03/07/2017 MEAN FIELD GLASSY SYSTEMS - GIULIO BIROLI.

Illustrating how many problems have rough energy landscapes -

↳ we are looking at problems with so many different local minima that one should study the statistical properties of the energy landscapes → this is what mean field is doing for you.

1 ROUGH ENERGY LANDSCAPE I: THE RANDOM ENERGY MODEL. (Derrida 1981)

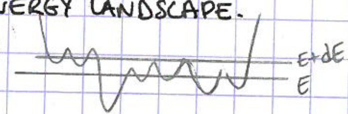
In spirit, let's be as simple as possible.

- Ising spins - configuration space: $\mathcal{C} = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$ $(\# \mathcal{C} = 2^N)$.
- disordered energy landscape: $E(\mathcal{C}) \sim \mathcal{N}(0, \frac{N}{2})$ iid.

$\frac{N}{2}$ ↑ { • necessary for everything to be well defined • is also found for more principled models (e.g. 2d Ising model)

1.1 STUDY OF THE ENERGY LANDSCAPE.

Pictorial view:



of $\mathcal{C}$ with energy between E et E+dE: $\mathcal{N}(E)dE$

Let's try to compute the average: $\langle \mathcal{N}(E)dE \rangle = \langle \sum_{i=1}^{2^N} Y_i \rangle$ $\quad Y_i = \begin{cases} 1 & \text{if } E(\mathcal{C}_i) \in [E, E+dE] \\ 0 \end{cases}$

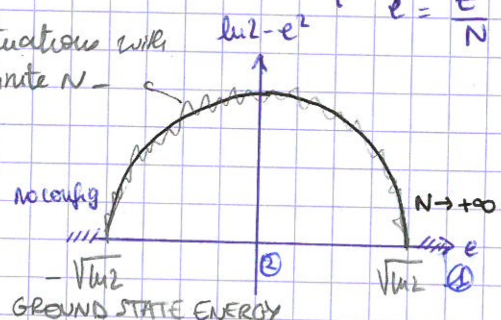
over the choice of the energies of configurations

$$= \sum_{i=1}^{2^N} \langle Y_i \rangle$$

$$= 2^N \int_E^{E+dE} \frac{e^{-E'^2/N}}{\sqrt{\pi N}} dE'$$

$$\overset{dE \ll 1}{=} 2^N \frac{e^{-E^2/N}}{\sqrt{\pi N}} dE$$

which we will rewrite: $\begin{cases} \langle \mathcal{N}(E)dE \rangle = \underbrace{\frac{1}{\sqrt{\pi N}}}_{\text{subleading}} \exp\left[N\ln 2 - Ne^2\right] dE \simeq \exp N\left[\ln 2 - e^2\right] dE \\ e = \frac{E}{N} \end{cases}$



entropy density:

$$\frac{\ln \mathcal{N}(E)}{N} = s(e) = \ln 2 - e^2 \qquad |e| < \sqrt{\ln 2}$$

↳ fluctuations are killed by the $N \to +\infty$ - really equal to the mean -

Two important regimes:

① No configurations for $|e| > \sqrt{\ln 2}$: $\langle \mathcal{N}(E)dE \rangle = \exp(-\overset{\text{constant}}{c}N) \xrightarrow[N \to +\infty]{} 0$

what is the probability to have at least one configuration in this region?

$$P(\text{at least one } |e_i| \in [e, e+de]) = \sum_{n=1}^{\infty} P(n) \leq \sum_{n=0}^{\infty} n P(n) = e^{-cN} \xrightarrow[N \to +\infty]{} 0$$

② Fluctuations are killed between $-\sqrt{\ln 2}$ and $\sqrt{\ln 2}$

$$\langle (W(E)dE)^2\rangle - \langle W(E)dE\rangle^2 = \langle (\sum_i Y_i)^2\rangle - (\sum_i \langle Y_i\rangle)^2$$

$$= \sum_{ij} \langle Y_i Y_j\rangle - \langle Y_i Y_j\rangle$$

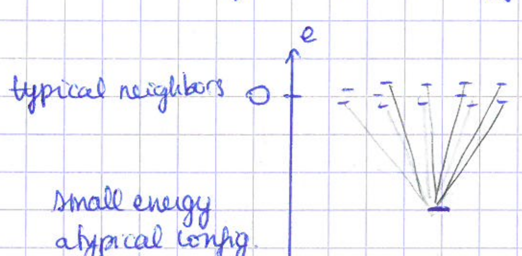
$$\overset{iid}{=} \sum_i \langle Y_i^2\rangle - \langle Y_i\rangle^2 \overset{Y_i=0,1}{=} \sum_i \langle Y_i\rangle - \langle Y_i\rangle^2$$

$$= 2^N \left[\frac{e^{-E^2/N} dE}{\sqrt{\pi N}} - \underbrace{\cancel{\frac{e^{-2E^2/N} dE^2}{\pi N}}}_{\text{subleading}} \right] = \langle W(E)dE\rangle$$

<u>conclusion:</u> $\langle (W(E)dE)^2\rangle_{\substack{c \\ \text{connected}}} \sim \langle W(E)dE\rangle$ and $W(E)dE = \langle W(E)dE\rangle + \sqrt{\langle W(E)dE\rangle}$

And fluctuations are much smaller than the average, so that they can be neglected. The system is <u>self averaging</u>.

