next to the object bind to it and do not slip, so $\underline{v}_\parallel = \underline{0}$ too. Hence $\underline{v} = \underline{0}$.



velocity profile in laminar flow near a boundary. There is a boundary layer of thickness $\delta(Re)$ which governs how far away from the boundary the effect of the boundary causes the $\underline{v}$ to deviate from $\underline{U}$, the velocity far from the wall.

For Re ~ 20–100, the boundary layer can separate from the object, so that there is a turbulent region of low pressure behind the object. This low pressure region effectively sucks the object and causes there to be a lot of drag. The amount of drag is related to how big the wake is. At lowish Re, the boundary layer separates near the top of the body:

As the Re ↑, the boundary layer itself oscillates and becomes turbulent. Now, a turbulent boundary layer separates from the object further along the object:



Hence the turbulent wake is narrower, there is less "sucking" and so there is less drag than when there was a laminar boundary layer. The drag crisis is a result of the boundary layer becoming turbulent.

So we now need to understand why a turbulent boundary layer separates less readily than a laminar boundary layer. The laminar boundary layer essentially gets sucked off the object by the fast moving fluid outside the boundary layer, and is virtually a static object. A turbulent boundary layer has momentum mixed-in from the faster-moving fluid outside it, and so the momentum makes it go round the object further before it gets sucked off. So turbulence delays boundary layer separation and therefore reduces the drag!

[This is why golf balls are dimpled: it increases the turbulence in the boundary layer, so reduces the drag.]

(iii) Turbulent diffusion

Suppose one considers a fluid in which heat is diffusing. If the fluid is at rest or flowing in a laminar fashion, the heat transport will have a certain diffusion coefficient. If the fluid is turbulent, however, the velocity fluctuations also spread the heat. As a result the thermal diffusion coefficient appears to be greatly enhanced. Consider two markers placed in a fluid and allowed to undergo Brownian motion. The distance between them, $R = |\underline{x}_1 - \underline{x}_2|$, will grow as

$$R \sim (Dt)^{1/2}$$

or

$$\frac{d}{dt} R^2 = \text{constant}.$$

$$\frac{d}{dt}\Delta v = \Delta F(t), \quad \langle \Delta F \Delta F' \rangle = 6\epsilon\,\delta(t-t')$$

$$\langle \Delta v(t_1)\Delta v(t_2)\rangle = \int_0^{t_1}\int_0^{t_2} \langle \Delta F \Delta F'\rangle\, dt\, dt' = 6\epsilon \min(t_1, t_2)$$

$$\rightarrow \langle \Delta v^2 \rangle = 6\epsilon t$$

$$\rightarrow \langle \Delta r^2(t)\rangle = \int_0^t \int_0^t \langle \Delta v(t_1)\Delta v(t_2)\rangle\, dt_1\, dt_2 = 2\epsilon t^3$$

In 1926, Richardson observed that if the same experiment is done in a turbulent fluid, then

$$\frac{d}{dt} R^2 \propto R^{4/3} \qquad \text{i.e.} \qquad \frac{dR}{dt} \sim R^{1/3}$$

This observation immediately tells us something interesting: writing

$$\dot{\underline{x}} = \underline{v}(t, \underline{x})$$

we have

$$|\underline{v}(\underline{x}_1, t) - \underline{v}(\underline{x}_2, t)| \sim |\underline{x}_1 - \underline{x}_2|^{1/3}$$

We will shortly look at moments of velocity differences averaged over the flow i.e.

$$v_r \equiv [\underline{v}(\underline{x}+\underline{r}) - \underline{v}(\underline{x})] \cdot \frac{\underline{r}}{|\underline{r}|}$$

where the scalar product ensures that we are looking at the component parallel to the vector connecting $\underline{x}$ and $\underline{x}+\underline{r}$.

Then, defining the structure functions

$$S_n \equiv \langle v_r^n \rangle$$

we anticipate

$$S_3 \sim r.$$

② Concepts.

(1) Cascade.

There are several key ideas in turbulence, which essentially form a dogma that is pervasive in all the literature. The first is the idea of a turbulent <u>cascade</u>. To talk about the cascade, we first must mention the word <u>eddy</u>. An eddy is to a fluid mechanic what a quasi-particle is to a solid state physicist: a term describing an object that is hard to define, but intuitively obvious. In fluid mechanics, an eddy is a swirling fluid motion with a characteristic length, velocity and time scale. As early as Leonardo da Vinci (arguably) and certainly by 1926 when Richardson wrote on the subject, it was noticed that turbulent fluids consisted of a spectrum of different size eddies.

Leonardo wrote:

"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to the random and reverse motion .....

.... the small eddies are almost numberless and large things are rotated only by large eddies and not by small ones, and small things are turned by both small eddies and large."

Richardson, on the other hand, was a little more whimsical.

August de Morgan, parodying Jonathan Swift, had written:

Big fleas have little fleas
Upon their backs to bite 'em
And little fleas have lesser fleas
And so, ad infinitum.

And the great fleas themselves, in turn,
Have greater fleas to go on
While these again have greater still,
And greater still and so on.

