Introduction to Goldenfeld hydrodynamics
course for Boulder summer school 2011

This is a course of three lectures on aspects of hydrodynamics, ranging from low Reynolds number flows to fully-developed turbulence. Many aspects of fluid flow, pattern formation, non-equilibrium and spatially-extended dynamical systems will be covered by other lectures. However, these lectures will have a distinctive focus on the connections with statistical mechanics. These connections arise in two ways that are non-trivial and reflective of the way we do condensed matter theory. Moreover, they transcend hydrodynamics per se. So we learn about general aspects of physics from studying hydrodynamics.

Connection #1: Singular perturbations, Critical Phenomena, Boundary Layers

Low Re number flow introduced the concept of boundary layers. We are going to see how they arise and learn that they are a manifestation of a surprising and general statement:

"Small terms are not always negligible".

More precisely we will see that if ϵ denotes some parameter in a physical theory,

$$\boxed{\lim_{\epsilon \to 0} \text{Physics}(\epsilon) \neq \text{Physics}(\epsilon = 0).} \qquad (1)$$

This idea was first encountered in hydrodynamics, but turns out to be important in many problems in continuum mechanics.

The generality of these notions was expressed by G. I. Barenblatt in a series of books, such as the classic "Scaling, Self-Similarity and Intermediate Asymptotics". The notions of boundary layers are a special case of what has come to be known as singular perturbation theory, e.g. as in equation (1). These problems took one hundred years to sort out mathematically, culminating in the development of matched asymptotic expansions, multiple scales analysis and boundary layer theory, during the 1950's and 1960's.

Remarkably, around the same time, another scientific revolution was going on. This was the development of quantum field theory and the techniques for regularising and renormalising field theories. Regularisation means to remove infinities from a theory. Renormalisation is the procedure for absorbing infinities into a redefinition of the parameters of a theory. Renormalisation group is a way to do this in a manner that is independent of the particular renormalisation scheme employed. To many physicists, including those who participated in these developments, the mathematics was disturbingly hokey, even by the standards of physicists, and the physical interpretation unclear.

The same could also be said about the mathematics of singular perturbation theory, often dismissed as "applied" mathematics and therefore "formal" or "heuristic" and lacking in rigorous justification, and defined only through an obscure and arcane collection of "recipes".

Remarkably, these two unwanted children of the union between physics and mathematics turned out not only to be similar in their notoriety, but in fact to be one and the same! Moreover, by understanding the process of renormalisation in the context of hydrodynamics — without any statistical element whatsoever — we find that is possible to appreciate its physical significance in a concrete, even experimentally realisable form. At the same time, we will see that the strange mathematical forms arising in singular perturbation theory can be derived and simplified and even improved by realising that they are expressions of renormalisation group ideas. Thus

$$\boxed{\begin{array}{c}\text{singular perturbation}\\ \text{theory}\end{array} \quad = \quad \begin{array}{c}\text{renormalisation}\\ \text{group theory}\end{array}} \qquad (2).$$

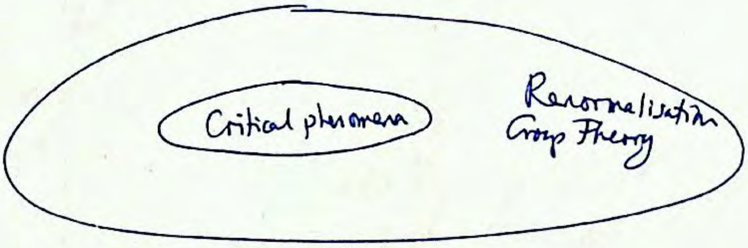
Renormalisation group theory is celebrated because of its connection not only to Quantum Field Theory (QFT) but also to statistical field theory and in particular the problem of critical phenomena. The critical phenomena problem is usually thought of as being about critical exponents, such as the way that the magnetisation in a magnet varies with relative temperature:

$$M \propto \left(\frac{T_c - T}{T_c}\right)^{\beta} \qquad T < T_c$$

where the exponent β would be $1/2$ in mean field theory, but in practice (in 3D) takes on a value $0.326\ldots$

We'll see that the critical phenomena problem is also an example of a singular perturbation problem, and thus an example of the application of renormalisation group.

Thus



Critical phenomena $\neq$ RG

<u>Connection #2</u>: <u>Turbulence, statistical mechanics, critical points.</u>

Turbulence is the generic state of flow of fluids. It seems to be something qualitatively different from low Reynolds number flow. First of all, the flow field $\underline{v}(\underline{r}, t)$ seems to be a stochastic variable, not a deterministic one. Second the fluctuations are not small: they are large. Thirdly the state shows presence of emergent properties, just as a magnetically ordered state is emergent, with universal properties that are independent of a lower level of description. In the turbulent state, this corresponds to universal scaling laws for velocity fluctuations, renormalised turbulent drag that does not depend upon fluid properties such as viscosity and other properties. Fourth, we will see that there is strong evidence for a version of a fluctuation–dissipation style relation, connecting microscopic fluctuations with large scale flow properties such as frictional drag. The evidence for this connection includes power law scalings, data collapse and even evidence for small anomalous dimensions. Turbulence is more complicated than standard critical phenomena however, because of the presence of multi-fractal scaling. In the third lecture of this course, I will explain why naive attempts to solve the turbulence problem using sophisticated field theoretic methods, Feynman diagrams etc. were unsuccessful, and present my own ideas as to how we can make progress. The key is to look at experiment, and not focus on a theoretical idealisation that inadvertently removes a key element of the solution: the boundaries of the flow domain.

Drag.

Goals:
1. Formulate the drag problem.
2. Expose Stokes' paradox
3. Explain boundary layers
4. Sketch of RG solution.