The ground state energy is $-\sqrt{\ln 2}$, most of the configurations have energy $0 + \sqrt{\frac{1}{N}}$



N neighbors with gaussian r.v, so that the statistic of the min is: $E_{min} = -\sqrt{\frac{N}{2}\ln N} +$ fluctuation
(on variance)

$\hookrightarrow e_{min} = -\sqrt{\frac{\ln N}{2N}} \rightarrow$ which is not far from 0

We say that we have a golf <u>course landscape</u>. Need to cross big energy barrier to escape low energy states.

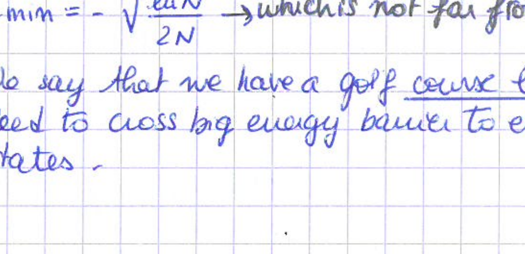
12 THERMODYNAMICS

The partition function is $Z = \sum_c e^{-\beta E_c} = \int_{-\sqrt{\ln 2}}^{\sqrt{\ln 2}} de\, W(e)\, e^{-\beta N e} \equiv e^{-N\beta f}$

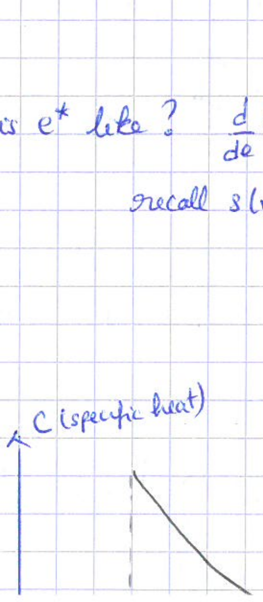
$$= \int_{-\sqrt{\ln 2}}^{\sqrt{\ln 2}} de \exp\left(N(s(e) - \beta e)\right) \approx \begin{cases} e^{N(s(e^*) - \beta e^*)} & \text{if } |e^*| < \sqrt{\ln 2} \\ e^{N(\underbrace{\overset{=0}{s(e_{GS})}}_{\text{ground state}} - \beta e_{GS})} & \text{otherwise.} \end{cases}$$

What is e^* like? $\frac{d}{de}(s(e) - \beta e) = 0 \Rightarrow s'(e^*) = \beta \Rightarrow -2e^* = \beta \Rightarrow \boxed{e^* = -\frac{1}{2T}}$

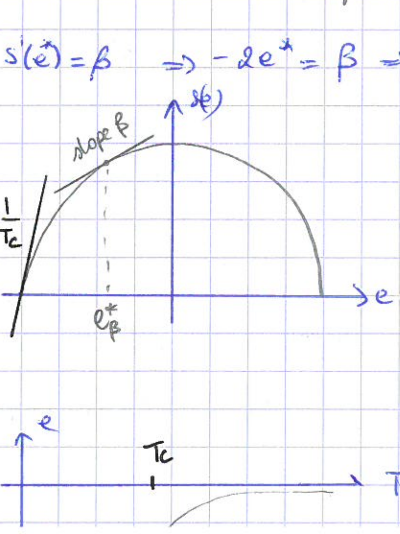
recall $s(e) = \ln 2 - e$



$\Rightarrow T_c = \frac{1}{2\sqrt{\ln 2}}$



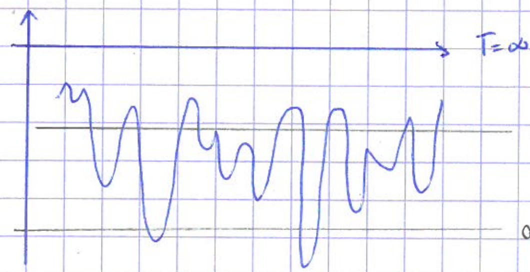
and



what is finally the free energy: $f = \begin{cases} -1/\beta \left[(e^*) - \beta e^* \right] = 1/\beta \left[\ln 2 + \frac{\beta^2}{4} \right] & T > T_c \\ -1/\beta \left[-\beta\, e_{GS} \right] = e_{GS} = -\sqrt{\ln 2} \end{cases}$

1.3 FEW WORDS ON DYNAMICS

To equilibrate at temperature T, the system should reach energy $e^* = \max(-\sqrt{\ln 2}; -\frac{1}{2T})$.
But the dynamics is very slow as it needs to go up some barrier to get to the rare state of the given energy (single spin flip MC).

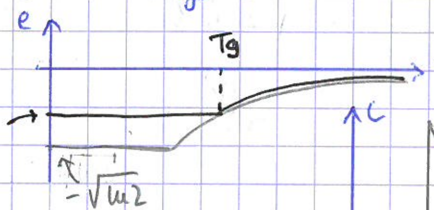


sample exponential number of states.

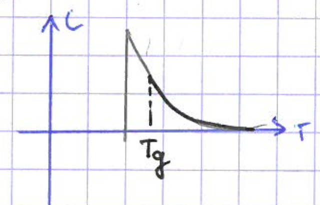
after transition sample finite number of states, $S(e) = 0$

ENTROPY VANISHING TRANSITION.