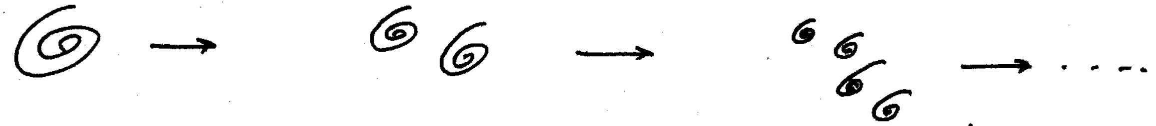
This might sound like a poetic vision of scaling. However, Richardson described the cascade as

Big whorls have little whorls
Which feed on their velocity
And little whorls have lesser whorls
And so on to viscosity.

(in the molecular sense).

The last line tells us something important physically: the process of creating the little whorls stops when they get so small that viscosity prevents their existence as separate, long-lived objects.

So what is the process by which big eddies break up into small eddies?



Richardson, and later Kolmogorov, hypothesized that this process is Hamiltonian. i.e. is a result of the eddy dynamics, and only when the length scale gets down to the smallest scales does viscosity affect the dynamics. This is a far reaching proposal, as we'll see.

(ii) Fully-developed turbulence

The great fluid mechanic G. I. Taylor proposed the notion of *fully-developed turbulence*. Loosely speaking, the concept is that at sufficiently large Re, and between a range of length scales which are Re dependent, all turbulent flows are identical statistically, with local isotropy and homogeneity. This limiting state may or may not exist. Certainly, reasonable doubts can be cast (see G. I. Barenblatt and NG, Phys. Fluids. 7, 3078 (1995).). The assumption of fully-developed turbulence has allowed fluid mechanicians to compare the statistical results of flows that are very different on large scales.

What is the appropriate range of scales? Clearly the scales must occur at short distances, where somehow the details of how the turbulence has been created are irrelevant. If the turbulence was due to (e.g.) a propellor of size L, then we are focusing on scales $\eta \ll r \ll L$, where η is the scale where friction dissipates energy. We have the picture then of the cascade of eddies sending energy from $L \rightarrow \eta$ where dissipation occurs. Another way to think of this is: *universality*. We are postulating universal statistical properties at small scales independent of the details of the forcing at the scale L. Note that this is the opposite of what happens near a critical point, where it is the long wavelength physics which is universal and independent of the short range properties.

(iii) Coherent structures.

Coherent structures are long-lived large scale fluid motions on the scale of L. They show that one cannot simply model turbulence as noise. Their origin and connection to the statistical properties of turbulence are poorly understood. How they self-organise would be an interesting topic to study.

3 Exact results.

There are very few exact statements that can be made about turbulence. Here I'll present two that are especially powerful: the so-called von Karman-Howarth relation, which is usually expressed in a form known as the Kolmogorov $\frac{4}{5}$ law. And the Doering-Constantin bound on the energy dissipation rate.

(i) Kolmogorov 4/5 law.

We already saw heuristically that

$$S_3(r) \sim r$$

from the Richardson observation of tracer trajectories. Kolmogorov showed that if we assume that a turbulent flow is isotropic and homogeneous, that

$$S_3(r) = \left\langle \left[(\underline{v}(\underline{x}+\underline{r}) - \underline{v}(x)) \cdot \frac{\underline{r}}{|\underline{r}|} \right]^3 \right\rangle = - \frac{12}{d(d+2)} \bar{\epsilon} r$$

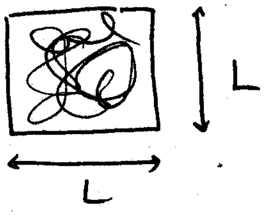
in the limit $Re \to \infty$. In $d=3$, this reduces to

$$S_3(\underline{r}) = -\frac{4}{5} \bar{\epsilon} r.$$

$\bar{\epsilon}$ is the mean energy dissipation rate. The $\frac{4}{5}$ law is derived from the Navier-Stokes equations. However, we will see that it can also be understood from scaling arguments — but one does not know from the scaling argument that the result is correct (scaling may not apply). So it is valuable that we have a rigorous derivation.

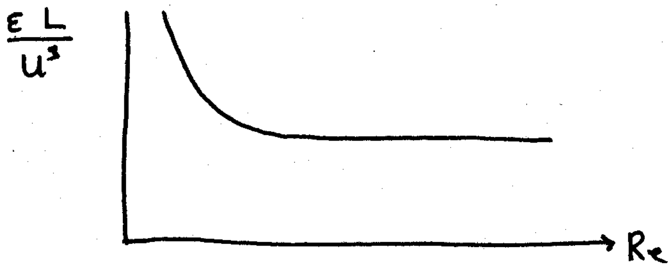
(ii) Doering-Constantin bound.

This bound discovered only ten years ago and proved by elementary means is a little gem. It concerns the energy dissipation rate ε.



$$\varepsilon \leq c \frac{U^3}{L}$$

where U is a characteristic velocity (e.g. $\frac{3}{2}U^2 = \int_0^\infty E(k)\,dk$ where E is the energy power spectrum). Empirically it is found that



i.e. the bound saturates. This result has been useful in (e.g.) understanding wall-bounded turbulent shear flows. Extensions of it have been derived for a variety of flow situations.

Notice something very strange about the bound. It does not depend on the viscosity!