I. Formulation.

(i) Begin with steady state Navier-Stokes, setting $\partial_t \underline{u} = \underline{0}$

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla} p + \nu \nabla^2 \underline{u}$$

$\nu = \frac{\mu}{\rho}$ = kinematic viscosity

μ = molecular viscosity

Estimate

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = O(U^2/L)$$
$$\nu\nabla^2\underline{u} = O(\nu U/L^2)$$

Ratio of inertial to viscous term

$$Re = \frac{UL}{\nu}$$

For viscous flows $Re \ll 1$ and the governing equations are

$$0 = -\underline{\nabla} p + \mu \nabla \underline{u}$$
$$\underline{\nabla}\cdot\underline{u} = 0$$

(ii) Two problems of classic interest: sphere and cylinder. Let's start with the sphere, where we expect axisymmetric steady flow. Note that this is an assumption, but may break down through spontaneous symmetry breaking. The solutions to a differential equation do not need to exhibit all the symmetries of the equation. In fact, the assumption works for low enough Re, but breaks down through a sequence of cascades involving trailing vortex streets, ultimately leading to turbulence as $Re \to \infty$.

Thus $\underline{u} = (u_r(r,\theta),\, u_\theta(r,\theta),\, 0)$ in standard spherical polars.

Stokes introduced the stream function $\Psi(r,\theta)$

$$\underline{u} = \underline{\nabla} \times \frac{\Psi}{R}\hat{\underline{\phi}}$$

$$= \underline{\nabla} \times \frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}$$

$R = r\sin\theta =$ projection onto cylindrical plane (x,y).

$$\Rightarrow \quad u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}$$

Exercise: verify that Ψ is called a stream function because it is constant along a streamline. A streamline is a curve $\underline{x}(s)$ which at any time t has the same direction as $\underline{u}(\underline{x},t)$.

$$\frac{\frac{dx}{ds}}{u} = \frac{\frac{dy}{ds}}{v} = \frac{\frac{dz}{ds}}{w} \qquad \text{where } \underline{u} = (u,v,w)$$

Let the local direction of the streamline be $\hat{\underline{e}}_s$.
Then

$$\underline{u}\cdot\underline{\nabla}\underline{u} = |\underline{u}|\,\underline{e}_s\cdot\underline{\nabla}\,\underline{u} = |\underline{u}|\frac{\partial}{\partial s}\underline{u}$$

Similarly any quantity F defined on a streamline obeys

$$\underline{u}\cdot\underline{\nabla}F = |\underline{u}|\,\underline{e}_s\cdot\underline{\nabla}F = |\underline{u}|\frac{\partial F}{\partial s}$$

If F is constant along the streamline, the $\frac{\partial F}{\partial s} = 0$.
So we need to see if the stream function Ψ obeys

$$(\underline{u}\cdot\underline{\nabla})\Psi = 0. \qquad \text{Does it ?}$$

There is an equation for Ψ found by using the identity

$$\underline{\nabla}\times\underline{\nabla}\times\underline{a} = \underline{\nabla}(\underline{\nabla}\cdot\underline{a}) - \nabla^2\underline{a}$$

to write

$$\underline{\nabla}p = -\mu\,\underline{\nabla}\times(\underline{\nabla}\times\underline{u})$$

and then

$$\underline{\nabla}\times\underline{u} = \left(0, 0, -\frac{1}{r\sin\theta}E^2\Psi\right)$$

where we define

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

Knowing the (r,θ) components of $\underline{\nabla} p$, we then use

$$\frac{\partial}{\partial r}\frac{\partial p}{\partial\theta} = \frac{\partial}{\partial\theta}\frac{\partial p}{\partial r}$$

to obtain

$$E^4 \psi = 0.$$

with boundary conditions obtained from no-slip on $\underline{u}$ at $r=a$:

$$\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \quad \text{at } r = a$$

and at $r \to \infty$ we expect the flow to be uniform

$$\begin{aligned} u_r &= U\cos\theta \\ u_\theta &= -U\sin\theta \end{aligned} \qquad \text{as } r \to \infty.$$

<u>Exercise</u>. <u>Solve for ψ</u>.

Strategy: assume $\psi = f(r)\sin^2\theta$ because this is of the form of the boundary condition at $r \to \infty$ (you can check that $\psi = \frac{1}{2}Ur^2\sin^2\theta$ as $r\to\infty$).

The equation satisfied by $f(r)$ has solutions of the form $f \sim r^n$ where $n = -1, 1, 2, 4$. Use the boundary conditions to determine the 4 constants of integration to get

$$\psi = \frac{1}{4}U\left(2r^2 + \frac{a^3}{r} - 3ar\right)\sin^2\theta.$$

Our solution for ψ means that we can calculate the drag. This is defined by the following steps.

<u>Step 1</u>: Force per unit area $P_i = -\sigma_{ik} n_k$ n_k = unit normal vector component

<u>Step 2</u>: $$\sigma_{ik} = -p\delta_{ik} + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

Step 3: $\underline{F}$ = force on body $= -\oint p\, d\underline{S}$.

This gives:

$$\sigma_{rr} = -p + 2\mu \frac{\partial u_r}{\partial r}$$

$$\sigma_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial u_r}{\partial \theta}$$

$$\sigma_{r\phi} = 0$$

and $p = p_\infty - \frac{3}{2}\frac{\mu U a}{r^2}\cos\theta$ obtained from calculating

$$E^2\psi = \frac{3}{2}\frac{Ua}{r}\sin^2\theta.$$

The net drag is then

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right)\Big|_{r=a} \times a^2 \sin\theta = 6\pi\mu U a$$

See Veysey + Goldenfeld for details.

(iii) Paradoxes.

(a) Cylinder at $Re = 0$

Let's see what happens if we follow the same analysis for a cylinder. We're not going to do a brute force calculation, because we can see the problem by dimensional reasoning.

