2. Laminar - Turbulent Transition in Convection

Goal:
* Systems that are far from eqm and have emergent phenomena as some control parameter is varied.

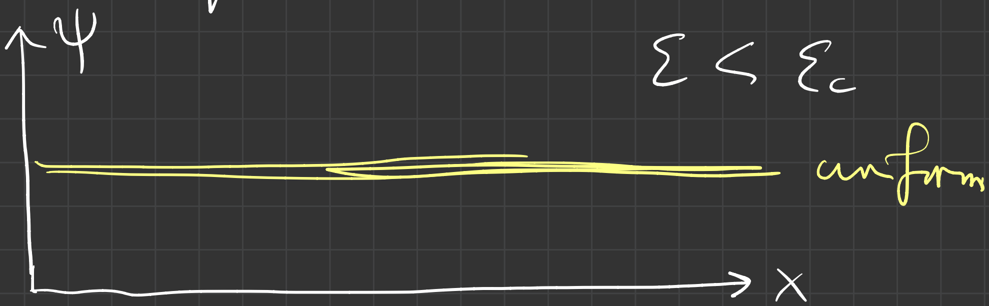
* Non-eqm systems form patterns in space and time.

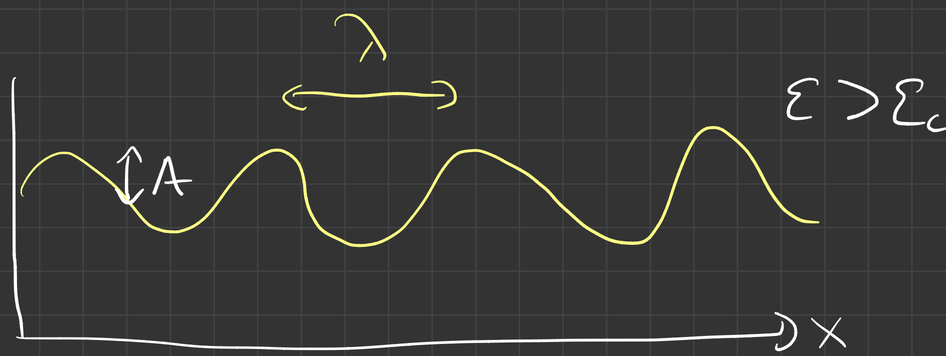
* Eqm systems can also form patterns.
  - ↳ Electrons in cuprate superconductors
    - ↳ Spontaneously break translational symmetry → charge order "Stripe" phases.

* Non-eqm fluid systems.
  - ↳ Sun spots
  - ↳ Miso soup

  } convection.

Fundamental question.





System spontaneously breaks continuous translational invariance $\longrightarrow$ discrete translational invariance.

Here: thermal convection: Rayleigh-Bénard convection (RBC)

* Fluid heated from below in a gravitational field.
* When does a pattern form? i.e. the phase diagram.

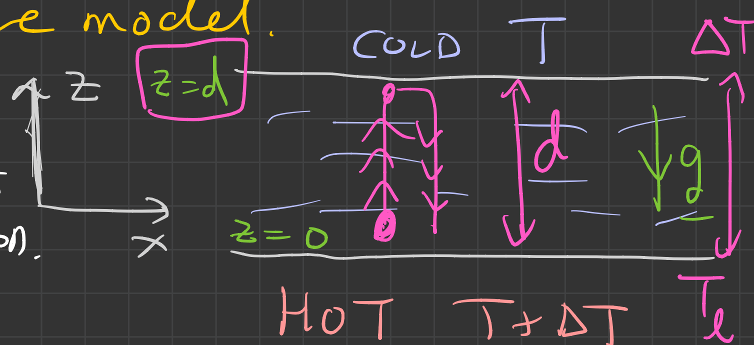
* Describe this pattern and its spatial and temporal fluctuations and correlations?

* Amazingly: the outcome is a description very close to Landau and Ginzburg–Landau theory! Scaling, universality and even the same equations that we've seen in eqm. stat. mech.

2. Formulate the model.

Fluid is viscous

Non-zero coefficient of thermal expansion.



Cartoon picture.

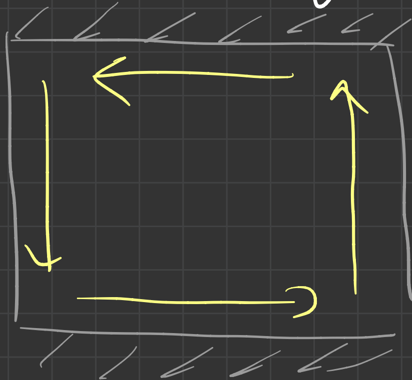
Bottom: parcel of fluid = hot.

$$\begin{array}{ccccc} \text{rise} & \leftarrow & \text{lower density} & \leftarrow & \text{expand} \\ \downarrow & & & & \\ \text{cool} & \rightarrow & \text{higher density} & \rightarrow & \text{descend} \end{array}$$

Q/ Overall: how does the fluid self-organize into rising and falling currents?

A/

*



Large-scale circulation

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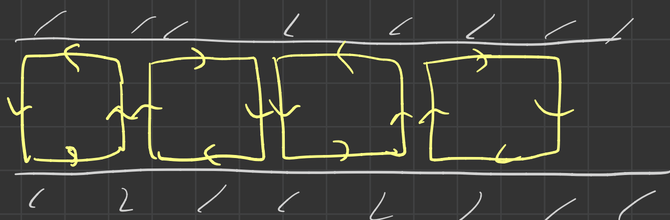
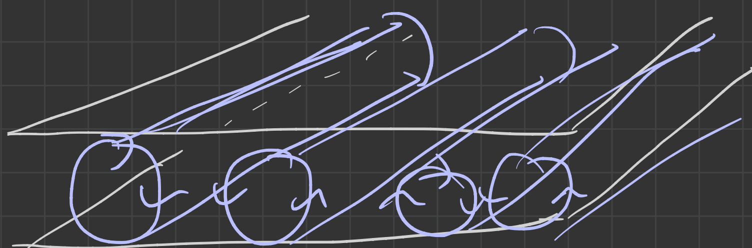
Cellular state.

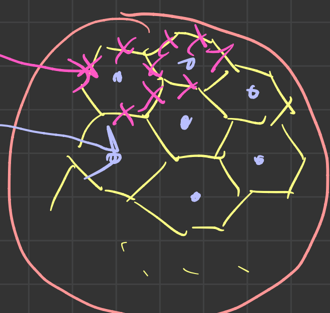


Convection rolls



Convection cells

down



up

above

What state occurs depends on boundary conditions $\underbrace{0 < \Delta T < \Delta T_c}$

uniform state stable

heat transferred from bottom to top plate simply by conduction. Fluid stationary, but a current of heat in $\hat{\underline{z}}$ -direction

in x-y plane. but may vary in z.

$\Delta T_c < \Delta T$ pattern forms.

Q/ How do we formulate this?

A/ Conservation laws and Symmetries
Constitutive relations e.g. $\rho = \rho(T, p)$
e.g. eqn of state.

Conservation laws: mass
momentum

Energy balance.

Let's look at the density $\boxed{\underline{\rho} = \bar{\rho}\left(1 - \alpha(T - \bar{T})\right)}$

* $\rho(\bar{T}) = \bar{\rho}$
* $\bar{T}$ is some reference temperature, such as the bottom plate.
* α = coefficient of thermal expansion.

Q/ What factors compete with temperature-induced buoyancy?

A/ * Thermal conductivity, κ. If κ large, then no hot spots

* Viscosity stop fluid from rising.

Mass conservation.

$$\partial_t \rho + \underline{\nabla} \cdot \underline{J} = 0$$

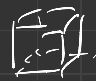
$$\underline{J} = \rho \underline{u}$$

Assume that variation in ρ small, so that approximately the fluid is incompressible

$$\underline{\nabla} \cdot \underline{u} = 0$$

Conservation of momentum:

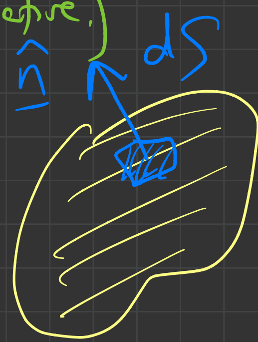
$$\boxed{\boxed{\rho}\,\boxed{\frac{d\underline{u}}{dt}} = -\nabla p} \qquad \boxed{\boxed{m}\,\boxed{a} = F}$$



$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}$$

(Total or advective derivative)

$$F = -\oint_S p\,\hat{n}\cdot d\underline{S}$$



So far: Newton II

But we neglected
* Gravitation
* Viscosity.

$F = \underline{u}(\underline{x},t)$

non-linear

$$\rho\left(\partial_t + \underline{u}\cdot\underline{\nabla}\right)\underline{u} = -\underline{\nabla} p + \rho\,\underline{g} + \eta\nabla^2\underline{u}$$

Viscosity: penalises velocity gradients so we need a term that is built from derivatio of $\underline{u}$. But must be rotationally covariant and must be a vector. $\Rightarrow$ lowest order in spatial gradients $\nabla^2 \underline{u}$

effective coarse-grained description of system.

Linear in $\underline{u}$ $\Rightarrow$ linear response theory.

<u>Next:</u> temperature field, obey the diffusion equation. <u>But</u> we must remember that the fluid is allowed to move.

$F(\underline{x}, t)$ is some property:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial F}{\partial \underline{x}}$$

$$= \left(\partial_t + \underline{u} \cdot \underline{\nabla}\right) F$$

Now $F = T(\underline{x}, t)$.