So during the simulation, doing an annealing, at some point the system falls out of equilibrium, as its relaxation time goes to large...
So that the actual observed picture is:



out of equilibrium at low temperature.



Rk: How crazy this model may seem, these properties are also observed experimentally in real world system. Hence the interest in such dum looking models.

REMARKS ON THE REPLICA METHOD: $\frac{\overline{\ln Z}}{N} = \beta f$ is the quantity of interest, difficult to compute.

The trick relies on the observation that moments of the partition function are easier to compute;

and $\overline{\ln Z} \overset{n \to 0}{=} \frac{1}{n} \ln(\overline{Z^n}) \overset{\text{because}}{=} \frac{1}{n} \ln\left(\overline{e^{n \ln Z}} \right) = \frac{1}{n} \ln\left(\overline{1 + n \ln Z + O(n^2)} \right)$

$= \frac{1}{\cancel{n}} \left(\cancel{n} \overline{\ln Z} + \cancel{O(n^2)} \right)$

2 ROUGH ENERGY LANDSCAPE II SPHERICAL P-SPIN

Model: $H = \sum_{i_1 \ldots i_p} J_{i_1 \ldots i_p} S_{i_1} \ldots S_{i_p}$ $\quad \begin{cases} \sum_{i=1}^{N} S_i^2 = N & \text{normalized} \\ J_{i_1 \ldots i_p} \sim \mathcal{N}\left(0, \frac{1}{2 N^{p-1}}\right) \end{cases}$

$i = 1 \ldots N$

p fixed ≥ 3

All fully-connected mean field models need proper scaling of J.

Rk: the fact that we left the self int. is not changing results while simplifying computations.

Rk: The property of the landscape of this apriori completely different model, are the same as the one from the structural glass transition.
The p-spin is some how the simplest instance of a class of models sharing properties.

21 PROPERTY OF THE ENERGY LANDSCAPE

We are looking at the "stationary" configurations, critical points: $\vec{\nabla}_s H = 0$

density of minima + saddles at energy E : $\mathcal{N}(E)\,\cancel{dE} = \int_{\text{sphere of radius } 1} d\vec{s}\ \underbrace{\delta[\vec{\nabla} H_\perp]\det \nabla^2 H(\vec{s})}_{\#\text{ properly the number of critical point}}\ \delta(H(\vec{s})-E)\,\cancel{dE}$

Rk. The normalization constraint forces up to consider gradient only along the sphere!

in 1d count zeros of $f(x)$: $\int dx\ \delta(f(x))\,|f'(x)|$

change of variable: $\begin{cases} \delta(f(x)) \underset{x\sim x_0}{\simeq} \delta(f'(x_0)(x-x_0)) \\ df(x) \quad f'(x_0)\,dx \end{cases}$

3 steps: ① rotation invariance:

$P(\vec{s}) = \langle \delta[\vec{\nabla} H_\perp]\,\delta(H(s)-E)\det\nabla^2 H(\vec{s})\rangle$ = probability that $\vec{s}$ is a critical point with $H(\vec{s}) = E$.

rotational invariance $\Rightarrow P(\vec{s}) = P(\vec{1})$ $\quad \vec{1} = (1, 1, \dots 1)$

$\Rightarrow \int d\vec{s}\,P(\vec{s}) = \left(\int_{\text{sphere}} d\vec{s}\right) P(\vec{1}) = S_N(\sqrt{N})\,P(\vec{1})$

why does it hold?

Let's consider the orthogonal matrix of rotation bringing $\vec{\ }$ to north pole:

$O\vec{s} = \vec{1} \quad = \sum O_{\cdot j} s_j = 1$ and $\vec{s} = O\vec{1}$ the -

* so we can rewrite: $H(\vec{s}) = -\sum_{\substack{i_1 \dots i_p \\ \bar{i}_1 \dots \bar{i}_p}} \underbrace{J_{i_1 \dots i_p} O_{i_1} \cdots O_{i_p \bar{i}_p}}_{J'} 1 \dots 1$ → new coupling interpretation

$= -\sum_{\bar{i}_1 \dots \bar{i}_p} J'_{\bar{i}_1 \dots \bar{i}_p}\, 1111$

↳ now what is left to show is that J' has the same statistical properties of J, so that $P(\vec{s}) = P(\vec{1})$.

* $J' = J\underbrace{O \dots O}_{p\text{ times}}$ and $P(J) \propto e^{-\sum_{i_1 \dots i_p} J_{i_1 \dots i_p}^2 / 2\left(\frac{1}{2N^{p-1}}\right)}$

$P(J') \rightarrow \begin{cases} \text{Jacobian } dJ' = dJ \\ JJ = \quad J'O^T \dots O^T O \dots O J'^T = J'J' \end{cases}$

actually tensor, but works all fine if we were to put all of the indices in -

↳ which proves the rotational invariance

② $\langle \delta[\vec{\nabla} H_\perp(\vec{1})]\det(\nabla^2 H(\vec{1}))\,\delta(H(\vec{1})-E)\rangle$

$\perp$ to $\vec{s}$

$\left.\frac{\partial H}{\partial s_{i_1}}\right|_{\vec{s}=1} = -\sum_{i_2 \dots i_p} (J_{i\, i_2 \dots i_p}\ 11\dots1 + J_{i_2\, i \dots i_p} 11\dots + \dots \quad J_{i_2 \dots i_p, i} 11\dots)$

$= -\sum_{i_2 \dots i_p} J_{i\, i_2 \dots i_p} + J_{i_2\, i \dots i_p} + \dots + J_{i_2 \dots i}$ = sum of gaussian v.