What would we have guessed for ε? From the definition of viscosity (in the Navier-Stokes equation $\frac{\partial \underline{v}}{\partial t} = \cdots + \nu \nabla^2 \underline{v}$) we expect that

$$\epsilon_{\text{viscous}} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\underline{v}^2\right) \sim \frac{\nu}{L^2} U^2$$

However, the Doering - Constantin bound gives

$$\epsilon_{DC} \sim \epsilon_0 \frac{U^3}{L}$$

The ratio

$$\frac{\epsilon_{DC}}{\epsilon_{\text{viscous}}} \sim \frac{U^3}{L}\cdot\frac{L^2}{\nu U^2} \sim \frac{UL}{\nu} = Re \gg 1$$

Thus, the energy dissipation process implied by the saturation of the Doering – Constantin bound is much more effective than regular friction. This process is the cascade: the eddies create smaller and smaller eddies until molecular viscosity can set in.

(4) Scaling laws.

We've already seen that there is a plausible argument for the cascade and the Doering – Constantin bound has hinted that the eddy dynamics is indeed Hamiltonian. Let's explore this more, following a celebrated argument of Kolmogorov (1941), usually known as K41.

Since the eddy dynamics is Hamiltonian, it is non-dissipative by definition, and therefore when we try to estimate statistical quantities relevant to turbulence, the viscosity should not enter. If we wish to estimate ϵ, we only have U, L (the scale of generation of turbulence) at our disposal. The only quantity we can create with the units of ϵ is U^3/L. Hence we predict

$$\epsilon \sim U^3/L.$$

Let's calculate the structure functions

$$S_n(\underline{r}) \equiv \left\langle \left[\left(\underline{v}(\underline{x}+\underline{r}) - \underline{v}(\underline{x}) \right) \cdot \frac{\underline{r}}{|\underline{r}|} \right]^n \right\rangle$$

The S_n have dimensions $\left(\frac{L}{T}\right)^n$. We only have $\bar{\epsilon}$ and $\underline{r}$ with which to construct the S_n.

$$[\bar{\epsilon}] = L^2/T^3$$

$$[r] = L$$

L = length unit
T = time unit.

$$\therefore \quad S_n(r) = C_n (\bar{\epsilon} r)^{n/3}$$

where C_n are dimensionless constants only dependent on the geometry of the flow.

In particular, note that:–

$$n=3: \quad S_3(r) \sim C_3(\bar{\epsilon} r)$$

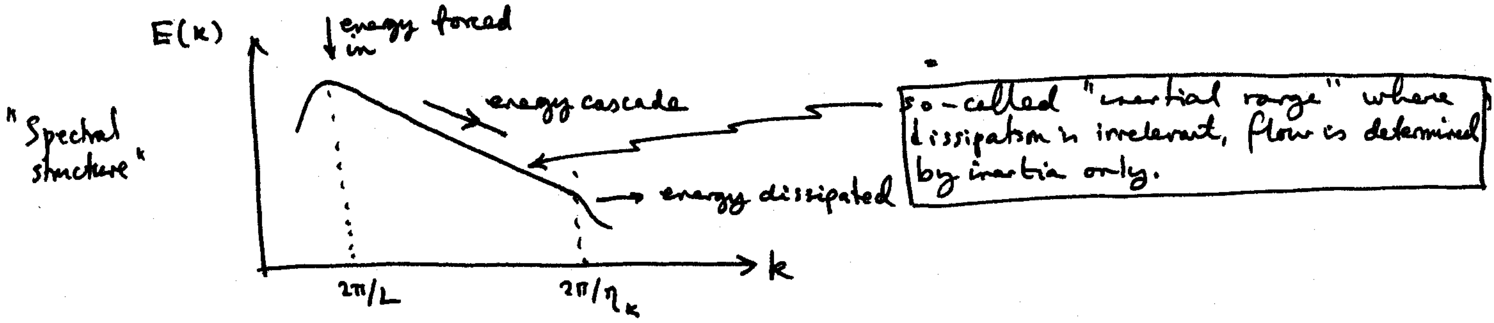
in agreement with the 4/5 law.

$$n=2: \quad S_2(r) = C_2(\bar{\epsilon} r)^{2/3}$$

Sometimes this is written in terms of the velocity power spectrum:

$$E(k) = 4\pi k^2 \langle |V_k|^2 \rangle \sim \bar{\epsilon}^{2/3} k^{-5/3}$$

The scaling should exist on a scale intermediate between that of L and that where molecular viscosity sets in, which we'll call η_K.



<u>Extended self-similarity</u>: some workers have tried to improve their ability to observe scaling in turbulent correlation functions. Suppose you have to try and measure $S_n(r) = C_n(\bar{\epsilon} r)^{\zeta_n}$ to determine the exponent ζ_n. K41 predicts $\zeta_n = n/3$ but what about experiment? Estimating high order correlation function is tricky from finite data. Ideally, you would plot $S_n(r)$ as measured against $(\bar{\epsilon} r)^{\zeta_n}$ as measured, and vary ζ_n until you got a straight line. The data are usually not turbulent enough that the inertial range is very well defined. Only if you have data in huge atmospheric storms or huge ocean tides do you have a high enough Re that there is a decent power law exhibited. So one can do the following: replace $\bar{\epsilon} r$ by $S_3(r)$! Then one

More scaling laws....