Heat diffusion equation is

$$\partial_t T + \underbrace{(\underline{u} \cdot \underline{\nabla})T}_{\text{advection}} = \kappa \nabla^2 T \qquad (*)$$

advection
couples motion of
fluid $\underline{u}$, to the
temperature T.

Strategy:

* Construct steady states of the equations and their boundary conditions
* Linear stability analysis.
  ↳ when do patterns first form?
* Non-linear stability analysis
  ↳ "Landau theory"
  ↳ Description close to onset of patterns.
* Behaviour well away from onset of patterns

Landau theory $T \lesssim T_c$ ←

"Phase modes" $T \ll T_c$ ←

Steady state

$\boxed{\underline{u} = \underline{0}}$ $\quad d_t T = 0$

$$\kappa \frac{d^2 T}{dz^2} = 0 \Rightarrow T(z) = T_0(z)$$
$$= T_l - \frac{z}{d}\Delta T$$

Temp. at lower plate.

$$\rho_0(z) = \bar{\rho}\left(1 - \alpha\left(T_0(z) - \bar{T}\right)\right)$$

Hydrostatic pressure:

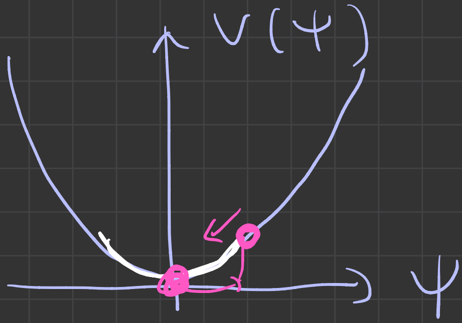
$$\boxed{0 = -\frac{dP_0}{dz} - \rho_0(z)\, g}$$

Next time: * small perturbations about this simple steady state.

* e.g. susceptibility.

Analogue of $\chi_T > 0 \quad T > T_c \quad M = 0$

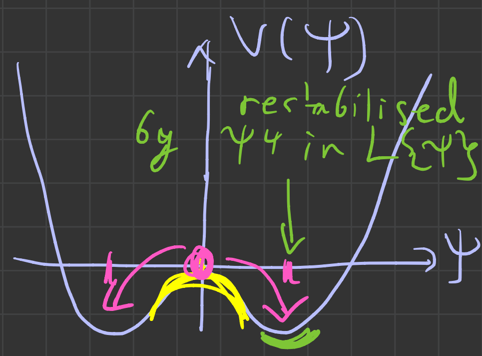
$\chi_T < 0 \quad T < T_c \quad M \neq 0$



$T > T_c$

$\chi_T > 0$

Linear stability of $\psi = 0$ solution $T > T_c$



$\chi_T < 0$

$T < T_c$

Linear instability of $\psi = 0$ solution $T < T_c$

2.1 Instability of the uniform state.

<u>Goal:</u>

* Uniform steady state that conducts heat by diffusion and no spatial patterns are present.

* Perturbation: what happens?
  Small perturbation $\Rightarrow$ linearise the non-linear eqns of motion.

* In Landau theory: measure susceptibility of the trivial $\Psi = 0$ state.

Terminology:

Time evolution of a small perturbation grows $\longrightarrow$ state about which we perturb is linearly unstable.

If the perturbation does not grow, then we say state is linearly stable.

[State may be non-linearly unstable] $\longleftarrow$ higher orders in perturbation theory.

A state that is linearly unstable: perturbation will grow until large enough that the linear perturbation theory is not valid. Linear approx. breaks down. Higher order non-linearities can restabilise the solution.

$$L\{\psi\} = \frac{1}{2} r_0 \psi^2 + \frac{1}{4} \boxed{u_0} \psi^4$$

$\hookrightarrow$ $r_0 < 0$ $\quad \psi = 0$ unstable

Evolves to $\quad \psi = \pm \left(\frac{-r_0}{\boxed{u_0}} \right)^{1/2}$

Return to convection

$$\left.\begin{aligned} T &= T_0(z) + T_1 \\ \rho &= \rho_0(z) + \rho_1 \\ p &= p_0(z) + p_1 \\ \underline{u} &= \underline{0} + \underline{u}_1 \end{aligned}\right\}$$

Take eqns of motion and expand to first order in perturbations

$$\rho_1 = -\alpha \bar{\rho} T_1$$

$$\underline{\nabla} \cdot \underline{u}_1 = 0$$

$$\rho_0(z) \partial_t \underline{u}_1 = -\underline{\nabla} p_1 + \eta \nabla^2 \underline{u}_1 + \rho_1 g$$

$$\boxed{\partial_t T_1 + w_1 \frac{dT_0}{dz} = \kappa \nabla^2 T_1 \quad (*)}$$

$$\underline{u} = (u, v, w)$$

Next: review the general formalism.

Suppose that we have a field $\psi(\underline{x},t)$

$$\partial_t \psi(\underline{x},t) = N\{\psi(\underline{x},t)\}$$

Some space-dependent non-linear operator on ψ, $\partial_{\underline{x}}\psi$, etc.

Steady state $\psi_0(\underline{x})$: $\partial_t \psi = 0$

$$N\{\psi_0(\underline{x})\} = 0$$

$$\psi = \psi_0(\underline{x}) + \psi_1(\underline{x},t)$$

$$\partial_t \psi_1(\underline{x},t) = N\{\psi_0 + \psi_1(\underline{x},t)\}$$

$$= N\{\psi_0\} + \frac{\partial N}{\partial \psi}\{\psi_0\} * \psi_1(\underline{x},t) + O(\psi_1^2)$$

$$\boxed{\partial_t \psi_1(\underline{x},t) = L\,\psi_1(\underline{x},t)}$$

Stability: $\Psi_1 = A \exp(i\underline{k}\cdot\underline{x} + \omega(\underline{k})t)$

dispersion relation

Satisfies: $\omega(\underline{k})\Psi_1 = L\Psi_1$

Growth rate of the perturbation $\propto e^{\omega(k)t}$

$\omega(k) \longleftarrow$ eigenvalue spectrum of linear operator L.

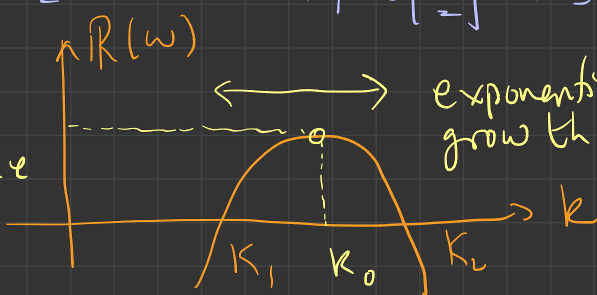
A number of scenarios:

I: $\mathbb{R}(\omega(\underline{k})) < 0 \quad \forall\, \underline{k}$

Any Fourier component will decrease in amplitude $\rightarrow$ $\Psi_0(\underline{x})$ is linearly stable.

II: $\mathbb{R}(\omega(k)) > 0 \quad k_1 < |\underline{k}| < k_2$

fastest growing mode

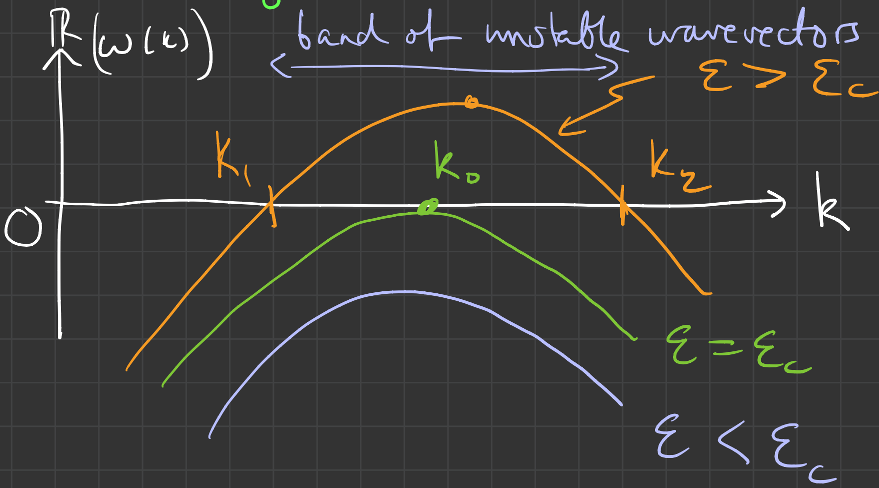


exponential growth

ψ will grow until the neglect of $O(\psi_1^2)$ is no longer justified, and the pattern may form predominantly around the fastest growing mode, and be restabilised by non-linearities.

III : $\Re(\omega(\underline{k})) \leq 0$

Marginal mode with $\omega(k) = 0 \implies$ cannot neglect $O(\psi_1^2) \implies$ non-linear stability.



Near threshold of instability, $\varepsilon \gtrsim \varepsilon_c$

$$|\underline{k}_1 - \underline{k}_2| \ll \underline{k}_0$$

Emergent state will have a pattern whose wavelength $\lambda = 2\pi / k_0$

A random initial condition will amplify modes $\underline{k} \to \underline{k}_0$, where $\omega(\underline{k}_0)$ is max will be selected.

Basic idea: simple, but execution can be technical.

<u>Next</u>: apply this formalism to convection.

Multiple steps, goal is to write the system of equations in terms of one variable.