↳ gaus v.

↳ enough to compute mean + variance.

⚠ change of naming for factorization

scalar product

$f_\alpha = \sum_i v_i^\alpha \frac{\partial H}{\partial s_i} = -\sum_{i, i_2 \dots i_p} (J_{i\, i_2 \dots i_p} v_i^\alpha + J_{i\, i_2 \dots i_p} v_{i_2}^\alpha + \dots + J_{i\, i_2 \dots i_p} v_{i_p}^\alpha)$

$= -\sum_{i_1 \dots i_p} J_{i_1 \dots i_p} (v_{i_1}^\alpha + \dots v_{i_p}^\alpha)$

non zero terms for $(i_1, .. i_p) = (e_1, .. e_p)$

$$\langle f_\alpha f_\beta \rangle = \sum_{i_1 .. i_p} \langle J_{i_1..i_p}^2 \rangle \, (v_{i_1}^\alpha + .. \,)^2$$

double products

$$= \sum_{i_1 .. i_p} \frac{1}{2N^{p-1}} \left(v_{i_1}^\alpha v_{i_1}^\beta + .. + v_{i_p}^\alpha v_{i_p}^\beta + v_{i_1}^\alpha v_{i_2}^\beta + \ldots \right)$$

$i_1 \neq i_p$ though...

$$= \frac{1}{2N^{p-1}} \, p \underbrace{\left(\sum_i v_i^\alpha v_i^\beta\right)}_{\delta\alpha\beta} \underbrace{\sum_{i_2} \ldots \sum_{i_p} \ldots}_{N^{p-1}} + \frac{1}{2N^{p-1}} \left[\underbrace{\sum_{i_1} v_{i_1}^\alpha \sum_{i_2} v_{i_2}^\beta}_{=0} \right] \sum_{i_3} .. \sum_{i_p} \ldots + \ldots$$

orthogonal

$$= \delta\alpha\beta \, \frac{p}{2}$$

* $\langle H^2 \rangle = \frac{N}{2}$ $\quad$ H uncorrelated from f_α $\langle f_\alpha H \rangle = 0$, gaussian variable.
* $\langle (\nabla^2 H)_{ij} = G_{ij} - p \frac{H}{N} \delta_{ij}$ $\quad$ G_{ij} uncorrelated, gaussian random entries GOE.

would require same computat°

$\langle G_{ij}^2 \rangle = \frac{p(p-1)}{2N}$, $\langle G_{ii}^2 \rangle = \frac{p(p-1)}{N}$

$$\langle \delta[H, (\vec{1})] \, |\det \nabla^2 H| \, \delta(H(\vec{1}) - E) \rangle$$

$$= \int dH \prod_{\alpha=1}^{N-1} df_\alpha \, \frac{e^{-f_\alpha^2/2(p/2)}}{\sqrt{2\pi \, p/2}} \prod_\alpha \delta(f_\alpha) \, \frac{e^{-H^2/2(N/2)}}{\sqrt{2\pi (N/2)}} \, \delta(H-E) \int \prod_{ij} dP(G_{ij}) \left|\det\left(G_{ij} - p\left(\frac{E}{N}\right)\right)\right|$$

$$= \left(\frac{1}{\sqrt{2\pi \, p/2}}\right)^{N-1} \frac{e^{-E^2/N}}{\sqrt{\pi N}} \left\langle \left| \det \left[\hat{G} - pe \mathbb{1} \right] \right| \right\rangle \qquad e = \frac{E}{N}$$

③ Let's consider $\hat{G} - pe\mathbb{1} \rightarrow$ spectrum $\lambda_\alpha - pe$.

$$\langle |\det[\hat{G} - pe\mathbb{1}]| \rangle = \langle \prod_\alpha |\lambda_\alpha - pe| \rangle = \langle e^{\sum_\alpha \ln \lambda_\alpha - pe} \rangle$$

↳ introduce $\rho(\lambda) = \frac{1}{N} \sum_\alpha \delta(\lambda - \lambda_\alpha)$ density spectral

$$= \langle e^{N \int d\lambda \ln(\lambda - pe) \rho(\lambda)} \rangle$$

$$\simeq e^{N \int \lambda \ln(\lambda - pe) \langle \rho(\lambda) \rangle}$$

→ concentration property of the spectral density of random matrix.