Q. How does the width of the inertial range scale with Re?

A/ To answer this, we look at the eddies on scales r within the inertial range. The eddy turnover time is

$$\tau_r = \frac{r}{v_r}$$

where $v_r = (\bar{\epsilon} r)^{1/3}$, which is an estimate of the time for the energy to be transfered between scales. The scale dependent Reynolds number is then

$$Re_r = \frac{v_r \cdot r}{\nu} = \frac{(\bar{\epsilon} r)^{1/3} r}{\nu}$$

For $r = L$ we get

$$Re_L = \frac{\bar{\epsilon}^{1/3} L^{4/3}}{\nu}$$

The dissipation scale η_K is defined as the scale where the flow gets so slow that $\epsilon \sim \nu u^2/\ell^2$ i.e. $Re = O(1)$.
Using the definition

$$Re_{\eta_K} \equiv 1 \Rightarrow 1 = \frac{\bar{\epsilon}^{1/3} \eta_K^{4/3}}{\nu}$$

and thus $\eta_K = (\nu \bar{\epsilon}^{-1/3})^{3/4}$.

In particular

$$\boxed{Re = \frac{Re_L}{Re_{\eta_K}} = \left(\frac{L}{\eta_K}\right)^{4/3}}$$

This shows that $L/\eta = Re^{3/4}$ and that the number of degrees of freedom active in turbulence in 3D is $\left(\frac{L}{\eta}\right)^3 \sim Re^{9/4}$.

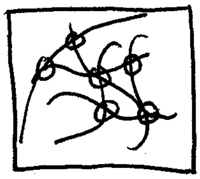
plots $S_n(r)$ vs $S_3(r)^{\zeta_n}$ and tries to determine the exponents ζ_n. This works much better, empirically, and exponents have been determined.

<u>Exp</u> : $\zeta_3 = 1.0$ $\quad \zeta_2 = 0.70$ $\quad \zeta_4 = 1.28$ $\quad \zeta_5 = 1.53$ $\quad \zeta_6 = 1.77$ $\quad \zeta_7 = 2.01$

<u>K41</u> : $\zeta_3^K = 1.0$ $\quad \zeta_2^K = 0.67$ $\quad \zeta_4^K = 1.33$ $\quad \zeta_5^K = 1.67$ $\quad \zeta_6^K = 2.0$ $\quad \zeta_7^K = 2.33$

As you can see there are deviations from K41. The reason for the deviations is believed to be fluctuation in the energy dissipation. We'll see that computer simulations show that the regions of high vorticity and dissipation are focused on vortex tubes that fluctuate around. Dissipation occurs strongly when they intersect

$Re \gg 1$



o = regions of intense dissipation

~ = regions of intense vorticity.

You might speculate that as $Re \to \infty$, there are more and more vortex tubes, more and more spatially homogeneous dissipation,

$Re \ggg 1$

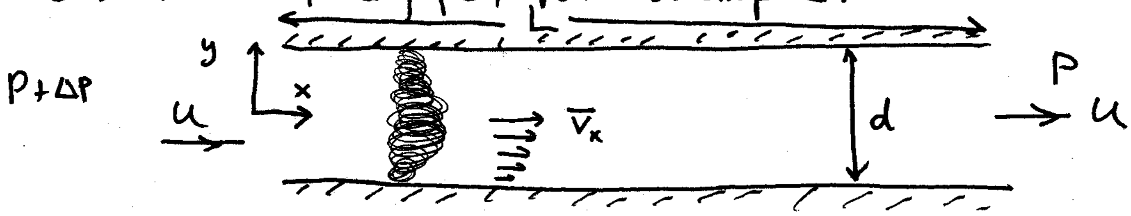
~ region of intense vorticity

• regions of intense dissipation

and so perhaps K41 is exact as $Re \to \infty$.

⑤ Wall-bounded turbulent shear flow: the Law of the Wall.

There is a nice analogy between the spectral structure of turbulence and the velocity as a function of distance from a wall, in a turbulent flow. Let's think of a pipe, for example.



We are interested in knowing how the average velocity in the x-direction $\bar{V}_x$, averaged over time, varies with vertical distance y. Let's use scaling à la K41.

The natural scale for velocity is $u^* = \sqrt{\tau/\rho}$ where τ is the shear stress exerted on the wall of the pipe and ρ is the fluid density.

$$\tau = \frac{\Delta P}{L}\frac{d}{4}$$

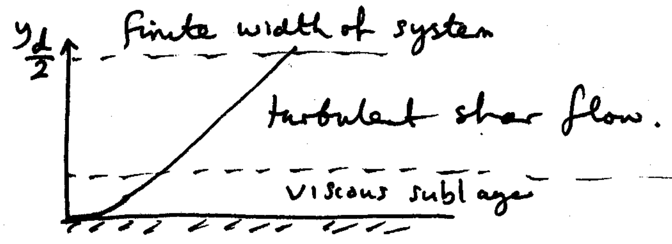
where ΔP is the pressure drop across the pipe of length L and diameter d. We'll write $\phi = \frac{u}{u^*}$

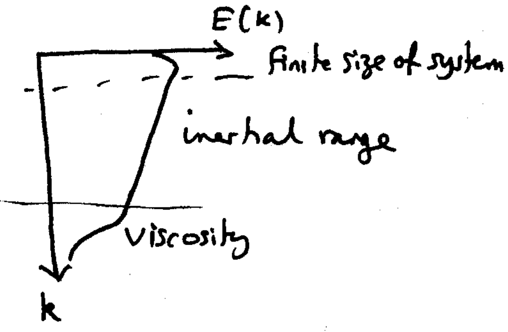
We can make a dimensionless measure of distance y by writing

$$\eta = \frac{u^* y}{\nu}$$

Lastly we have the Reynolds number $Re = \frac{Ud}{\nu}$ where U is the mean velocity averaged over the cross-section.