<u>Step 1.</u> <u>Equation for w_1</u>

$$\underline{\nabla} \times \left[\rho_0 \partial_t \underline{u}_1 = \boxed{-\underline{\nabla} p_1} + \eta \nabla^2 u_1 + \rho_1 \underline{g} \right]$$

$$\left(\partial_t - \nu \nabla^2 \right) \underline{\nabla} \times \underline{u}_1 = -\alpha \underline{\nabla} T_1 \times \underline{g}$$

$$\nu \equiv \frac{\eta}{\rho_0}$$

$$\rho_1 = -\alpha \bar{\rho} T_1$$

$$\rho_0 \equiv \bar{\rho}$$

Take another $\underline{\nabla} \times$!

$$(\partial_t - \nu\nabla^2)\, \nabla^2 \underline{u}_1 = \alpha \left[(\underline{g} \cdot \underline{\nabla}) \underline{\nabla} T_1 - \underline{g} \nabla^2 T_1 \right]$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{u}_1) = \underline{\nabla}(\underline{\nabla} \cdot \underline{u}_1) - \nabla^2 \underline{u}_1$$

$$\underline{g} = (0, 0, -g)$$

$$(\partial_t - \nu\nabla^2)\, \nabla^2 w_1 = \alpha g \left[-\cancel{\frac{\partial^2 T_1}{\partial z^2}} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cancel{\frac{\partial^2}{\partial z^2}} \right) T_1 \right]$$

$$= \alpha g\, \partial_\perp^2 \boxed{T_1} \quad (*)$$

$$\partial_\perp^2 = \partial_x^2 + \partial_y^2$$

$w_1 \; (T_1)$ ✓

Next: eliminate T_1 and so get an eqn just involving w_1.

$$(\partial_t - \kappa\nabla^2)\, T_1 = -w_1 \frac{dT_0}{dz}$$

$$\boxed{(\partial_t - \kappa\nabla^2)(\partial_t - \nu\nabla^2)\nabla^2 w_1 = -\alpha g \frac{dT_0}{dz}(\partial_\perp^2 w_1)}$$

* 6th order PDE for w_1!
* Rotationally covariant in x-y plane.

Step 2 Eigenvalues

Method: seperation of variables.

$$w_1(\underline{x}, t) = \boxed{W(z)}\, f(x,y)\, e^{\omega t}$$

Insist that $\boxed{\partial_\perp^2 f(x,y) = -a^2 f(x,y)}$

Remember that the basic idea of separation of variables is

$$\left(\text{function of } z\right) = \left(\text{function of } x, y\right)$$

$\Longrightarrow$ Both LHS and RHS must be constant

$$\left[\nu\left(D^2-\boxed{a^2}\right)-\boxed{\omega}\right]\left[\kappa\left(D^2-\boxed{a^2}\right)-\boxed{\omega}\right]\left(D^2-\boxed{a^2}\right)W = \alpha g \frac{dT_0}{dz}\boxed{a^2}\,W \qquad (*)$$

$$D \equiv \frac{d}{dz} \implies \text{we have an ODE for } W.$$

The eigenvalues come from the boundary conditions.
Require perturbations vanish on boundaries.

$$u_1 = v_1 = w_1 = \boxed{T_1 = 0} \qquad z=0 \quad z=d$$

$$\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y} = 0 \qquad z=0 \quad z=d$$

$$\implies \boxed{\frac{\partial w_1}{\partial z} = 0} \qquad z=0,\ z=d.$$

from $\underline{\nabla}\cdot\underline{u}_1 = 0$

$T_1 = 0$ at boundary:

$$\left(\nu(D^2-a^2)-\omega\right)(D^2-a^2)W = 0$$

at $z=0 \quad z=d$

$$D^4 W - \left(2a^2 + \omega/\nu\right)D^2 W = 0 \quad \text{at } z=0,\ z=d.$$

Step 3: Simplify Boundary Condition.

Idea: understand basic structure by making a caricature with slightly different boundary conditions.

$$\left.\begin{array}{l} W = 0 \\ D^2 W = 0 \\ D^4 W = 0 \end{array}\right\} \quad z = 0 \quad z = d.$$

Solution: $W(z) = \sin\left(\frac{n\pi z}{d}\right)$ $n = 1, 2, 3, \ldots$

satisfy the b.c.'s and (*) as long as:

$$\left(\boxed{\omega} + \nu a_*^2\right)\left(\boxed{\omega} + \kappa a_*^2\right) a_*^2 + \alpha g \frac{dT_0}{dz} a^2 = 0$$

$$a_*^2 \equiv a^2 + \frac{n^2\pi^2}{d^2}$$

Solve for ω:

$$\omega = -\frac{1}{2}(\nu+\kappa) a_*^2 \pm \left\{\frac{1}{4}(\nu+\kappa)^2 a_*^4 + \left(\frac{\alpha g \Delta T}{d} \cdot \frac{a^2}{a_*^2} - \nu\kappa a_*^4\right)\right\}^{1/2}$$

$\omega \gtrless 0$ as we vary $k = \frac{n\pi}{d}$ and a, which is the physical wavenumber.

For instability we need $\omega > 0 \Rightarrow$ ~~need~~ the $+$ sign in front of $\sqrt{}$.

For this to happen

$$\frac{\alpha g \boxed{\Delta T}}{\nu \kappa d} > \frac{1}{a^2}\left(a^2 + \frac{\pi^2 n^2}{d^2}\right)^3$$

The smallest value for ΔT to give instability is when RHS has its minimum value.

$n = 1$.

What about a? $a = a_c = \dfrac{\pi}{d\sqrt{2}}$

$$\Rightarrow \underbrace{\frac{\alpha g \Delta T d^3}{\nu \kappa}} > \frac{27\pi^4}{4} \equiv R_c$$

Pure number $\equiv Ra \equiv$ Rayleigh number.

Destabilise uniform state in multiple ways.

Increase ΔT | Decrease ν
most sensitive $\rightarrow$ d | κ
α

Increase LHS and eventually cause uniform state to become destabilised.

Note: Increasing ν or κ stabilises system.

prevents easy convection flow

prevents hot spots giving buoyancy.

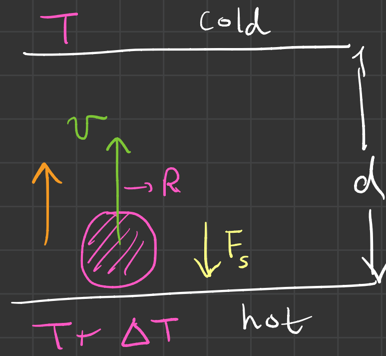
In summary: * Patterns form Ra exceeds a threshold $R_c = 1708$ expt.

Emphasise: this is not transition to turbulence

* Depends on ν, κ. Prandtl number $P_r \equiv \nu / \kappa$.

Next: What are the patterns we get?
What is the phase diagram?
Are patterns stable?

Simple explanation for the $R_c \sim d^3$ behaviour.



① We need the blob of heat to not have the temperature difference ΔT not diffuse away during the time t_c of convecting the distance d.

$$\tau_{diffusion} = d^2 / \kappa$$

where κ = heat diffusion coefficient

So $\tau_{diffusion} \ll t_c$ for convection to occur.

② What is the time t_c? If the blob moves with a velocity v, then

$$t_c = \frac{d}{v}.$$

③ For the blob to move at a constant velocity v, the forces on it must balance out.

The buoyancy force upwards is

$$F_B = \alpha \left(\begin{matrix}\text{volume of} \\ \text{blob}\end{matrix}\right) \times \Delta T \times g.$$

$$= \alpha (\rho R^3) \Delta T g. \quad (R = \text{radius of hot blob})$$

The drag force opposing buoyancy is Stokes drag, i.e. proportional to velocity

$$F_s = 6 \pi R \eta v$$

Equating

$$F_B = F_S$$

$$\Rightarrow \alpha (\rho R^3) \Delta T g = 6\pi R \eta v$$

$$\Rightarrow v = \frac{\alpha g \Delta T \rho R^2}{6\pi \eta} \quad (*)$$

(4) Calculate t_c.

Then $$t_c = \frac{6\pi\eta d}{\alpha g \Delta T \rho R^2}. \quad (*)$$

(5) Require $t_c \ll \tau_{\text{diffusion}}$.

$$\frac{6\pi\eta d}{\alpha g \Delta T \rho R^2} \ll \frac{d^2}{K} \quad (*)$$

This yields

$$\boxed{6\pi} \ll \frac{\alpha g \Delta T \rho (R^2 d)}{K\eta} = \boxed{\frac{\alpha g \Delta T \boxed{R^2 d}}{\nu K}}$$

(6) The blob is essentially a structure whose dimensions are $R = O(d)$.

So then we get $O(1) \cdot 6\pi \ll \frac{\alpha g \Delta T \boxed{d^3}}{\nu K}$

The explanation of the d^3 is really $d R^2$ with $R \sim d$.

2.2. Linear Stability Analysis + Transition to Turbulence

In the case of turbulent convection, the supercritical instability of the laminar state is separated from turbulence by a series of instabilities to different types of pattern.

(i) Pipe flow

Pressure gradient $\frac{dp}{dx} < 0$

$$\underline{v} = \left(1 - \frac{r^2}{a^2}\right) U_c \, \hat{\underline{x}}$$

$$\frac{dp}{dx} = -\nu \frac{4 U_c}{a^2}$$

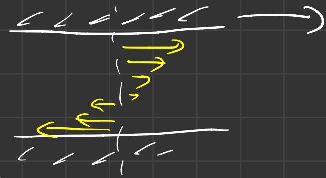
Stability about this solution: stable up to $Re = 10^7$

Belief that smooth pipes are linearly stable.