For a cylinder, we want the force per unit length. This can depend on

$$F_{net} = F_{net}\begin{pmatrix} \text{radius} & a \\ \text{viscosity} & \mu \\ \text{density} & \rho \\ \text{velocity at infinity} & U \end{pmatrix}$$

Using Buckingham's π-theorem we write

$$\Pi_0 = Re = \frac{Ua}{\nu} \qquad \nu = \mu/\rho$$

$$\Pi_1 = F_{net}/\rho\nu U$$

With two dimensionless groups, we must have

$$\Pi_1 = f(\Pi_0)$$

where f is some function to be determined. For $R=0$, we have

$$\Pi_1 = f(0) = \text{constant}$$

$$\Rightarrow \quad \boxed{F_{net} \propto \rho \nu U}$$

This result is absurd, because it is independent of the radius of the cylinder!!

<u>Exercise</u>. why did this argument not apply to the sphere? In other words, we obtained the drag on a sphere at $Re=0$ with no problems, because this argument did not apply.

<u>(b) Sphere at $Re \neq 0$.</u>

Our governing equations were valid for $Re=0$ because they ignored the $(\underline{u}\cdot\underline{\nabla})\underline{u}$ terms in Navier-Stokes. This is why our solution strategy involving the stream function worked. It satisfies $(\underline{u}\cdot\underline{\nabla})\psi = 0$. Let's now include this term by perturbation theory, with our perturbation parameter

$$\epsilon = \frac{|(\underline{u}\cdot\underline{\nabla})\underline{u}|}{|\nu\nabla^2\underline{u}|} \sim \frac{\frac{U^2}{r}}{\nu\, U/r^2} \sim \frac{Ur}{\nu} = Re\left(\frac{r}{a}\right)$$

where r = radial coordinate.

Near $r \gtrsim a$, ϵ may be small, but far from the surface of the sphere, the ratio ϵ becomes arbitrarily large. This happens as soon as $Re \neq 0$. In short there is no parameter small uniformly over the domain. The ratio of inertial to viscous forces can not be taken as small anywhere.

$$\text{Hydro}(Re=0) \neq \lim_{Re\to 0} \text{Hydro}(Re), \text{ for cylinder.}$$

while for sphere, in some sense,

$$\Delta \text{Hydro}\,(Re = 0) \neq \lim_{Re \to 0} \Delta \text{Hydro}\,(Re)$$

where Δ denotes the deviation from the zeroth order solution.

The problem with the cylinder calculation is known as Stokes' Paradox. The problem with the correction to the sphere calculation is known as Whitehead's paradox.

To solve this in a deep way, we need to go back to basics, and study intermediate asymptotics and RG.

References:

N.G. Lectures on Phase Transitions and the Renormalization Group, (1992), Ch.10
G.I. Barenblatt, Scaling, Self-Similarity and Intermediate Asymptotics, Cambridge (1995).
L.Y. Chen, NG, Y. Oono, Phys Rev E 54, 376 (1996)
Q. Hou, NG, A. McKane, Phys Rev E (2001).

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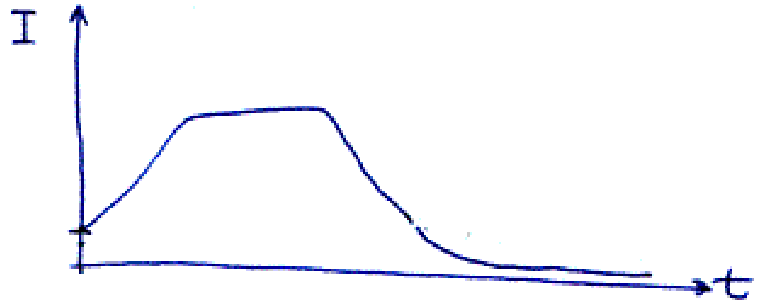
Outline.

I. Similarity Solutions and Travelling Waves.
- diffusion eqn
- dimensional analysis — type I / type II
- Barenblatt eqn.
- Heuristics of Barenblatt eqn.
- Sketch of RG
- Meaning of solution + comparison with critical phenomena
- Geometric interpretation + Wilson RG.

II. Global asymptotics
- Boundary layers
- WKB/matching
- Reductive perturbation theory.

III. Applications
- Pattern formation (e.g.) convection, particle beams,
- Focusing solutions
- GR, BH.
- Phase field models
- Stochastic differential equations / turbulence
- Generalized CLT / EVT.

Indigestibility index



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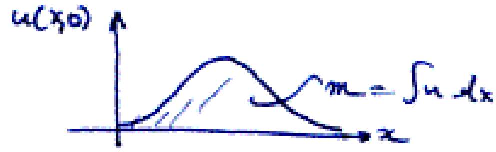
1) Diffusion.



$t=0$ $\qquad \partial_t u = \frac{1}{2}\kappa\,\partial_x^2 u$

$$u(x,0) = \frac{m}{\sqrt{2\pi l^2}}\, e^{-x^2/2l^2}$$

$t>0$
$$u(x,t) = \frac{m\, e^{-x^2/(2(l^2+\kappa t))}}{\sqrt{2\pi(l^2+\kappa t)}}$$

$t\to\infty$
$$u(x,t) \underset{t\to\infty}{\sim} \frac{m\, e^{-x^2/2\kappa t}}{\sqrt{2\pi\kappa t}}.$$

Same as t fixed, $l\to 0$. Asymptotic similarity solution, from degenerate initial condition (a delta fn).

2) Dimensional analysis.

$\Pi, \Pi_0, \Pi_1, \Pi_2 \cdots \Pi_n$ dimensionless groups.

DA $\Rightarrow$ $\Pi = f(\Pi_0, \Pi_1, \cdots \Pi_n)$.

Suppose $\Pi_0 \sim 0$. ¿ $\Pi = f(0, \Pi_1, \Pi_2, \cdots \Pi_n)$?