And this because one can show (challenge)

$P[\rho(\lambda)] \propto \exp[N^2 F(\rho(\lambda))] \Rightarrow$ large deviation function, with N^2!

x, $P(x) = e^{N^2 f(x)}$

$\langle x \rangle = \int dx \, e^{N^2 f(x)} x \simeq x_{max}$ of $f(x)$

$\langle e^{Nh(x)} \rangle = \int dx \, e^{N^2 f(x) + Nh(x)} = e^{Nh(x_{max})} = e^{Nh(\langle x \rangle)}$

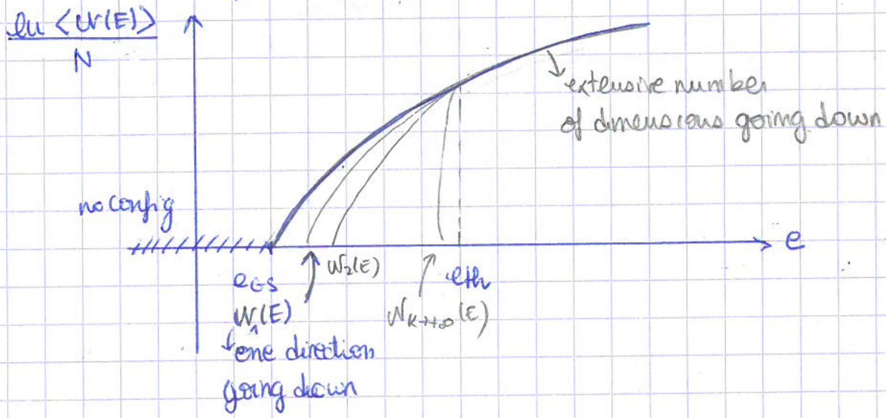
①+②+③: $\langle \mathcal{N}(E) \rangle = \overset{S_N(\sqrt{N})}{\exp}\left[\frac{N}{2} \ln \pi - \frac{N}{2} \ln \frac{1}{2} + \frac{N}{2} \right] - \frac{N}{2} \ln \pi p - Ne^2$

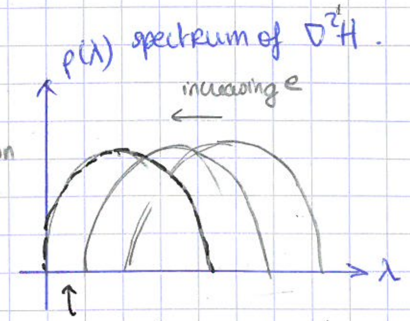
$$+ N \int d\lambda \, \langle \rho(\lambda) \rangle \, |\lambda - pe|$$

$\langle \rho(\lambda) \rangle$ coming from Wigner semi circle

$\langle \rho(\lambda) \rangle = \mathcal{N} \sqrt{\frac{p(p-1)}{2} - \lambda^2}$

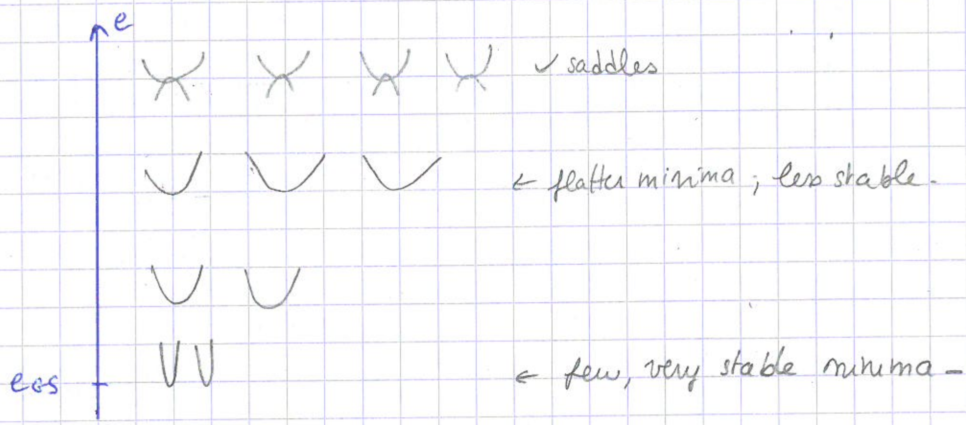
Now what is the physics coming from all of this?





e_{th}, starts to have extensive flat/going down directions $\equiv$ minima disappear.

pictorial view



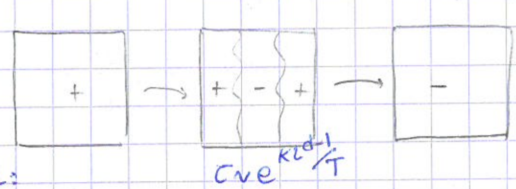
Physics: p-spin spherical model $\equiv$ interacting particles on random graphs
C.S: K-SAT $K \geq 4$.

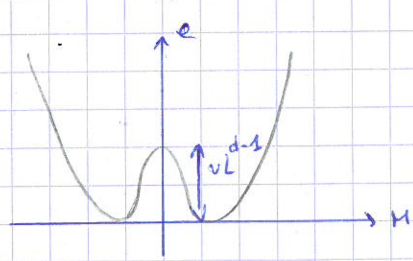
Rk: This computation technic have been used for spin glasses, as well for systems with random forces not deriving from potential (ecology, machine learning...)
ref Toubout?

05/07/2017

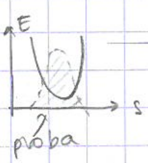
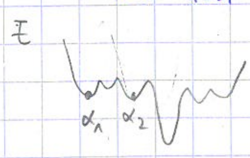
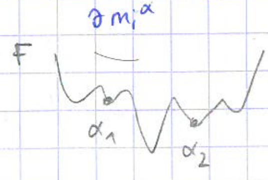
22 FROM ENERGY LANDSCAPE TO FREE ENERGY LANDSCAPE. THERMODYNAMICS.