2021: Hall & Ozcakir: Roughness of scale $\epsilon \to R_c \sim C \epsilon^{-3/2} |\log \epsilon|^{-3/4}$

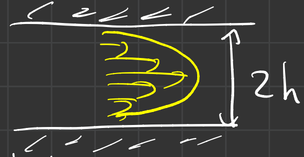
$C = O(30)$

(ii) Plane Couette Flow



Linearly stable (Romanov 1973) $\leftarrow$
but expt $\rightarrow$ turbulence $Re > 350$

(iii) Plane Poiseuille Flow



Pressure driven flow.
Linear instability: $R_c = 5772 = U_c h/\nu$
$\rightarrow$ discovered by Heisenberg!
But: turbulence occurs at $R_c \sim 1500$, expt.

Conclusion: * Finite amplitude instabilities.
Critical amplitude of disturbance
$\epsilon_c \sim 1/Re$ (Hof)
PRL 2003

* We need to systematically study sub-critical transitions!

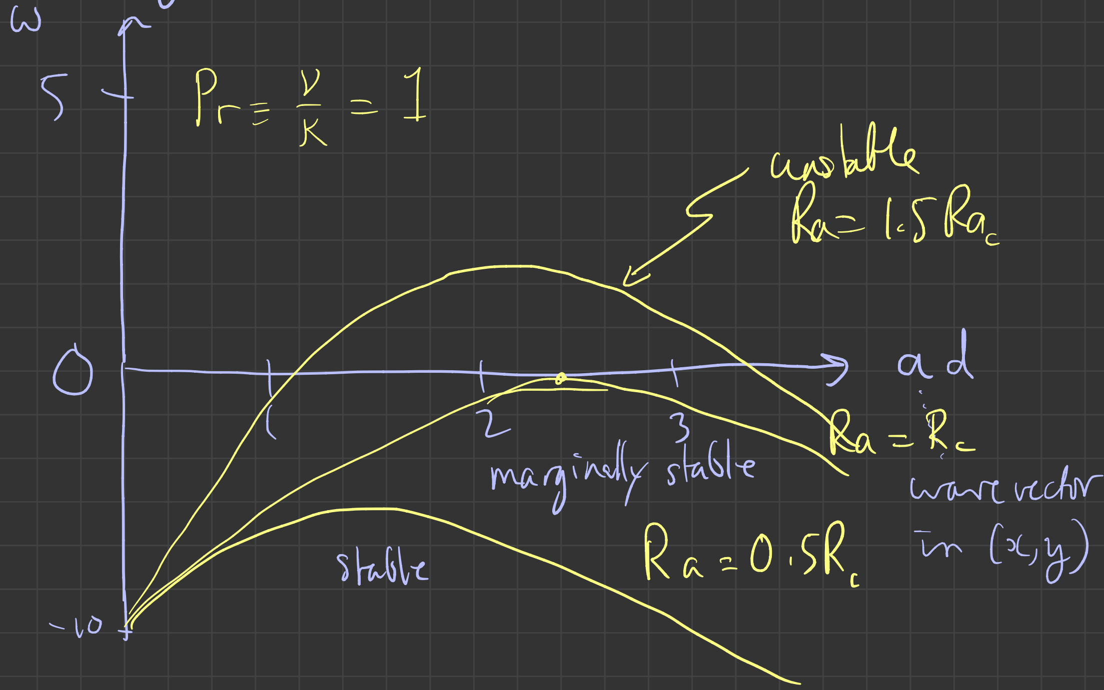
May 12 2021

* Neutral stability curve.
* Dimensional analysis

Complete Similarity
Incomplete Similarity
Intermediate asymptotics of
1st & 2nd Kind.

* Types of Patterns
* Stability.

Stability criterion was $\mathbb{R}(\omega(a))$



<u>Next:</u> want to reduce the number of parameters that we need to consider for the phase diagram.

<u>Boussinesq approximation:</u>

$$\rho\left(\partial_t + \underline{u}\cdot\underline{\nabla}\right)\underline{u} = -\underline{\nabla} p + \rho \underline{g} + \eta \nabla^2 \underline{u}$$

ignore the temperature dependence.

temperature dependent

Dimensionfull form of equations for Raleigh-Benard convection in the Boussinesq approximation.

$$\bar{\rho}\left(\partial_t + \underline{u}\cdot\underline{\nabla}\right)\underline{u} = -\underline{\nabla} p + \eta \nabla^2 \underline{u} - \bar{\rho}\, g\left(1 - \alpha\left(T - \bar{T}\right)\right)\hat{\underline{z}}$$

$$\underline{\nabla}\cdot\underline{u} = 0$$ incompressibility.

$$\left(\partial_t + \underbrace{\underline{u}\cdot\underline{\nabla}}\right) T = \kappa \nabla^2 T$$

advection: couples temperature and velocity field.

4. Dimensional analysis.

Basic idea: * reduce the parameter space of a problem.

* useful $\because$ it can allow us to relate different versions of a problem at different scales.

  ↳ example: airplane, car, helicopter design using scale models and scaled density working fluids.

* Sometimes people mistake dimensional analysis as including looking at limiting behaviour.

Statement: Dimensionless function can not depend on dimensionfull quantities. It can only depend on dimensionless quantities

Ex: $x_1, x_2, \ldots x_m$ a quantity x

$$x = F(x_1\ x_2 \ldots x_m)$$

Identify dimensionless groups of variables made up from combinations of $\{x_i\}$

$$\Pi = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_m^{\alpha_m} x^{\beta}$$

$$\left.\begin{aligned} \Pi_1 &= x_1^{\alpha_1^1} x_2^{\alpha_1^2} x_3^{\alpha_1^3} \ldots x_m^{\alpha_1^m} \\ \Pi_2 &= x_1^{\alpha_2^1} x_2^{\alpha_2^2} x_3^{\alpha_2^3} \ldots \\ &\vdots \end{aligned}\right\} n \text{ equations.}$$

DA $\Rightarrow$ $\Pi = f(\Pi_1, \Pi_2, \Pi_3, \Pi_n)$ $\quad n < M$

Buckingham's Π theorem $\rightarrow$ how many dimensionless groups.
(1913)

No unique way to construct or choose the Π's

Second step is the one everyone tries:
For some physical reason one of these variables might be very small $\Pi_1 \ll 1$

Tend to expect:

$$\Pi = f(0, \underbrace{\Pi_2, \cdots \Pi_n}_{n-1 \text{ variables.}}) + o(\Pi_1)$$

$$\lim_{\Pi_1 \to 0} f(\Pi_1, \Pi_2 \cdots \Pi_n) = f(0, \Pi_2, \cdots)$$

Intermediate asymptotics of first kind, complete similarity.

In many problems, this limit process doesn't exist.

E.g. $$f(\Pi_1, \Pi_2 \cdots \Pi_n) = \Pi_1^{\alpha_1} g(\Pi_2 \cdots \Pi_n)$$

Suppose $\alpha_1 > 0 \Rightarrow$

$$\lim_{\Pi_1 \to 0} f = 0$$

Suppose $\alpha_1 < 0 \Rightarrow$

$$\lim_{\Pi_1 \to 0} f = \infty$$

In asymptotic limit

$$\Pi_1 \to 0 \quad f \sim \Pi_1^{\alpha_1} g(\Pi_2 \cdots \Pi_n)$$

Incomplete similarity or Intermediate asymptotics of 2nd kind

This class of problems includes second order phase transitions, many problems in EM, mechanics, QM, ...

Typically in these cases, the statement is that

$$\Pi \underset{\Pi_1 \to 0}{\sim} \Pi_1^{\alpha}\, g\left(\frac{\Pi_2}{\Pi_1^{\beta_1}}, \frac{\Pi_3}{\Pi_1^{\beta_2}}, \ldots \frac{\Pi_n}{\Pi_1^{\beta_n}}\right)$$

$\alpha, \beta_1, \beta_2, \beta_3 \ldots \beta_n$ cannot be determined by dimensional analysis

$\Rightarrow$ compute these exponents by understanding the problem at least in the asymptotic limit.

Ex: statistical mechanics

$$M(t, H) = M_0 (-t)^{\beta}\, \Phi\left(\frac{H}{t^{\Delta}}\right)$$

$$t \equiv \frac{T - T_c}{T_c}$$

Scaling law for magnetization in terms of external field H and reduced temperature.

This functional form is observed only very close to the critical point $t \to 0$ $H \to 0$.

Suggest a connection between critical point phenomena and asymptotic analysis!

All of asymptotic analysis — singular perturbation theory (boundary layer theory, WKB analysis, matched asymptotic expansions, multiple scales analysis, reductive perturbation theory) are unified by RG.

<u>Next:</u> application of DA to convection.