No! $\lim \Pi_0 \to 0$ may not exist.

Alternatives:

(1) $\Pi = f(0, \Pi_1, \Pi_2 \cdots \Pi_n)$ "Common sense", Type I

(2) $\Pi = \Pi_0^{\alpha} f\left(\frac{\Pi_1}{\Pi_0^{\alpha_1}}, \frac{\Pi_2}{\Pi_0^{\alpha_2}} \cdots \frac{\Pi_n}{\Pi_0^{\alpha_n}}\right)$ Type II.

(3) None of the above.

Example. Diffusion eqn.

$$\Pi = \frac{u}{m}\sqrt{\kappa t} \qquad \Pi_1 = \frac{x}{\sqrt{\kappa t}} \qquad \Pi_0 = \frac{l}{\sqrt{\kappa t}}$$

For $\Pi_0 \sim 0$ as $t\to\infty$ or $l\to 0$ we use common sense

$\Pi = f(\pi_0, \pi_1) \sim \tilde{f}(\pi_1) \Rightarrow u = \frac{m}{\sqrt{\kappa t}} f(x/\sqrt{\kappa t})$.

Solve for $\tilde{f}'' + \xi \tilde{f}' + \tilde{f} = 0$ $\quad \tilde{f} \to 0 \; |x| \to \infty$. $\quad \xi \equiv x/\sqrt{\kappa t}$

$\tilde{f} = e^{-\xi^2/2}$

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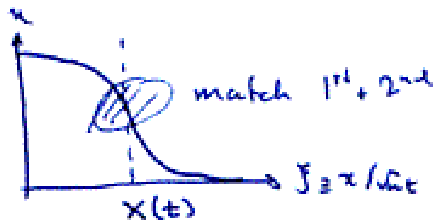
3 Barenblatt equation

$$\partial_t u(x,t) = D\kappa\,\partial_x^2 u \qquad D = \begin{cases} \frac{1}{2} & \partial_t u \geq 0 \\ \frac{1}{2}(1+\epsilon) & \partial_t u < 0 \end{cases}$$

DA: $\Pi = \frac{u}{m}\sqrt{\kappa t}$ $\quad \Pi_0 = \frac{l}{\sqrt{\kappa t}}$ $\quad \Pi_1 = \frac{x}{\sqrt{\kappa t}}$ $\quad \Pi_2 = \epsilon.$

Guess: $u = \frac{m}{\sqrt{\kappa t}}\tilde{f}\left(\frac{x}{\sqrt{\kappa t}}, \epsilon\right) \qquad t \to \infty.$



match 1st + 2nd derivs. Can't do it. Silly eqn? No! Kamenomostkaya (1958) proved existence and uniqueness with continuous 2nd derivs.

Actual solution.

$$u(x,t) = \frac{1}{t^{\frac{1}{2}+\alpha(\epsilon)}}\bar{f}\left(\frac{x}{\sqrt{\kappa t}}, \epsilon\right)$$

$\alpha(\epsilon)$ = anomalous dimension

4 Heuristic solution.

Let's understand physics of Barenblatt term. B-eqn is not local conservation law

$$\partial_t u = -\nabla\cdot\underline{J}$$

$m \equiv \int_{-\infty}^{\infty} u(x,t)\,dx$ not constant of motion. i.e. $m = m(t)$.

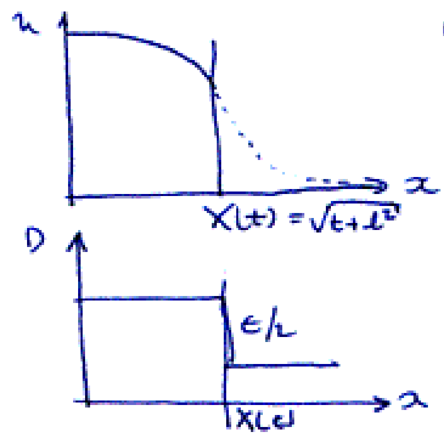
Suppose mass removed slowly w.r.t. spreading. Then in this adiabatic limit, profile adjusts to distribution when D = constant and

$$u = \frac{m(t)\,e^{-x^2/2(\kappa t+l^2)}}{\sqrt{2\pi(\kappa t+l^2)}}.$$

How to find $m(t)$?

4

$$m(t) = \int_{-\infty}^{\infty} u \, dx$$



$$\partial_t m = \int \partial_t u = \int D(x) \partial_x^2 u$$

$$= -\int \partial_x D \, \partial_x u = \epsilon \kappa \partial_x u(X(t), t)$$

$$= -\frac{\epsilon X m(t)}{\sqrt{2\pi}} \frac{e^{-1/2}}{(\kappa t + l^2)}$$

Solution: $m(t) = m(0) \dfrac{l^{2\alpha}}{(\kappa t + l^2)^{\alpha}}$ $\quad \alpha = \epsilon/\sqrt{2\pi e}$.

Long time behavior.

$$\boxed{u(x,t) \sim \frac{m(0)\, l^{2\alpha}}{(\kappa t)^{1/2 + \alpha(\epsilon)}} e^{-x^2/2\kappa t} \qquad \alpha = \epsilon/\sqrt{2\pi e}}$$

Comments.

$\epsilon = 0$ $\quad u(x,t) \underset{t \to \infty}{\sim} \dfrac{m(0)\, e^{-x^2/2\kappa t}}{(\kappa t)^{1/2}}$ $\quad$ independent of l.

$\epsilon \neq 0$ $\quad u(x,t) \underset{t \to \infty}{\sim} \dfrac{m(0)\, l^{2\alpha}\, e^{-x^2/2\kappa t}}{(\kappa t)^{\frac{1}{2} + \alpha(\epsilon)}}$ $\quad$ depends on l. Long-term memory.

c.f. critical phenomena

$$G(k) = \langle |\phi|_k^2 \rangle \sim k^{-2+\eta}$$

But $[\phi] = L^{1-d/2} \Rightarrow [G(k)] = L^2$.