Near the wall, there is a viscous boundary layer, but beyond that is a turbulent shear flow.





Dimensional analysis.

$$\partial_y u = F(y, \tau, d, \nu, \rho) = \frac{u^*}{y} F_1(\eta, Re)$$

i.e. $$\partial_\eta \phi = \frac{1}{\eta}\Phi(\eta, Re)$$

For large η, large Re, we replace $\Phi(\eta, Re) \to \Phi(\infty, \infty) = \frac{1}{\kappa}$.

Then $$\partial_\eta \phi = \frac{1}{\eta\kappa} \implies \boxed{\phi = \frac{1}{\kappa}\ln\eta + B}$$

This is the "Law of the Wall" — a logarithmic dependence of the velocity away from the wall. It works well and

$$0.36 < \kappa < 0.44$$
$$5 < B < 6.3$$

Huh? κ and B should be universal constants according to the derivation. These large variations indicate that perhaps there is a weak Re or flow dependence, both of which would be violation of K41 and the assumption therein.

It can be shown that a power law form as used for decades by engineers:

$$\phi \propto \eta^{\alpha}$$

violates (e.g.) the Doering-Constantin rigorous bound. Engineers empirically used an α that is Re dependent. In fact, one can show that the leading asymptotic form of $\alpha(\mathrm{Re})$ consistent with the Doering-Constantin bound is

$$\alpha \propto \frac{1}{\log_e \mathrm{Re}}$$

and a very detailed and extensive analysis by Barenblatt and Chorin has shown that this provides an excellent fit to data and explains in a beautiful way the deviation from the law of the wall, and the values of κ and B.

<u>Summary</u>. The classical turbulence studies, K41 and law of the wall and other simple scaling results do not precisely agree with experiment and more advanced methods are needed to explain the anomalies & the breakdown of simple scaling. We'll see in my RG lecture that this is closely related to the breakdown of mean field theory at critical points.

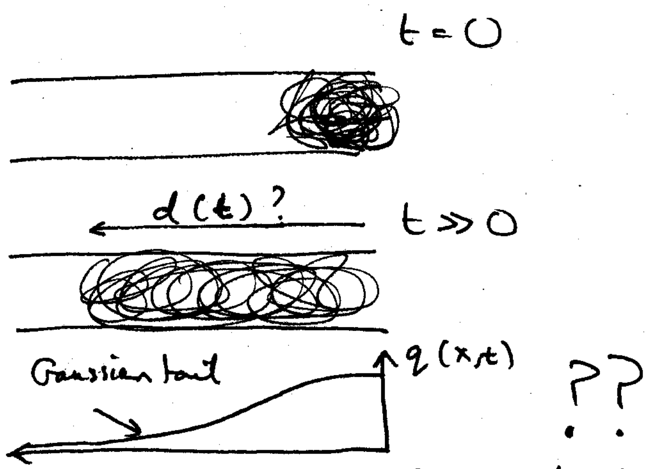
6 Other topics

(1) Propagation of turbulence.

Here's an interesting question for which the answer is not as well known as it should be. How does turbulent energy spread in a system which is turbulent at one end but quiescent at the other? How fast does it spread?

A first guess might be that the energy diffuses so that

$$d(t) \sim t^{1/2} \quad ?$$



In fact, if this were true, we'd also expect that the turbulent energy distribution over space followed a Gaussian.

Let $q(x,t) = \langle \frac{1}{2} u^2(x,t) \rangle$ where the averaging is over time long compared to the eddy turnover time but $\ll t$ governing the motion, and over a coarse graining volume whose size Δ is $\eta_* \ll \Delta \ll L$. Then diffusion guesses are as shown in the picture above. Let's approach this from the K41 point of view.

(a) Decay of turbulence.

First let's ask an even simpler problem. If the turbulence

is homogeneous and spatially uniform, i.e. $q(x,t) = q(t)$.
How does $q(t)$ vary in time? K+L told us that because we cannot use ν in the dimensional analysis,

$$\epsilon = \frac{dq}{dt} = -\epsilon_0 \frac{u^3}{L} = -\frac{\epsilon_0 q^{3/2}}{L}$$

This equation has been tested experimentally, using superfluid helium as a test fluid. (See Smith et al, PRL 71, 2583 (1993)).

Now let's add space. We expect the energy to have an associated current $\underline{J} = K \underline{\nabla} q$ and so

$$\frac{\partial q}{\partial t} = -\underline{\nabla} \cdot \underline{J} - \epsilon_0 q^{3/2}/L$$
$$= \underline{\nabla} \cdot (K \underline{\nabla} q) - \frac{\epsilon_0 q^{3/2}}{L}$$

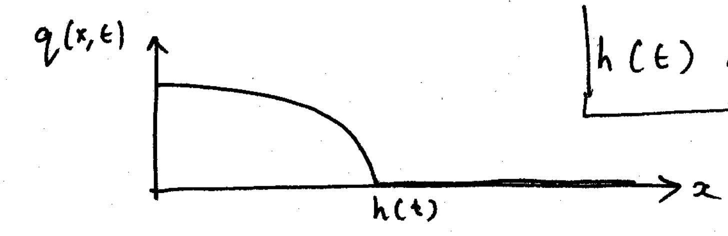
Now what is the energy diffusion constant K? By dimensional analysis, it can only depend on $L\sqrt{q}$.

$$K = K_1 L \sqrt{q}.$$

$$\therefore \quad \boxed{\partial_t q = K_1 L \underline{\partial} \cdot (\sqrt{q}\, \underline{\partial} q) - \frac{\epsilon_0 q^{3/2}}{L}} \qquad (*)$$

This is a non-linear diffusion equation and it can be solved by RG techniques (L. Chen + NG. Phys. Rev. A 45, 5572 (1992)).
In the turbulence burst problem, the right value for L is the size of the turbulent burst, i.e. $L \propto h(t)$. That is we solve Eqn (*) in the interval $0 < x < h(t)$ where $h(t)$ is to be determined — another moving boundary problem.