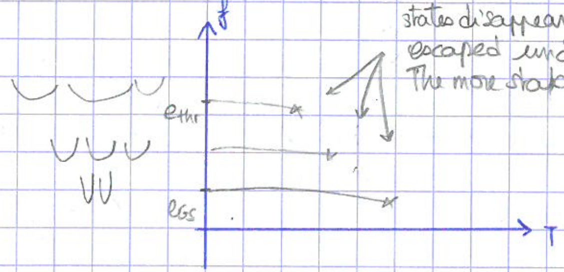
→ Ferromagnetic Ising model in finite dimension:





→ More generally:

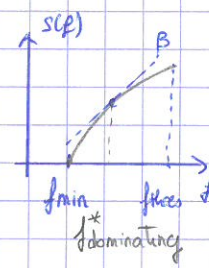
	$T=0$	$T\neq 0$ → fluctuations
state α	$\{s_i^\alpha\}$  $\left.\frac{\partial E(s_i)}{\partial s_i}\right\vert_{s_i=s_i^\alpha}$ 	$\langle s_i\rangle_\alpha = m_i^\alpha = s_i^\alpha\sqrt{q}$ ← overlap to be defined later. fluctuations within one state; zero temperature. $\frac{\partial F(m_i^\alpha)}{\partial m_i^\alpha}=0$ → TAP free energy. 



states disappear (spinodal) when they are easily escaped under the thermal fluctuations. The more stable / deeper, last for the longest.

probability concentrates around states for $N \to \infty$

so what is the thermodynamics looking like: $Z = \sum_{C} e^{-\beta H(C)} = \sum_{\alpha} e^{-\beta f_\alpha N}$

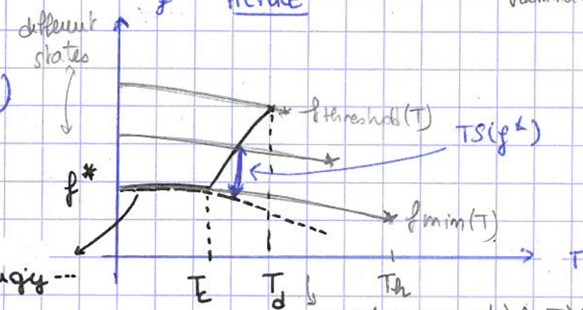


the configurational entropy is $S(f) = \frac{\ln \mathcal{N}(f)}{N}$

Which is used to rewrite the partition function $Z = \int_f df \, e^{N(S(f) - \beta f)}$

PICTURE

$\Rightarrow$ saddle point $\begin{cases} \frac{\ln Z}{N} = S(f^*) - \beta f^* \\ f^* = \underset{f \in [f_{min}, f_{thres}]}{\quad} \left(S(f) - \beta f \right) \end{cases}$

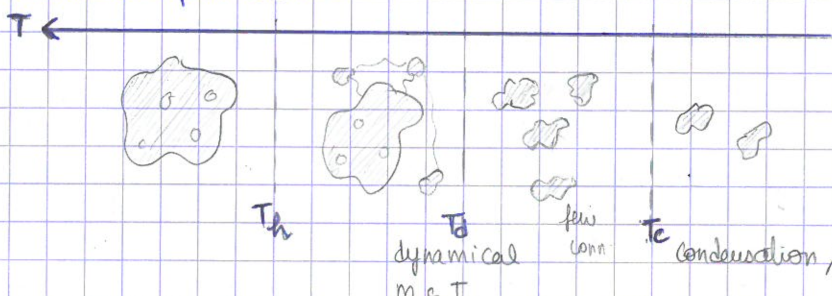


$\begin{cases} \cdot\ T_d = \text{partition} = \text{many different states} \\ \cdot\ T > T_d = \text{paramagnet contribution overwhelming.} \end{cases}$

$m_i = 0$ (= paramagnet) for $T > T_d$.

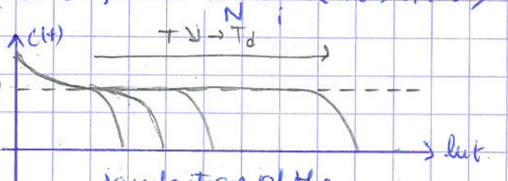
region where the interplay f $S(f)$ is effective.

which could be represented:



23 DYNAMICS

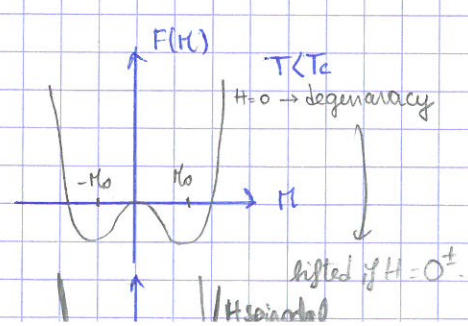
→ Whichever the precise dynamics one chooses (Metropolis, Langevin...)
Consider $C(t) = -\frac{1}{N} \sum_i \langle S_i(t) S_i(0) \rangle$



rarefaction of the paths between forming clusters making relaxation harder.

→ $\tau_{relax} \sim \frac{1}{(T - T_d)^{\delta}}$ cf D. Reichman.