$\overline{O}$ will denote the dimensionless form of O.

$$t = \frac{d^2}{\kappa}\,\overline{t} \qquad \underset{\sim}{x} = d\,\overline{\underset{\sim}{x}}$$

$$\underline{u} = \frac{\kappa}{d}\,\overline{\underline{u}} \qquad T = \frac{\nu\kappa}{\overline{\rho} g \alpha d^3}\,\overline{T}$$

$$p = \frac{\overline{\rho}\,\nu\kappa}{d^2}\,\overline{p}$$

<u>Verify!</u>

Substitute into the differential equation.

$$Pr\left(\partial_t \underline{\bar{u}} + \underline{\bar{u}}\cdot\underline{\bar{\nabla}}\,\underline{\bar{u}}\right) = -\underline{\bar{\nabla}}\,\bar{p} + \bar{\nabla}^2 \underline{\bar{u}} + \bar{T}\,\underline{\hat{z}}$$

$$\partial_t \bar{T} + \underline{\bar{u}}\cdot\underline{\bar{\nabla}}\,\bar{T} = Ra\,\bar{u}_z + \bar{\nabla}^2 \bar{T}$$

$$\bar{\nabla}\cdot\underline{\bar{u}} = 0$$

$$Ra = \frac{\rho g \alpha d^3 \Delta T}{\nu \kappa} \qquad \text{(Rayleigh \#)}$$

↳ driving force

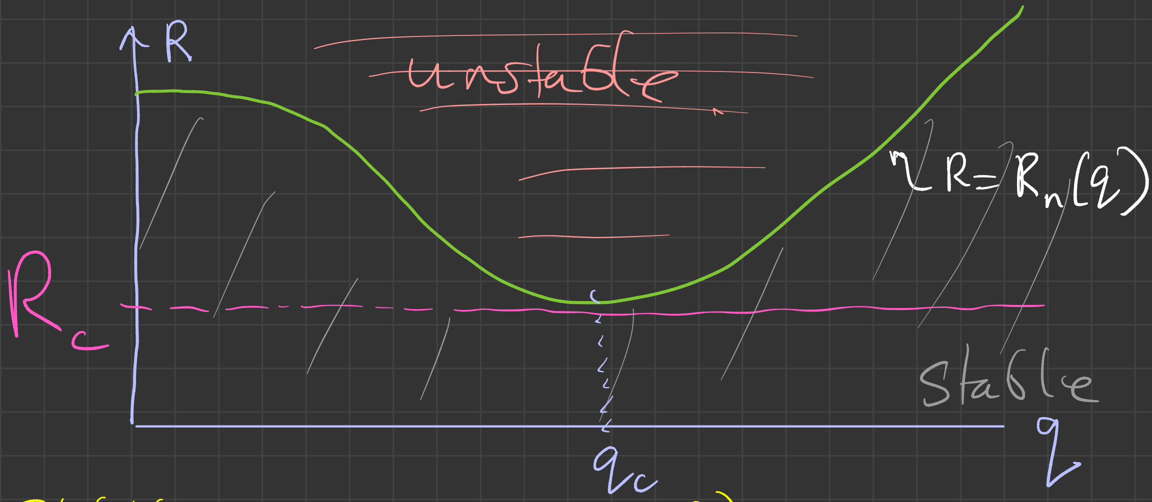
relative strength of dissipative mechanisms

$$P_r = \nu / \kappa \qquad \text{(Prandtl)}$$

From now on, work in dimensionless variables and drop the $\bar{\ }$.

5. Phase diagram

Goal:
- $\omega(q) \rightarrow$ neutral stability curve
- What are some of the pattern forming states?



Stability analysis $\omega(q; R)$

Neutral stability curve:

$$\mathbb{R}\left(\omega(q); R_n(q)\right) = 0$$

Control parameter
$R = Ra$
at fixed Pr.

R value R_n satisfies this eqn. $n \equiv$ neutral

For $(q, R) < R_n(q)$ fluid is linearly stable.

$R_c =$ smallest value of control parameter where instability occurs at some critical wave number q_c.

Q/ If the spatially uniform state is unstable, what does it eventually become once the non-linearities have restabilised the system?

A/ Hard to answer!

Next: write down some simple non-linear steady states that are not spatially uniform.

(a) Stripes.

$$\underline{u}(\underline{x}) = A\,\underline{u}_q(z)\left[e^{iq_c x} + c.c.\right]$$

Roll patterns in the x-direction.

(b) Squares

$$\underline{u}(\underline{x}) = A\,\underline{u}_q(z)\left[e^{iq_c x} + e^{iq_c y} + c.c.\right]$$

(c) Orthorhombic state.

$$\underline{u}(\underline{x}) = A\,\underline{u}_q(z)\left[e^{iq_c(x\cos\theta + y\sin\theta)} + e^{iq_c(x\cos\theta - y\sin\theta)} + c.c.\right]$$

$$\underline{u}(\underline{x}) = A\,\underline{u}_q(z)\; 4\cos(q_1 x)\cos(q_2 y)$$

$$q_1 = q_c\cos\theta \qquad q_2 = q_c\sin\theta.$$

(d) <u>Hexagons</u>

$$\underline{u}(\underline{x}) = A\,\underline{u}_q(z)\left\{ e^{iq_c\left(\frac{x}{2}+\sqrt{3}\underline{y}\right)} + e^{iq_c\left(\frac{x}{2}-\sqrt{3}y/2\right)} + \text{c.c.} \right\}$$

$$+ A\,\underline{u}_q(z)\, e^{i\phi_3 + iq_c y} + \text{c.c.}$$

$$\phi_3 = 0 \text{ or } \pi$$

; many other trial states can be envisioned.

Q/ Which of these possible states will be observed?

A/ Any state that is linearly unstable will not be observed

$\Longrightarrow$ Need to understand linear stability about these spatially extended states

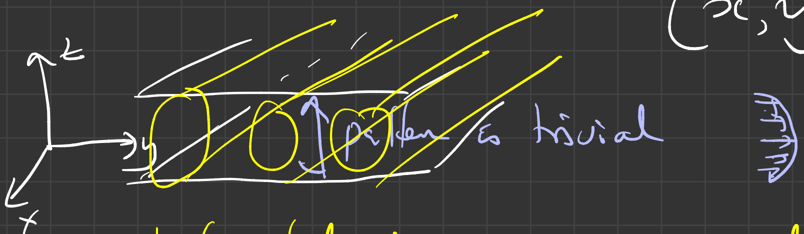
These instabilities of patterns are called secondary instabilities.

$$\text{uniform} \xrightarrow{1^{st}} \text{pattern} \xrightarrow{2^{nd}} \text{different pattern or dynamical state}$$

Let's suppose that we have a primary instability of uniform state to a pattern

$$\underline{u}_{\underline{q}} = \underline{u}_{\underline{q}}(\underline{x}_{\perp}, z)$$

$$(x, y)$$



velocity profiles

Interested in $\underline{x}_{\perp}$ coordinate pattern.

$$\underline{u} = \underline{u}_{\underline{q}} + \delta\underline{u}$$

$$\delta\underline{u} = \underline{u}_1(\underline{x}_{\perp}, z)\, e^{i\underline{Q}\cdot\underline{x}_{\perp}}\, e^{\omega(\underline{q},\underline{Q})t}$$

wave vector of pattern

wave vector of perturbation

Basic state is periodic, all we need to do is consider first Brillouin Zone.

Ex: $d=1$ stripe pattern

$$-\frac{q}{2} < Q_x < \frac{q}{2}$$

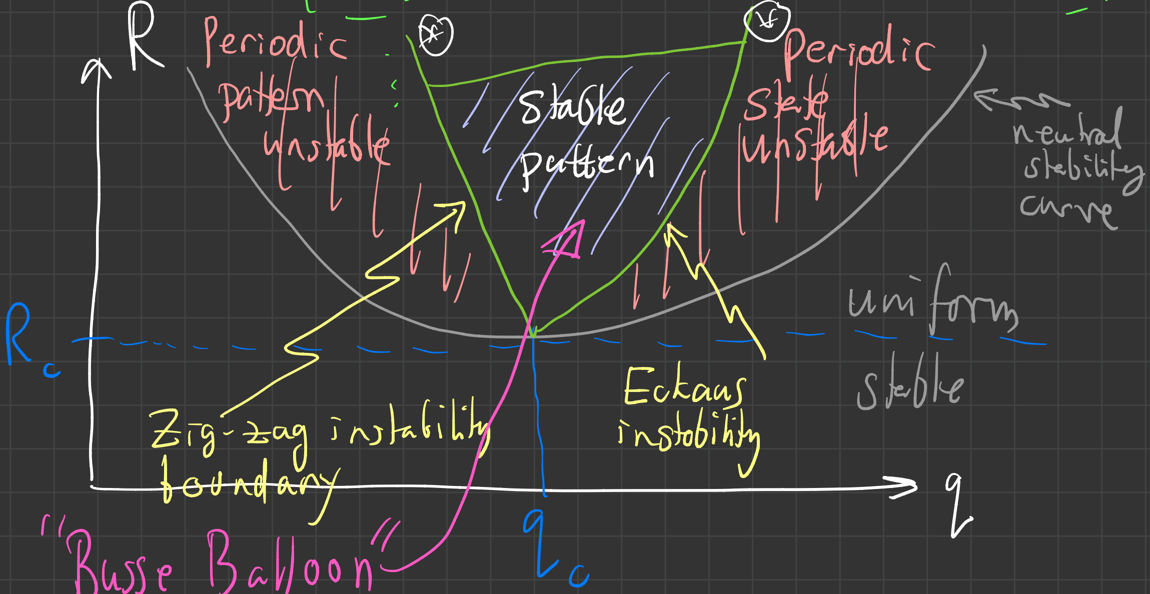
Many possibilities:

$\underline{Q} = \underline{0}$

$\underline{Q}$ is on zone boundary.