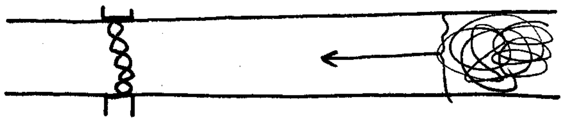
The solution turns out to be



$$\boxed{h(t) \sim t^{2/3+\alpha}}$$

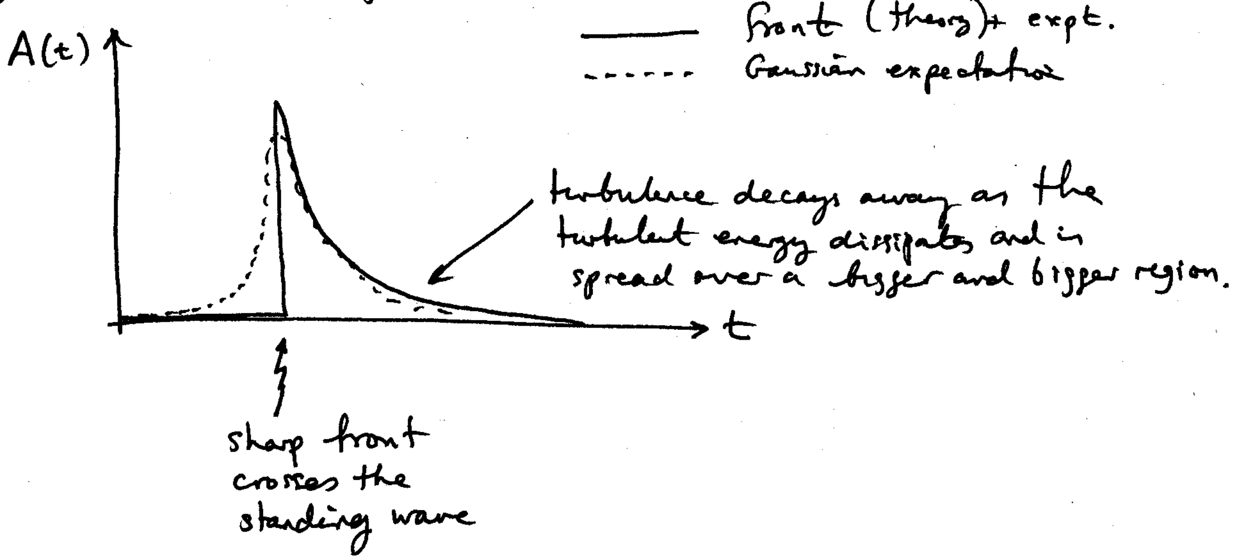
where α is an anomalous exponent, $\alpha = O(\epsilon_0)$, that can be calculated by RG. The important point is that $h \sim t^{2/3}$ and that there is no turbulent tail: in fact there is a sharp front! Turbulence bounds itself with sharp fronts. When you run into bona fide air turbulence, it happens suddenly!

Here's how we tested this qualitatively. We did measurements of second sound attenuation in He II, a superfluid. At large Re, the superfluid behaves like water, but with a viscosity 1/100 that of water. The finite viscosity derives from dissipative interactions between quasi-particles and quantum vortex lines. A detailed discussion is in W. Vinen, PRB 61, 1410 (2000). Second sound is attenuated by quantum vortex lines, which follow the regions of high vorticity in the fluid.



second sound standing wave

When the turbulence crosses the standing wave, the attenuation goes up. So we measure attenuation vs. time, and using the solution for $q(x,t)$ one can predict $A(t)$:



If we had a Gaussian tail, the rise would not have been abrupt.

Turbulence in 2D.

Goal: * explain the special simplifications of 2D hydrodynamics

↳ no vortex stretching

↳ Dual cascades

* explain the different cascade directions
* familiarise students with soap film turbulence

(1) Vorticity equation.

From N-S: $\frac{D\underline{u}}{Dt} \equiv \partial_t \underline{u} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}$

we define vorticity $\underline{\omega} = \underline{\nabla} \times \underline{u}$, and find the equation of motion

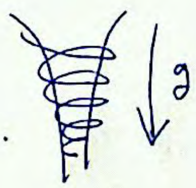
$$\frac{D\omega_i}{Dt} = (\underline{\omega}\cdot\underline{\nabla})u_i + \nu\nabla^2\omega_i$$

Suppose that at some instant in time $\underline{\omega} = (0,0,\omega_z)$.