24 ORDER PARAMETER, ORDERING FIELD, FRANZ-PARISI POTENTIAL



To motivate the discussion look at the Ising model first $\begin{cases} \text{order parameter: } M \\ \text{ordering field: } H \end{cases}$

let's sample an equilibrium configuration at T, $\mathbb{C}_{eq}$

define $q(\mathbb{C}, C_{eq}; x)$ → close to 1 if $\mathbb{C} \sim \mathbb{C}_{eq}$ at x ← position in space
→ close to 0 if $\mathbb{C} \neq \mathbb{C}_{eq}$ at x. ≡ ORDER PARAMETER.

transform the hamiltonian to be biased toward $\mathbb{C}_{eq}$: $H(\mathbb{C}) \leftarrow H(\mathbb{C}) - \sum_x \underbrace{\varepsilon_x}_{\equiv \text{ ORDERING FIELD}} q(\mathbb{C}, \mathbb{C}_{eq}, x)$

→ pushes down the energy of the state corresponding to $\mathbb{C}_{eq}$

Landau free energy has an equivalent in this context called the Franz-Parisi potential.

FIM → $F(M) = -\frac{1}{\beta N}\log\left[\sum_{\mathbb{C}} \frac{e^{-\beta H(\mathbb{C})}}{Z}\,\delta(M(\mathbb{C})-M)\right]$ (Landau free energy)

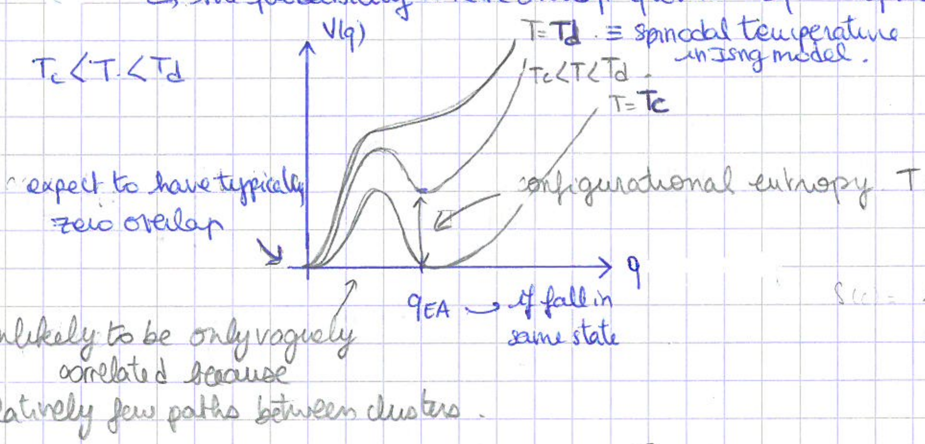
here, more generally → $V(q) = -\frac{1}{\beta N}\log\left[\sum_{\mathbb{C}} \frac{e^{-\beta H(\mathbb{C})}}{Z}\,\delta(q(\mathbb{C},\mathbb{C}_{eq})-q)\right]$ (Franz parisi)

$\sum_x \frac{q(\mathbb{C},\mathbb{C}_{eq},x)}{N}$

= how much it costs to have a given value of the O.P.

↳ COMPUTATION NEXT WEEK FLO.

let's just guess the result for now:
↳ the probability to have overlap q with C_{eq} $P(q) \asymp e^{-\beta N V(q)}$ → interpreted as a large deviation function



because there is only one such state, a

$S(c) = \ln \frac{W(q)}{N}$

Introducing the replica trick: $\overline{\ln[\ ]} = \lim_{n\to 0}\left(\frac{1}{n}\ln\left(\overline{[\ ]^n}\right)\right)$ ← $\sum_{C_{eq}}$

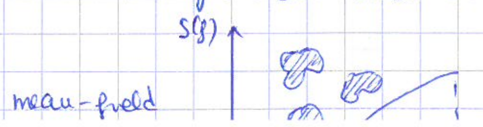
to compute $V(q) = \lim_{n\to 0} \ln\left(\sum_{\substack{C_{eq}\\ \mathbb{C}_1\\ \mathbb{C}_2}} \frac{e^{-\beta H(\mathbb{C}_{eq})}}{Z}\frac{e^{-\beta H(\mathbb{C}_1)}}{Z}\cdots\frac{e^{-\beta H(\mathbb{C}_{n-1})}}{Z}\prod_{a=1}^{n}\delta(q(\mathbb{C}^a,\mathbb{C}_{eq})-q)\right)$

06/05/2017

3 FROM MEANFIELD THEORY TOWARDS FINITE D.

→ standard 2^{nd} order phase transitions; uppercritical dimension: MF is correct
below upper crit. dim: exponents might change but qualitatively relevant

Unlike most standard 2^{nd} order phase transition, for the glass transition going from MF to finite dimension is not at all obvious

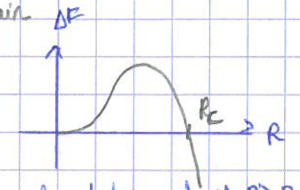


This picture of exponentially many different states is impossible in 3d.

nucleation argument

being at $T_K < T < T_d$: imagine having two states α, β at $f_\beta < f_\alpha$

can the state α survive at infinite time : droplet α, radius r: $\Rightarrow \Delta F = \underbrace{(f_\beta - f_\alpha) R^d}_{\text{bulk gain}} + \underbrace{Y r^{d-1}}_{\text{surface tension}}$



$\Rightarrow$ nucleation, droplet invade if $R > R_c$ on a finite time scale.