What happened? Needed ultraviolet cut-off (lattice) : $G(k) \sim l^{\eta} k^{-2+\eta}$

Renormalisation in space $\longleftrightarrow$ renormalisation in time.

5 RG: Gell-Mann Low

Step 1 $\quad \left(\partial_t - \frac{1}{2}\kappa \partial_x^2\right) u = \frac{\epsilon}{2} \Theta\left(X(t,\epsilon) - |x|\right) \partial_x^2 u$

$$\partial_t u(X(t), t) = 0$$

Step 2

$$u(x,t) = \int G(x-y,t)\, u(y,0) + \frac{\epsilon}{2} \int_0^t ds \int dy\, G(x-y,t-s)\, \Theta(-\partial_y u(y,s))\, \partial_y^2 u(\cdot$$

$$G(x,y) \equiv \frac{1}{\sqrt{2\pi y}}\, e^{-x^2/2y}$$

5

Step 3 Evaluate integrals and isolate divergences

$$u(x,t) = \frac{m_0}{\sqrt{2\pi t}} e^{-x^2/2t}\left[1 - \frac{\varepsilon}{\sqrt{2\pi e}}\log\frac{\kappa t}{\ell^2} + O(\varepsilon^2)\right] + O(\ell,\varepsilon)$$

Step 4 Perturbative renormalisation

- m_0 is initial mass.
- At time t, mass is lower because B-eqn doesn't conserve mass
- Diffusion operator does not recognise this. The divergence of pert. theory is its way of telling us that something singular is going on.
- m_0 is in fact m at time t:

$$m = Z^{-1}(\ell/\mu, \epsilon)\, m_0$$

Step 5 Power series expansion

$$Z = 1 + \sum_{n=1}^{\infty} a_n(\ell/\mu)\, \varepsilon^n$$

Chose a_n order by order in pert. theory to remove divergences.
Here we see that we should chose

$$a_1(\ell/\mu) = \frac{1}{\sqrt{2\pi e}}\log\left(\frac{C_1\mu^2}{\ell^2}\right) \qquad C_1 \text{ arbitrary} > 0.$$

$$u(x,t) = \frac{m}{\sqrt{2\pi\kappa t}} e^{-x^2/2\kappa t}\left(1 + \frac{\varepsilon}{\sqrt{2\pi e}}\log\frac{C_1\mu^2}{\ell^2} + O(\epsilon^2)\right)\left(1 - \frac{\varepsilon}{\sqrt{2\pi e}}\log t/\ell^2 + O(\epsilon^2)\right) + O(\ell,\epsilon)$$

Step 6 Cancel divergence.

$$[\ \ ] \times [\ \ ] = 1 - \frac{\varepsilon}{\sqrt{2\pi e}}\left(\log\frac{t}{\ell^2} + \log\frac{\ell^2}{C_1\mu^2}\right)$$

Step 7 RG eqn. Let's write this in time domain. This a family of solution. Let's specify that at some time t^*, $u(0,t^*)$ has value $Q(t^*)$. Then.

$$u = Q(t^*)\sqrt{\frac{t^*}{t}}\, e^{-x^2/2t}\left[1 - \frac{\varepsilon}{\sqrt{2\pi e}}\log t/t^* + O(\epsilon^2)\right]$$

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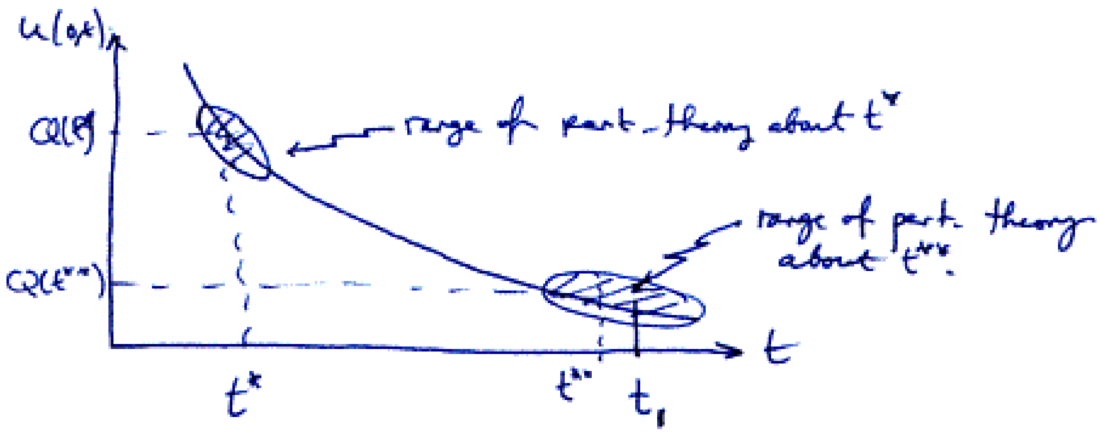


RG eqn: $t^* \dfrac{du}{dt^*} = 0$.

i.e. $\dfrac{\partial u}{\partial t^*} + \dfrac{dQ}{dt^*}\dfrac{\partial u}{\partial Q} = 0 \Rightarrow t\dfrac{dQ}{dt} = -Q\left[\frac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)\right]$

$\Rightarrow \quad Q(t) \sim t^{-\left(\frac{1}{2} + \alpha(\epsilon)\right)}$

Step 8. Plug into renormalized pert. expansion and set $t = t^*$.