Then

$$\frac{D\omega_z}{Dt} = \omega_z \frac{\partial u_z}{\partial z} + O(\nu)$$

If $\frac{\partial u_z}{\partial z} > 0$ ω_z grows, showing that vorticity is not conserved in 3D. Fluid circulating in a funnel accelerates and accumulates ω_z, so that $|\underline{\omega}|$ increases in direction of velocity gradient. This is called vortex stretching.



In 2D, $\underline{\omega} = (0,0,\omega_z)$, whereas $\underline{u} = (u_x, u_y, 0)$.

Thus $(\underline{\omega}\cdot\underline{\nabla})\underline{u} = \underline{0}$ and

$$\frac{D\omega}{Dt} = O(\nu).$$

Exercise: Assuming $\nu = 0$, show that $Z \equiv \frac{1}{A}\int_A \omega^2 d^2r$ is a constant of motion, where A = area of fluid domain. Show that all other moments of ω are conserved too when averaged over A.

Exercise: Show that $E = \frac{1}{2A}\int u^2 d^2\underline{r}$ is conserved in any dimension, in the $\nu = 0$ limit.

(2) Cascades in 2D.

Since a real fluid has viscosity, we know that kinetic energy gets dissipated through the terms of $O(\nu)$. So in a turbulent fluid, we must inject energy and in 2D enstrophy to keep the fluid in a non-equilibrium steady state. It is conventional to define injection rates

$$\epsilon = -\frac{dE}{dt} \qquad \text{(energy)}$$

$$\beta = \frac{dZ}{dt} \qquad \text{(enstrophy)}$$

Following the idea of a cascade in 3D, we might conjecture than in 2D there can be Hamiltonian dynamics of cascades with two conserved quantities: energy and enstrophy. The latter reflects the fact than angular momentum operators commute in 2D, and so there should be a conservation law reflecting this.

Exercise: What are the dimensions of Z, and thus of β?

Defining as before the longitudinal structure function as

$$S_n \equiv \left\langle \left[\left(\underline{u}(\underline{x}+\underline{r}) - \underline{u}(\underline{x})\right)\cdot \frac{\underline{r}}{r} \right]^n \right\rangle \equiv \langle \delta \underline{u}(r)^n \rangle$$

one finds that the vonKarman-Howarth relation becomes

$$S_3 = -\frac{3}{2}\epsilon r \qquad \text{(but see below)}$$

while assuming ν is not involved in the cascade of enstrophy yields by dimensional arguments

$$S_2(r) \propto \beta^{2/3} r^2$$

Q/ What happens to the enstrophy cascade at small scales?

A/ Eventually molecular viscosity cuts in, and the fluid consists of small patches of local shear, so that velocity differences $\sim r$, and $S_2 \propto r^2$.

Thus, the dissipation scale has the same scaling as the enstrophy cascade

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- Exercise: assuming that Re depends on scale, as we did in 3D, show than in 2D, the analogue of the Kolmorov scale η_K, below which scale the flow has dissipation is

$$\eta_K^{2D} \sim \nu^{1/2} \beta^{-1/6}$$

Summary: In 2D, there can in principle be two cascades, one for enstrophy, one for energy.

In Fourier space, their structure functions have different power laws, which are

$$E(k) \propto k^{-5/3} \quad \text{energy cascade}$$
$$\propto k^{-3} \quad \text{enstrophy cascade}$$

- (3) Direction of cascade, and the scale of forcing.

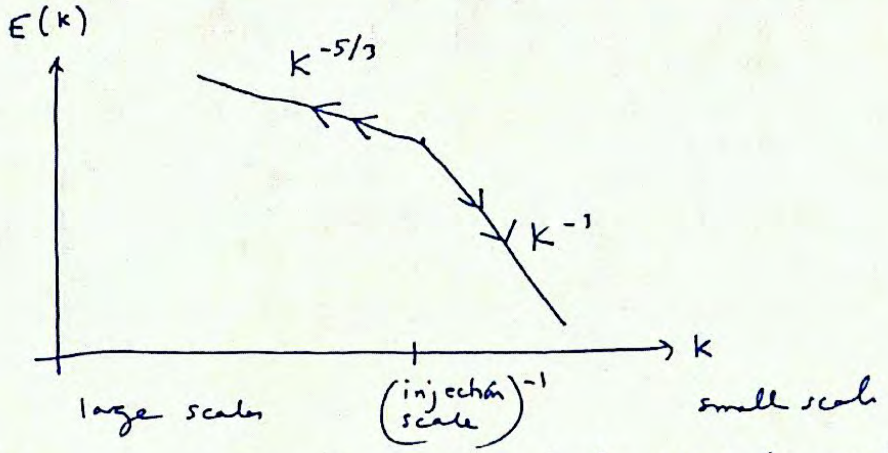
In this section we will see that the energy and enstrophy cascades have different directions. Let's consider the relation valid for homogeneous isotropic turbulence

$$\epsilon = \frac{1}{2}\nu\langle\omega^2\rangle = \frac{1}{2}\nu Z.$$

(It is derived in Landau + Lifschitz for example).

In 3D, as we decrease ν, i.e. increase Re, Z can diverge through vortex stretching, allowing ϵ to be constant. This is known as the dissipation anomaly, and turns out to be related to anomalies in QFT.

In 2D, Z is conserved as $\nu \to 0$, and so ϵ must decrease. This occurs through the organisation of eddies on scales larger than the injection scale, so that on such scales there is an energy cascade allowing there to be a statistical equilibrium.