$|\underline{Q}| \ll |\underline{q}|$ long-wavelength

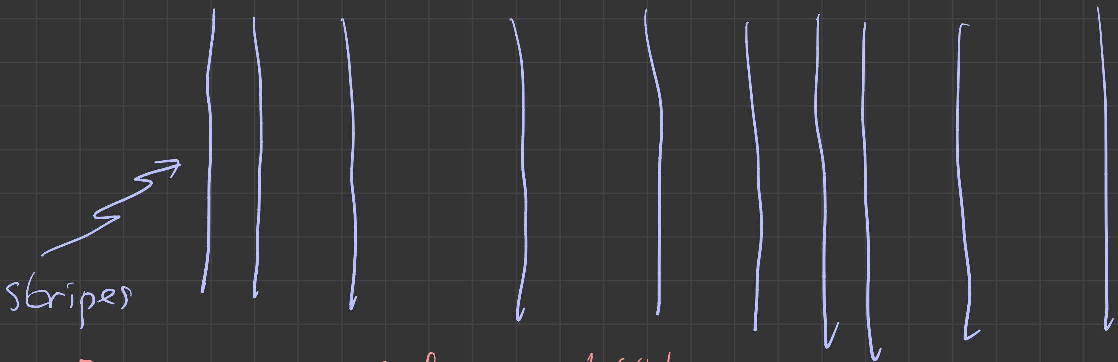
$|\underline{Q}|$ is incommensurate with $|\underline{q}|$



Common instabilities of stripe patterns ordering with

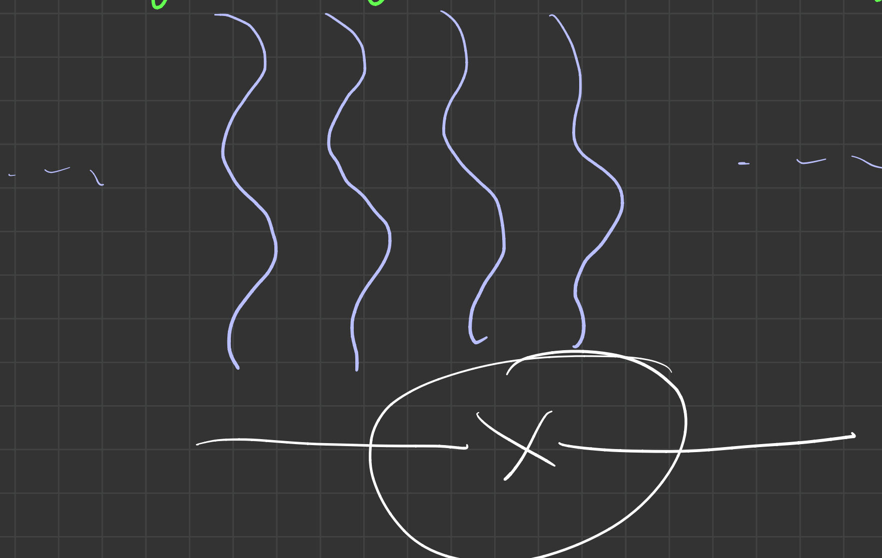
$$\underline{q} = (q_x, 0, 0)$$

Eckhaus instability: $Q_y = 0$ $\quad Q_x \ll q_x$



Period modulating instability. $\longrightarrow x$

Zig-zag instability, $Q_x = 0$ $\quad Q_y \ll q_x$



Wednesday May 19 2021

6. Non-linear theory for pattern dynamics.

Goal: How do various patterns equilibrate + compete?
Non-linear stabilisation of the patterns.

Strategy:
* bifurcation theory for one-degree of freedom
* couple the order parameter at different points in space by some trickery using the linear stability analysis.

Dynamical system:

$$\dot{u} = f(u, \epsilon)$$

Assume for control parameter $\epsilon < \epsilon_c$
$u = 0$ is a stable solution, steady state.

$$f(0, \epsilon) = 0 \qquad \epsilon < \epsilon_c$$

Near ϵ_c expand $f(u, \epsilon)$ — in the spirit of Landau theory

$$f(u) = a + bu + cu^2 + du^3 + \cdots$$

We know that $u = 0$ is a solution of $\dot{u} = 0$

$$\Rightarrow a = 0.$$

Next, we know that for $\epsilon > \epsilon_c$, we have an instability to a periodic state.

$\Rightarrow$ b must change sign at ϵ_c.

$b < 0 \qquad \epsilon < \epsilon_c$

$b > 0 \qquad \epsilon > \epsilon_c$

$$\Rightarrow \quad b = \boxed{b_0 \, (\epsilon - \epsilon_c)} + O\left((\epsilon - \epsilon_c)^2 \right)$$

$$b_0 > 0.$$

Next, suppose there is a symmetry which requires that the results are invariant under $u \to -u$.

$\Rightarrow \quad c = 0$

$\Rightarrow$ In vicinity of ϵ_c:

$$\dot{u} = r u + d u^3 + O(u^5)$$

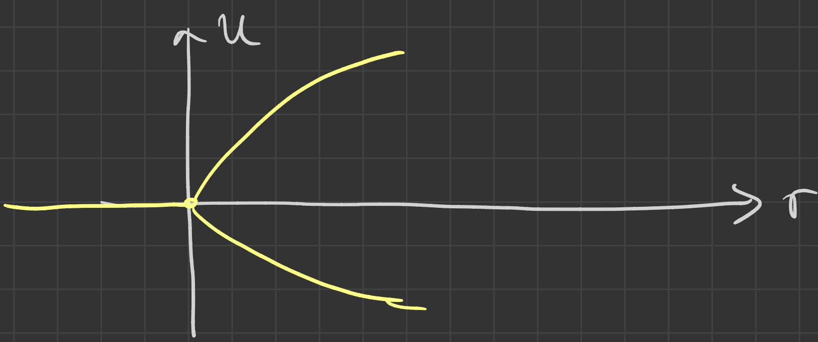
$$r = b_0 (\epsilon - \epsilon_c) \qquad d \gtrless 0$$

Steady states: $(d < 0)$

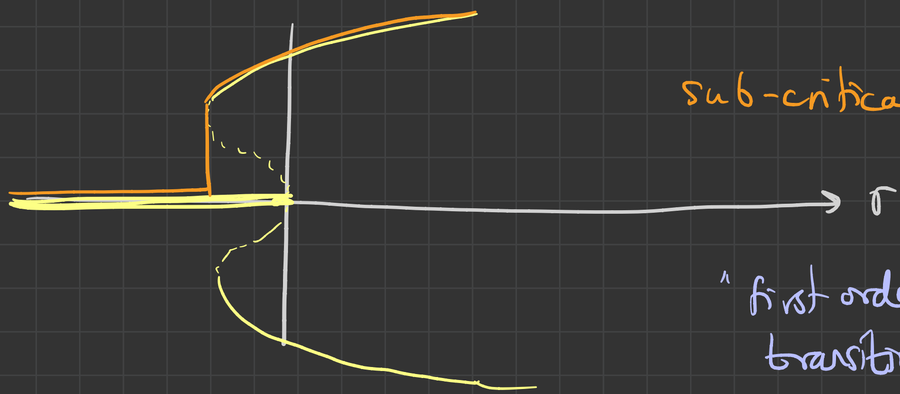
$$u = 0 \qquad r < 0$$

$$u = 0 \quad \pm (r/|d|)^{1/2} \qquad r > 0$$

If $d > 0$: $\begin{cases} u = 0 & (r > 0) \\ u = 0, \pm (r/-d)^{1/2} & (r < 0) \end{cases}$



supercritical.
"second order/ continuous transition"



sub-critical.
"first order transition"

Q/ Spatial extension?

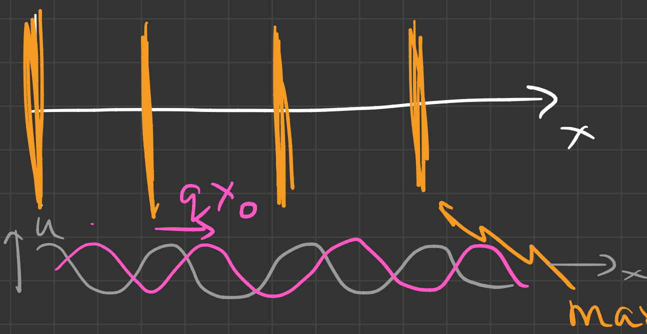
A/ $\underline{u}(\underline{x}, t)$ orders along $\hat{x}$ direction.

$$\underline{u} \propto \underline{A} \, e^{iqx} + c.c.$$

Every component of $\underline{u}$ will have an amplitude and a phase.

$$u(\underline{x},t) = |A| \exp(iqx + i\underline{\underline{\phi}}) + c.c.$$

Physical significance of ϕ is that it generates Galilean transformations



Shift in origin $x \to x + x_0$

This adds a term qx_0 to ϕ

Phase invariance $\equiv$ translational invariance.

$\Rightarrow$ Equation governing the complex amplitude $Ae^{i\phi}$ must be invariant under phase rotations if original system is translationally invariant $\Rightarrow$ global $O(2)$ symmetry.