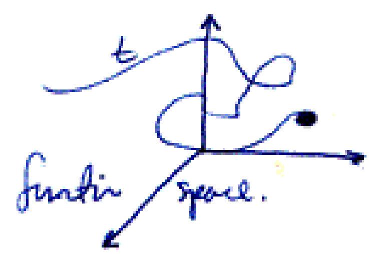
$$u \sim \frac{e^{-x^2/2\kappa t}}{t^{\frac{1}{2}+\alpha(\epsilon)}}$$

$$u = m_0 \left(\frac{\ell^2}{\kappa t}\right)^{\epsilon/\sqrt{2\pi e}} \frac{e^{-x^2/2(\kappa t)}}{\sqrt{2\pi\kappa t}}$$

6 Wilson RG

Define rescaling transformation

$$u'(x,t_0) \equiv R_{b,\phi}\left[u(x,t_0)\right]$$



Step 1: evolve forward in time using PDE to $t_1 = bt_0$ $\quad b>1$.

Step 2: rescale $x' = b^{-\phi}x$. Choose ϕ s.t. $\exists$ a fixed pt. $\phi = \frac{1}{2}$ here.

Step 3: rescale u: $u'(0,t_0) = u(0,t_0)$.

$\longrightarrow \quad u'(x,t_0) = Ru = Z(b)\,u(b^{\frac{1}{2}}x,\, bt_0)$

Semi-group: $R_{b_1} * R_{b_2} = R_{b_1 b_2} \Rightarrow Z(b) = b^y$ or $y = \dfrac{d\log Z}{\ldots}$.

Fixed pt: $u^* = R[u^*] \Rightarrow u^*(x,t) = b^{y} u^*(b^{1/2} x, bt)$. all log b

Set $b = \text{const}/t \rightarrow u^*(x,t) = t^{-y} f(x/\sqrt{\kappa t}$

In practice do step 1-> numerically. Very effective numerical scheme. (7)

Here, let's use our renormalized P.T. to do that step:

$$Z(b) = b^{-1/2}\left[1 - \frac{\epsilon}{\sqrt{2\pi e}} \log b\right] + O(\epsilon^2).$$

$\Rightarrow$ exponent $y = \frac{1}{2} + \frac{\epsilon}{\sqrt{2\pi e}} + O(\epsilon^2)$ and

$$u(x,t) = \frac{1}{t^{\frac{1}{2}+\alpha}} f(x/\sqrt{t}) \quad t \to \infty.$$

II. Global Asymptotics.

① Simple linear eqn $\to$ boundary layer concept.

$$\epsilon \ddot{y} + \dot{y} + y = 0 \qquad y(0) = 0 \quad y'(0) = 1$$

Naive perturbation theory:

$$y(t) = A(0)e^{-t} - \epsilon t A(0) e^{-t} + O(\epsilon^2), \quad \epsilon t \ll 1.$$

Evolve until δt, $\epsilon \delta t \ll 1$ $\quad y(\delta t) = A(0)(1 - \epsilon\delta t)e^{-\delta t} + O((\epsilon\delta t)^2)$.

Wilson: treat as initial condition and do it again.

$$n \text{ times}, \; n\delta t = t \quad \left\{ \begin{array}{l} y(2\delta t) = y(\delta t)(1-\epsilon\delta t)e^{-\delta t} + O((\delta t)^2) \\ \vdots \\ y(t) = A(0)(1-\epsilon\delta t)^{t/\delta t} e^{-t} + \dots \end{array} \right.$$

$$\lim_{\substack{n\to\infty \\ \delta t \to 0 \\ n\delta t = t}} y(t) = A(0)e^{-(1+\epsilon)t}$$

frequency renormalization.

But: Q where's second constant of integration?

A: Boundary layer at $t = 0$ of thickness $\delta = O(\epsilon)$

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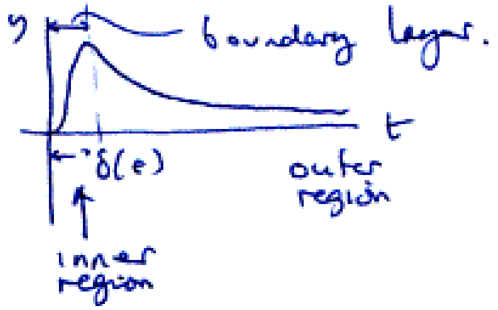
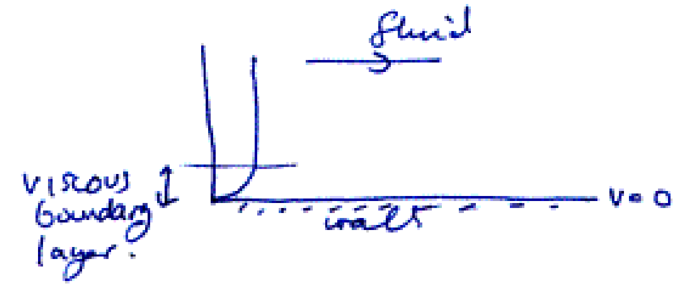
Actual solution looks like

$$y(t) = C_1 e^{-(1+\epsilon)t} + C_2 e^{-\frac{t}{\epsilon} + (1+\epsilon)t} + O(\epsilon^2)$$





(2) Problems with boundary layers example of singular perturbations.
ε multiplies highest derivative;

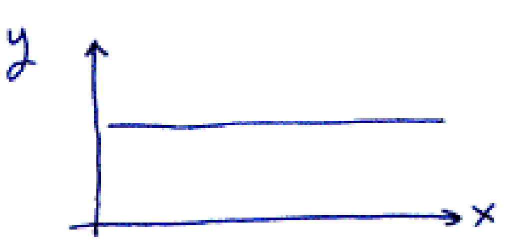
$$[\varepsilon = 0] \not\longleftrightarrow [\varepsilon \to 0]$$

CM QM.