Since the energy cascade and enstrophy cascade must coexist in stable stationary state, it would seem that the direction of energy cascade would need to be to large scales, so as to avoid interfering with the enstrophy cascade. The net effect of the two cascades is to give a constant rate of dissipation. However, the dissipation occurs at large scales, in fact through the interactions with the boundaries of the system!

Theoretical arguments for the inverse energy cascade were presented by Batchelor in his book from 1953 and in more detail by Kraichnan in 1967.

The inverse cascade has been observed in experiments in both driven 2D turbulent systems of electrolytes, powered electromagnetically. Recently we have also discovered how to observe the inverse cascade in turbulent soap films. (See next lecture).

Scaling theory of turbulence.

① Eddy turnover time.

τ_k = time for an eddy with velocity u_k to go distance $\sim 1/k$.

$E(k)$ = energy spectrum, so that $E(k)\Delta k$ = energy in wavenumber range $k \to k+\Delta k$.

Thus $u_k \sim (E(k)k)^{1/2} \implies \tau_k \sim \left(k^3 E(k)\right)^{-1/2}$

The energy flux is rate of energy input/unit volume

$$\epsilon \sim \frac{u_k^2}{\tau_k} \sim \frac{kE(k)}{\tau_k} \sim kE(k) \cdot k^{3/2} E(k)^{1/2}$$

$$\implies \epsilon \sim k^{5/2} E(k)^{3/2}$$

$$\implies \underline{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

Thus the K41 spectrum arises assuming ϵ = const. Thus we see that there is an energy flux from large scale, locally through smaller and smaller scales, until eventual dissipation due to molecular viscosity.

② The dissipation anomaly and intermediate asymptotics.

The scales at which dissipation become important is determined by looking at

$$\partial_t \underline{u} \sim \nu \nabla^2 \underline{u} \longrightarrow \frac{1}{\tau_\nu} \sim \nu k^2.$$

On the other hand, the inertial range has $\tau_k \sim \epsilon^{-1/3} k^{-2/3}$. The Kolmogorov scale η_K is the scale at which $\tau_k \sim \tau_\nu$ giving

$$\eta_K \sim (\nu^3/\epsilon)^{1/4}.$$

The mean dissipation rate of energy

$$\dot{E} \sim \frac{1}{V}\int_V \nu \underline{u} \nabla^2 \underline{u}\, d^3 r$$

$$\sim \nu k^2 u_k^2 \Big|_{k=2\pi/\eta_K} \sim \nu k^2 \frac{\epsilon^{2/3}}{k^{2/3}}\Big|_{k=\frac{2\pi}{\eta_K}} \sim \epsilon.$$

Note that this estimate is <u>independent of viscosity</u>.

Hence we have a peculiar situation. The Euler equations, which are N-S with $\nu = 0$ conserve energy, so $\dot{E} = 0$. On the other hand, the N-S equations, with $\nu \to 0$ have $\dot{E} \neq 0$. So

$$\lim_{\nu \to 0} NS(\nu) \neq NS(0).$$

Comment: NS has a ∇^2 term so needs more boundary conditions

Thus we have an example of intermediate asymptotics of the second kind!

<u>Exercise.</u> Estimate η_K for the turbulence in the Earth's atmosphere.
Hint: $\epsilon \sim u^3/L$, so guess $u \sim 1\,\mathrm{cm\,s^{-1}}$ for $L \sim 100\,\mathrm{m}$
so that $\epsilon \sim 10^{-8}\,\mathrm{m^2 s^{-3}} \rightarrow \eta_K \sim 10^{-3}\,\mathrm{m}$.
(See Vallis, p. 147)

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③ Scale invariance of Euler equations

Consider
$$\partial_t \underline{u} + \underline{u}\cdot\underline{\nabla} u = -\underline{\nabla} p$$

Scale transformation:
$$\underline{x} \to \underline{x}\lambda \qquad \text{pressure} \sim u^2.$$
$$\underline{u} \to \underline{u}\lambda^r$$
$$t \to t\,\lambda^{1-r}.$$

Then each term in the equation gets multiplied by λ^{2r-1}! Thus the Euler equation is scale invariant.

Next, suppose that for Navier-Stokes, not Euler, we assume $\epsilon = \text{const}$ and that the same scale invariance holds, on average, in the inertial range.

Q/ How does the energy flux scale?

A/ $\epsilon_K \sim \dfrac{u_K^3}{k^{-1}} \sim \lambda^{3r-1}$ when k is in the inertial range.

If ϵ_K is independent of scale requires that $r = 1/3$.

If $r = 1/3$ then $u_K \sim \epsilon^{1/3} k^{-1/3} \implies \underline{E(k) \sim u_K^2 k^{-1} \sim \epsilon^{2/3} k^{-5/3}}$.

<u>Exercise.</u> Assume that we are in 2D turbulence, and the conserved quantity is not ϵ, but β, the rate of enstrophy flux. Derive the enstrophy cascade energy spectrum

<u>Solution:</u>
$$\beta \sim \frac{u_K^3}{k^{-3}} \qquad (\because [\beta] = T^{-3})$$
$$\sim \lambda^{3r-3} \implies r = 1 \text{ for } \beta = \text{scale invariant.}$$
$$\implies u_K \sim \beta^{1/3} k^{-1} \implies \underline{\underline{E(k) \sim \beta^{2/3} k^{-3}}}$$