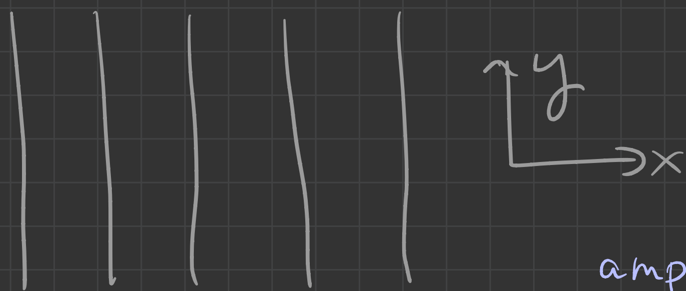
$$\partial_t A(\underline{x},t) = \underline{\underline{rA}} + dA|A|^2 \quad *$$

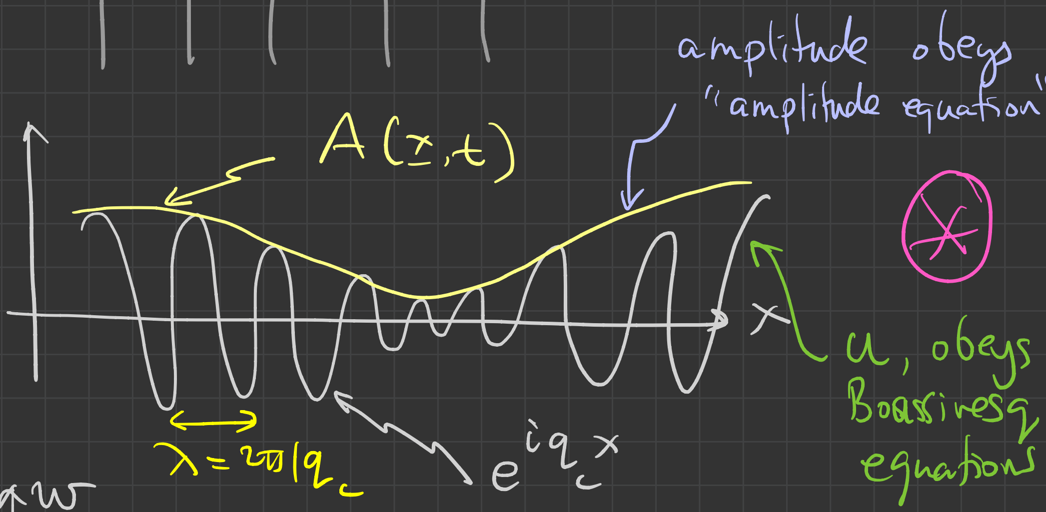
+ gradient terms

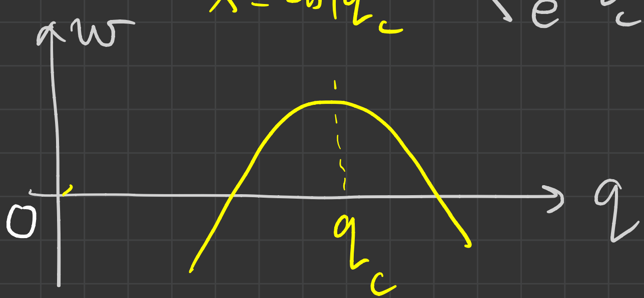
RG way to derive the equation for the pattern amplitude: Chen, NG, Oono PRE (1996)

Here use a heuristic method that uses only the dispersion relation from linear stability analysis about a uniform state.

Consider is the system with rolls $\parallel \hat{y}$







Consider the form of the dispersion relation, $\omega(\underline{q})$:

$$\omega \longrightarrow \partial_t$$

$$\underline{q} \longrightarrow i\underline{\nabla}$$

Knowing $\omega(\underline{q}) \longrightarrow$ write down a linearised PDE for small perturbations

$$\boxed{r = \epsilon - \epsilon_c}$$

$$\omega(q) = \tau^{-1}\left[r - \xi^2 (q - q_c)^2 + O(q q_c^3) \right]$$

τ: temporal scale

ξ: spatial scale

$$[\omega] = T^{-1}$$

$$[q] = L^{-1}$$

Strategy: * use the uniform dispersion relation

* take pattern with wavenumber q

* modulate by $|A| e^{iQ_x x} e^{iQ_y y}$

$$\underline{q} = (q_c, 0) \longrightarrow \underline{q} = (q_c + Q_x, Q_y)$$

* Plug into $\omega(\underline{q})$.

$$\tau\,\omega(\underline{q}) = r - \xi^2\left[\sqrt{(q_c+Q_x)^2+Q_y^2} - q_c\right]^2$$

$$\cong r - \xi^2\left\{\frac{Q_y^2}{2q_c} + Q_x\right\}^2 \quad (*)$$

$$u(x,y,t) = A(x,y,t)\,e^{iq_c x} + \text{c.c.}$$

$$= A_0\, e^{\omega(q)t}\, e^{i(Q_x x + Q_y y)}\, e^{iq_c x} + \text{c.c.}$$

Make identification

$$\omega(q) = \partial_t$$

$$Q_x = -i\,\partial_x$$

$$Q_y = -i\,\partial_y$$

dispersion relation (*)

$$\boxed{\tau\partial_t A = \underline{\underline{rA}} + \xi^2\left(\partial_x - \frac{i}{2q_c}\partial_y^2\right)^2 A}$$

Put all the terms together!

$$\boxed{\tau\partial_t A = rA + dA|A|^2 + \xi^2\left(\partial_x - \frac{i}{2q_c}\partial_y^2\right)^2 A}$$

(Newell - Whitehead - Segel)

Note: N-W-S eqn is not symmetric in (x, y)

* $\because$ we are expressly interested in rolls along y. $\longrightarrow$ breaks rotational invariance.

* There is a rotationally covariant form of this amplitude equation.
(Gunaratne, Swinney, Qi)
(Graham, RG)

Next: Scaling properties of the amplitude equation.

Scaling Properties of the amplitude equation.

All terms must be of "comparable order" in terms of $(\epsilon - \epsilon_c)$.

Balance of the various terms:

$$\tau\, \partial_t A \sim (\epsilon - \epsilon_c) A$$

$$\xi^2 \partial_x^2 A \sim (\epsilon - \epsilon_c) A$$

$$\frac{\xi^2}{q_c^2} \partial_y^4 A \sim (\epsilon - \epsilon_c) A$$

We can estimate that

$$\partial_x A \sim (\epsilon - \epsilon_c)^{1/2} A$$

$$\partial_y A \sim (\epsilon - \epsilon_c)^{1/4} A$$

Phase variations in x obey a diffusion equation. Phase variations along y obey a

$$\partial_t A \sim \partial_y^4 A \quad \longleftarrow \text{bending beam equation}$$

∴ rolls along y direction have emergent stiffness or rigidity.

Origin: energy bent beam $\sim (\text{curvature})^2$

<u>Goal</u>: look at space-time scaling near ϵ_c.

<u>Next</u>: rescale to get a universal eqn.

$$\tau \partial_t A = rA + dA|A|^2 + \xi^2 \left(\partial_x - \frac{i}{2q_c} \partial_y^2\right)^2 A$$

Dimensions of time: τ

space: ξ

A : $(r/d)^{1/2}$

Rescale

$$X \equiv \frac{\sqrt{r}}{\xi} x$$

$$Y \equiv \frac{r^{1/4}}{(\xi/q_c)^{1/2}} y.$$

$$T \equiv \frac{r}{\tau} t$$

$$\Psi \equiv (r/d)^{-1/2} A$$

$$\boxed{\partial_T \Psi = \Psi - \Psi|\Psi|^2 + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \Psi}$$

Universal eqn with no adjustable parameters

This is analogous to Gross-Pitaevskii eqn + time-dependent Ginzburg-Landau eqns

Spatial variations transverse to rolls relax with a characteristic length scale $O(1)$ in X

i.e. $O(\xi/r^{1/2})$ in physical coordinates

Remember $r \sim \underbrace{\epsilon - \epsilon_c}_{\delta\epsilon} \Rightarrow \xi(\delta\epsilon) \sim (\delta\epsilon)^{-1/2}$ diverges near point of instability

Spatial variations along the rolls have a characteristic length scale

$$\mathcal{O}\left(\frac{\xi^{1/2}}{q_c^{\frac{1}{2}} r^{1/4}}\right) \quad (\epsilon - \epsilon_c)^{-1/4}$$

Time scale relaxation $\mathcal{O}(\tau/r)$

i.e. $\delta\epsilon^{-1}$.

Order parameter scale $\mathcal{O}(r^{1/2})$

Divergence of these space and time scales near the critical point at the onset of convection $\sim$ MFT of critical phenomena!

<u>Comment:</u> * Very hard to constrain expts. enough to see thermal fluctuation corrections

* Has been done: Wu, Ahlers, Cannell PRL (1995).

7. Emergent Patterns Below Threshold.

* Behaviour near ϵ_c

* What about $\epsilon \gg \epsilon_c$ well into the patterned phase?

Guess: * Amplitude of pattern is saturated by non-linearities
$A =$ constant.

* Only remaining degrees of freedom are the phase degrees of freedom.

Exactly analogous to what we saw in eqm.