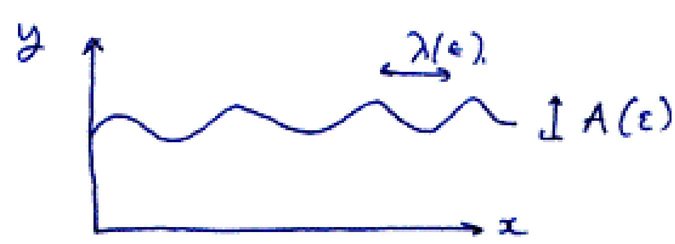
Other examples: WKB problems

Bifurcation near critical points

ε = control parameter in Rayleigh - Benard convection



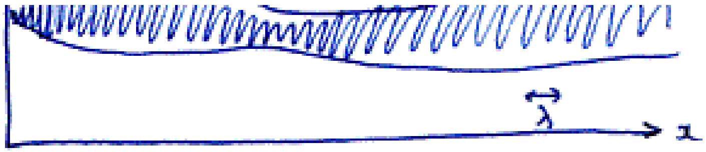
$\varepsilon < \varepsilon_c$



$\varepsilon > \varepsilon_c$

Amplitude can vary spatially (e.g.) near walls, topological defects.
$A(x,t;\epsilon)$. What governs?

$A(\epsilon)$ slowly varying on scale



)(ε).

(9')

Turns out that

$$\frac{\partial A(x,t)}{\partial t} = \nabla^2 A + \epsilon A(1-3|A|^2)$$

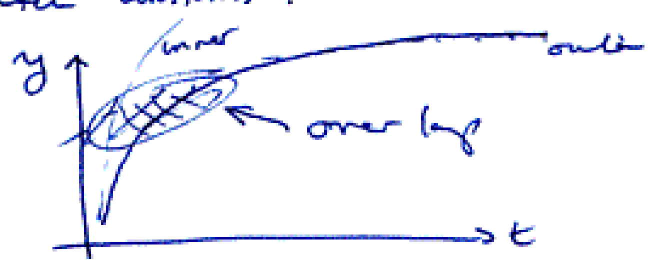
Time dependent Ginzburg-Landau eqn. Derived systematically via RG procedure. <u>Universal</u> long wavelength, low frequency behavior governing the PDE near bifurcation point.

(3) What happens when we use RG to solve boundary layers etc?

<u>Conventional</u>: $\epsilon \ddot{y} + \dot{y} + y = 0$. Outer expansion

Rescale $t = \epsilon\tau$ (easy to figure out). $y_{\tau\tau} + y_{\tau} + y\epsilon = 0$ inner expansion.

Solve inner + outer, find overlapping regime of validity in ϵ, t space match constants.



<u>Problems</u>:

(1) Inner + outer expansion naively of form
$y(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots$

(2) Find that you cannot match unless put in "non-intuitive terms"
e.g.
$y_0(t) + \epsilon y_1(t) + \epsilon^{3/2} y_{3/2}(t) + \epsilon^2 y_2(t)$
or <u>worse</u>.

E.g. $\epsilon y'' + xy' - xy = 0 \quad y(0) = 0 \quad y(1) = e$

— Matching requires and $\underline{\epsilon \log \epsilon}$ term to be introduced out of nowhere

— RG $\rightarrow$ $y(x) = e^x x^{-\epsilon}\left[1 - \sqrt{\frac{2}{\pi}}\int_{x/\sqrt{\epsilon}}^{\infty} ds\, e^{-s^2/2}\right]$

* just using inner expansion
* Just using naive perturbation expansion.
* origin of non-analytic terms in matching is exposed in RG method.

Take home message: RG makes singular perturbation theory mechanical!

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(4) How good are the results?

Ex: A 'terrible problem' (switchback).

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \epsilon u \frac{du}{dr} = 0 \qquad \begin{aligned} u(1) &= 0 \\ u(\infty) &= 1. \end{aligned}$$

RG
$$u(r,\epsilon) = 1 - \frac{e_2(\epsilon r)}{e_2(\epsilon)} + O\left(1/e_2(\epsilon)^2\right)$$

$$e_2(t) = \int_t^\infty dx\, x^{-2} e^{-x} \sim \frac{1}{t} + \ln t + (\gamma - 1) - \frac{t}{2} + \ldots$$

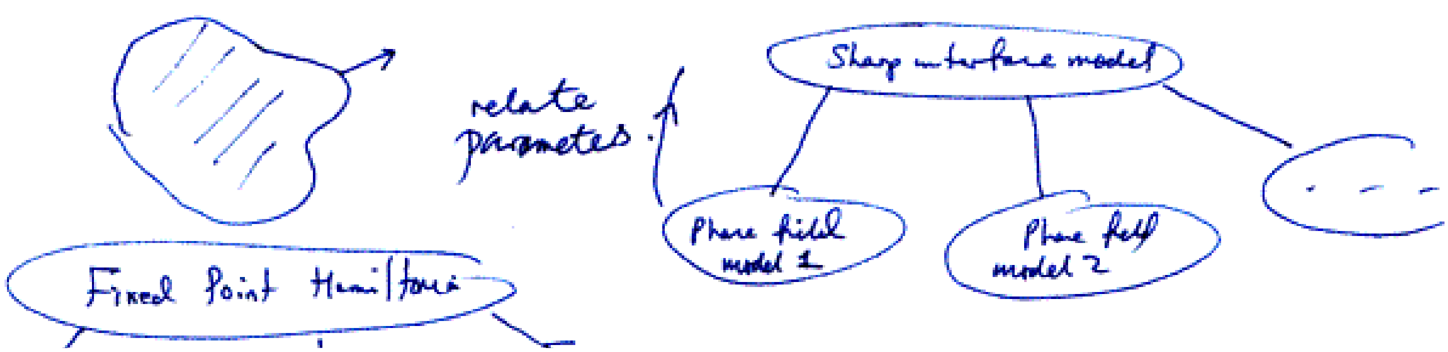
Work well, even when $\epsilon \sim O(1)$.