Summarise: phase dynamics

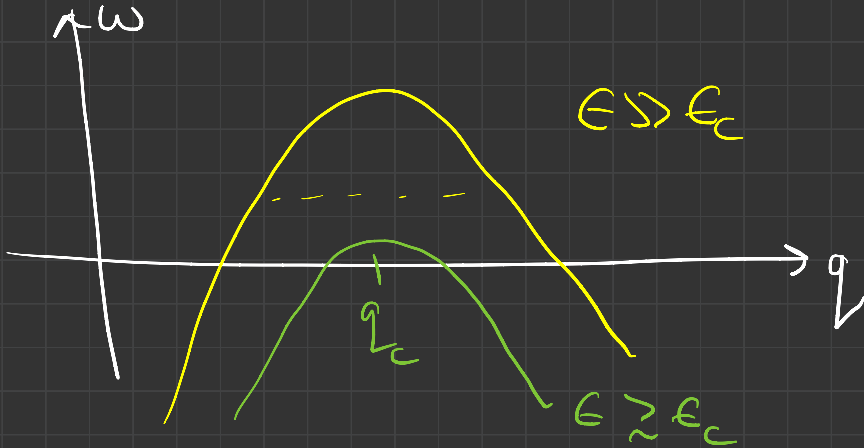
Near ϵ_c: Phase dynamics are diffusive

$$\partial_t \phi = D_{\parallel} \partial_x^2 \phi + D_{\perp} \partial_y^2 \phi$$

$$D_{\parallel} = \frac{\xi^2}{\tau} \frac{(\epsilon - 3\xi^2 k^2)}{\epsilon - \xi^2 k^2} \qquad D_{\perp} = \frac{\xi^2}{\tau} \left(k / q_c \right)$$

$k = q - q_c$ wavenumber of the pattern

[Remember : $\epsilon \gg \epsilon_c$]

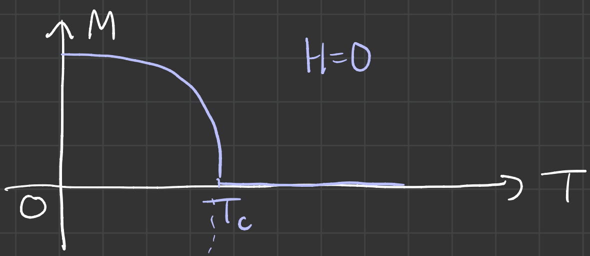


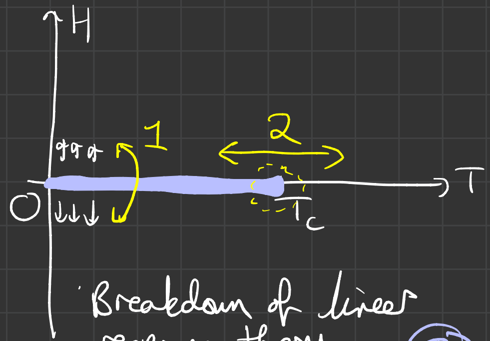
Well away from ϵ_c, dynamics is described by an equation describing the space-time variation of "local wavenumber"

See : Cross + Newell.

§2. Widom Scaling.

There are two stylized facts that were known about critical phenomena in the early 1960's.





Order parameter scaling

$$M = M_0 \left| \frac{T-T_c}{T_c} \right|^\beta \qquad T \to T_c^{\pm} \qquad (1)$$

$$t \equiv \frac{T-T_c}{T_c} \qquad h \equiv H/k_B T_c$$

Breakdown of linear response theory. (2)

$$M \sim H^{1/\delta} \qquad T = T_c$$

not $M \sim H/k_B T$ (Curie)

Widom and Kadanoff realised that together these results are equivalent to

$$M(t,h) = |t|^\beta F_m(h/t^\Delta) \qquad t<0$$

where Δ is a new exponent that we'll shortly calculate

Q/ What is the function F_M and the exponent Δ?

A/ • For (1) to hold, we need $F_M(z) = \text{const}$ for $z=0$

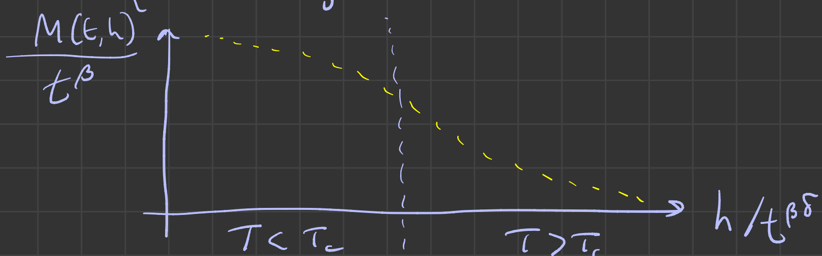
• For large z, i.e. $h \neq 0$ $t \to 0$ we need to recover (2) which means t must somehow cancel out. This can only happen if $F_M(z) \sim z^{1/\delta}$ as $z \to \infty$.

Then $t^{\beta - \Delta/\delta} = O(1) \Rightarrow \boxed{\beta\delta = \Delta}$

$$\Rightarrow \boxed{M(t,h) = t^{\beta} F_M\left(h/t^{\beta\delta}\right)}$$

$M(t,h)$ is ostensibly a function of two variables h, t, but near the critical point is actually a function of a "similarity" or "scaling" variable $h/t^{\beta\delta}$.

We can test this as follows. Take sets of data for $t = t_1, t_2, t_3, \ldots$ $h = h_1, h_2, \ldots$ and plot $\{M(t_i, h_i)\}$ as



$\rightarrow$ $\boxed{\text{Back to slides}}$

§ 3. Predator–Prey Model

3.1 Centers and neutral cycles

$A =$ predator density $\qquad B =$ prey density.

$$\dot{A} = pAB - dA$$
$$\dot{B} = bB - pAB$$

$$A + B \xrightarrow{p} A + A$$
$$A \xrightarrow{d} \emptyset$$
$$B \xrightarrow{b} B + B$$

We can easily calculate steady states and phase portrait.

$$\dot{A} = 0 \Rightarrow A(pB - d) = 0 \qquad A^* = 0 \text{ or } B^* = \frac{d}{p}$$

$$\dot{B} = 0 \rightarrow B(b - pA) = 0 \qquad B^* = 0 \text{ or } A^* = \frac{b}{p}$$

Coexistence

Steady states: $(A^*, B^*) = (0,0)$; $\boxed{(A^*, B^*) = (b/p, d/p)}$

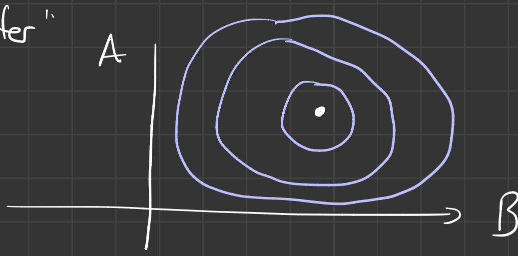
Long-time $A^* = 0$, $B \to \infty$ as e^{bt} as $t \to \infty$.

Linear stability: $A = A^* + \delta A e^{\omega t}$ $\quad B = B^* + \delta B e^{\omega t}$

$(0,0)$ unstable $\qquad (b/p, d/p) \rightarrow \omega = \pm i\omega_0$

$(0,\infty)$ unstable $\qquad \omega_0 = \sqrt{bd}$

These oscillations about the coexistence fixed point describe a "center"



The phase space is obtained from.

$$\frac{dA}{dB} = \dot{A}/\dot{B} = \frac{pAB - dA}{bB - pAB} = \frac{A(pB-d)}{B(b-pA)}$$

$$\Rightarrow \int_0^t \frac{dA}{A}(b-pA) = \int_0^t \frac{dB}{B}(pB-d)$$

$$\Rightarrow b\ln A(t) - pA(t) - b\ln A(0) + pA(0) = pB(t) - d\ln B(t) + d\ln B(0) - pB(0)$$

$$\Rightarrow b\ln A(t) + d\ln B(t) - p(A(t)+B(t)) = b\ln A(0) + d\ln B(0) - p(A(0)+B(0))$$

So $C(t) \equiv b\ln A(t) + d\ln B(t) - p(A(t)+B(t))$

$= C(0)$ i.e. a conserved integral of the motion.

This model predicts periodic oscillations, with trajectory (amplitude/phase) determined by $C(0)$.

However, this solution is unphysical because it is structurally unstable.

3.2 Finite carrying capacity.

The FP $A=0$, $B = B_0 e^{bt}$ is unphysical because in reality there is a finite amount of food for the prey. So B is bounded above by what is called "carrying capacity".

We model this as

$$\dot{B} = bB(1 - B/K) - pAB.$$

Fixed Points:

$A^* = B^* = 0.$ Extinction

$A^* = 0 \quad B^* = K$ Predator death, Prey saturate.

$A^* = \left(1 - \frac{d}{Kp}\right)\frac{b}{p} \; ; \; B^* = d/p$ Coexistence

Stability:

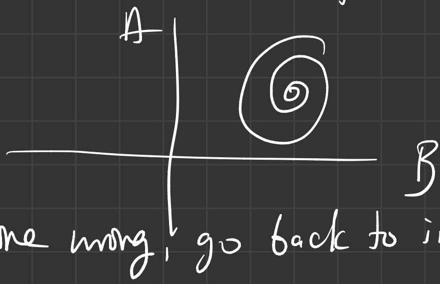
Extinction: unstable.

Prey saturation: stable $p < p_c = d/K$

Coexistence: $p > p_c$ and linearly stable with

$$\omega = -\frac{db}{2Kp}\left(1 \pm \sqrt{1 - \frac{4Kp}{b}\left(\frac{Kp}{d} - 1\right)}\right)$$

Summary: No persistent population cycles. The phase portrait has stable spiral



To see what has gone wrong, go back to individual level model.

Q/ How did ecologists deal with this embarrasment?

A/ Let's change the physical picture.

The predation term pAB only applies if the concentration of predator and prey is small.

But suppose prey concentration is large. Predator does not need to look far to find a prey to eat. In other words, prey concentration is not a limiting factor. So

$$pAB \longrightarrow \frac{pAB}{C+B}$$

where C is a constant.

For $B \ll C$, pAB is recovered.

For $B \gg C$ $\quad pAB \longrightarrow pA$ indep of B.

Thus

$$\dot{A} = \frac{pAB}{C+B} - dA$$

$$\dot{B} = bB(1-B/K) - \frac{pAB}{C+B}$$

This non-linear system <u>does</u> have limit cycles

<u>Back to slides!</u>