III. Applications.

(1) Pattern formation problems = convection
spatially extended dynamical system.
e.g. particle beams at Fermilab.

(2) BH collapse (Choptuik phenomenon).

(3) Pattern formation dynamics: snowflakes



n.n. Ising

next n.n. Ising

Ising + 3 spin

. . .

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Sharp interface

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$$\partial_t u = D\nabla^2 u \qquad u = \frac{T - T_m}{L/C}$$

$$V_n = -D(\partial_n u_L - \partial_n u_S)$$

$$u_{\text{interface}} = \Delta - d_0(\underline{n})\kappa - \beta(\underline{n})V_n .$$

Phase field.

$$\partial_t u = D\nabla^2 u + \frac{1}{2}\frac{\partial\phi}{\partial t}$$

$$\tau\,\partial_t\phi = W^2\nabla^2\phi - \frac{\delta}{\delta\phi}F(\phi,\lambda u)$$

$$F = \int f(\phi) + \lambda u\, g(\phi)$$

$$f = -\phi^2/2 + \phi^4/4$$

$$\frac{\partial g}{\partial\phi} = (1-\phi^2)^n \qquad n = 1, 2, 3, \ldots$$

Isotropic

Result

$$\lambda = \frac{W_0}{d_0}a_1 \qquad a_1 = \frac{1}{\sqrt{2}} \quad n=1$$

$$\frac{5}{4\sqrt{2}} \quad n=2$$

$$\tau = \frac{W_0^3 a_1 a_2}{d_0 D} \qquad a_2 = 5/6 \quad n=1$$

$$a_2 = \frac{47}{75} \quad n=2$$

$\vdots$

(4) Low Reynolds number flow

Drag on cylinder.

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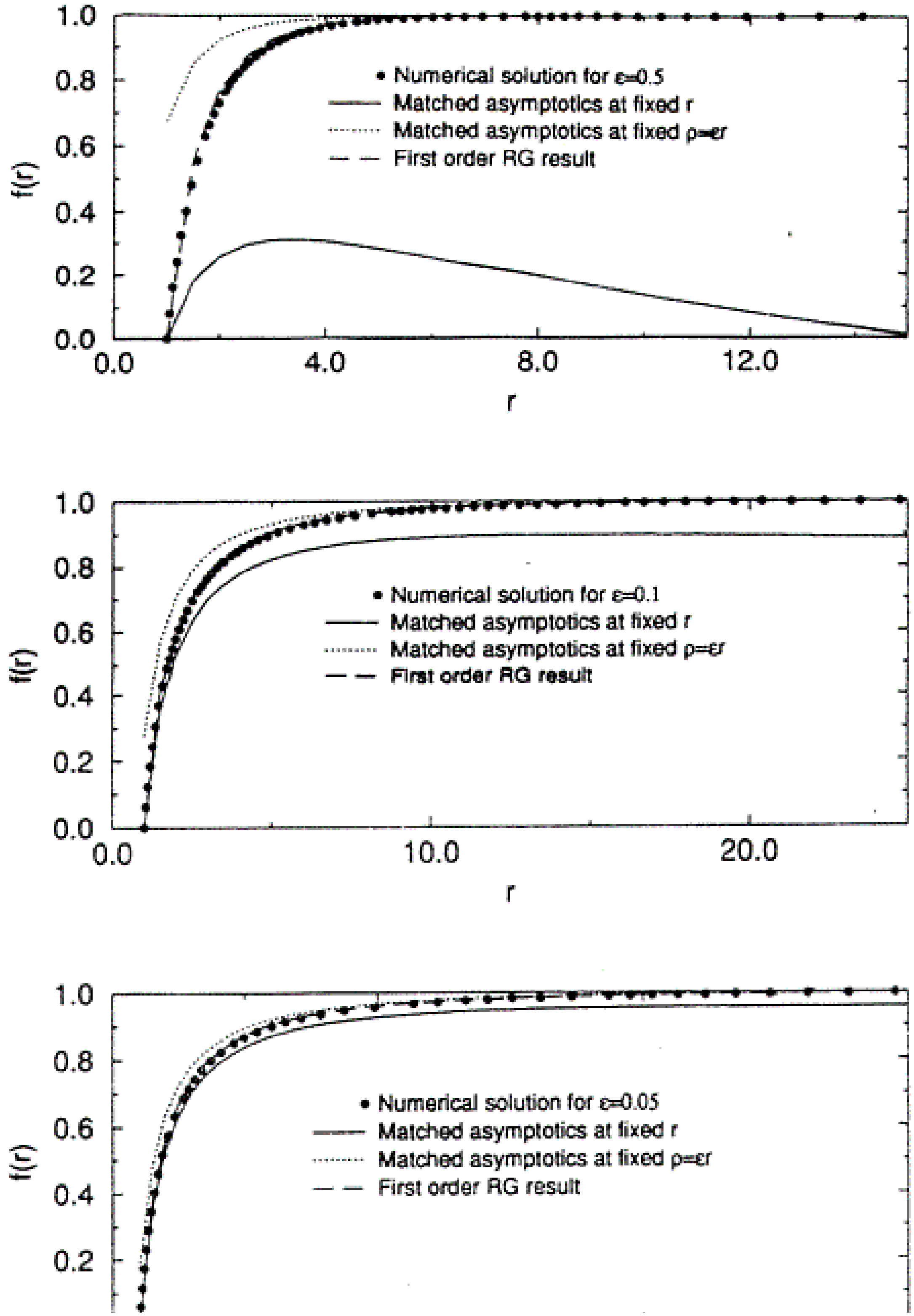
f(r)
r
• Numerical solution for ε=0.5
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result
• Numerical solution for ε=0.1
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result
• Numerical solution for ε=0.05
Matched asymptotics at fixed r
Matched asymptotics at fixed ρ=εr
First order RG result

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