$\rightarrow$ so the system is a liquid, not many different stable states.

QUESTIONS & ANSWERS :

Within the mean field theory, the different states are all stable, as the energy barriers diverge in the thermodynamic limit.

↳ in finite dimension, they will indeed become instable.

For a ferromagnet,

To study the finite dimension system, we consider magnetization profile $M(x)$.

$$Z = \int \mathcal{D}M(x) \sum_{\mathcal{C}} e^{-\beta H(\mathcal{C})} \delta[M(\mathcal{C}) - M(x)]$$

$$= \int \mathcal{D}M(x)\, e^{-F(M(x))}$$

idea:

↳ integrate all variables out but the interesting fluctuations of the order parameter $M(x)$.

↳ $F(x) = -\frac{1}{\beta} \ln\left(\sum_{\mathcal{C}} e^{-\beta H(\mathcal{C})} \delta[M(\mathcal{C}) - M(x)]\right)$.

↳ if you are Landau you write directly the field theory : $F(M(x)) = \int d^d x\, \frac{c}{2} (\nabla M)^2 + V(M)$

↳ otherwise :
1 - compute exactly
2 - coarse graining (R) $x, M(x)$
3 - Kac Model

Now for glasses, the order parameter is $q(\mathcal{C}, \mathcal{C}_{eq}; x)$:

~~does~~ is there no way to have patterns?

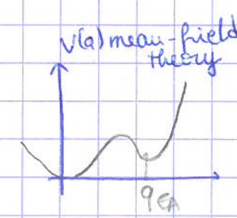
$\rightarrow$ what makes it a glass model $\rightarrow m(x) = 0$

$$Z = \int \mathcal{D}q(x) \sum_{\mathcal{C}} e^{-\beta H(\mathcal{C})} \delta[q(\mathcal{C}, \mathcal{C}_{eq}; x) - q(x)]$$

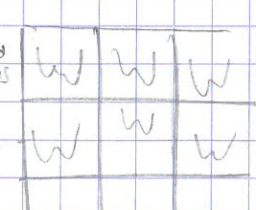
$$= \int \mathcal{D}q(x)\, e^{-\beta F(q(x))}$$

with $F(q(x)) = -\frac{1}{\beta} \ln \sum_{\mathcal{C}} e^{-\beta H(\mathcal{C})} \delta\left(q(\mathcal{C}, \mathcal{C}_{eq}; x) - q(x)\right)$.

And now ? there is no Landau...



local fluctuations



$\rightarrow$ if writing the naïf : $F(q(x)) = \int d^d x\, \frac{c}{2} (\nabla q)^2 + V(q)$ we are ignoring the fluctuation of the actual $V(q)$ in the different sub systems. (depending on the difference of the $\mathcal{C}_{eq}$ in the different sub systems).

If we at the fluctuations ~~to~~ the mean field theory, is there still a glass transition ?

3.1 without ~~the fluctions~~ fluctuations

In the ferromagnet we look for long range order.

For the glass, we'll fix a high overlap on the boundary of a domain. Does it force the overlap to be close to one inside the domain, when the domain size goes to infinity ?

For $T \to T_K$, $V(q_{EA}) = TS \ll 1$.

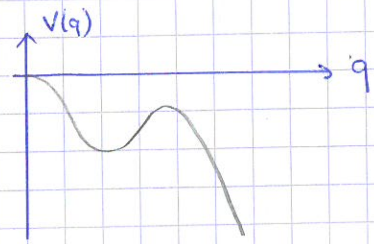
Solve $\dfrac{\delta F(q(x))}{\delta q(x)} = -c\,\Delta q(x) + V'(q(x)) = 0$

at distance r from the center: $c\left(\dfrac{d^2}{dr^2} + \cancel{\dfrac{d-1}{r}\dfrac{d}{dr}}\right) q(r) = V'(q(r))$ with $q(r) = q_{EA}$ $r > R$

for simplicity, and because we will see that the overlap drops close to R.

$\to c\,\dfrac{d^2 q}{dr^2} = V'(q(r))$ $\to$ newton particle: $q \leftrightarrow x$, $r \leftrightarrow t$, $c \leftrightarrow m$, $V \leftrightarrow -V$

$\to$ Newton particle in inverted potential:



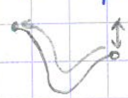
Solutions, equilibrium positions:

① $*$ $q(r) = q_{EA}$ $\forall r$ (mean field solution) $\Rightarrow F(q(x)) = \int d^dx\,(0 + V(q_{EA}))$
(no initial cinetique energy)
$= \frac{4}{3}\pi R^3 V(q_{EA})$

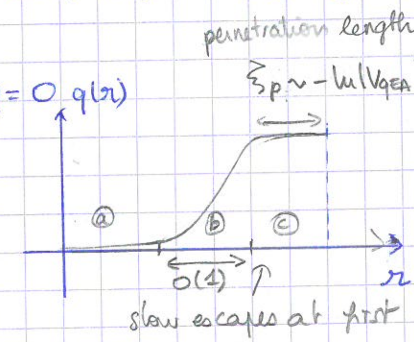
② $*$ start from q_{EA} with small $\vec{v}$ $\Rightarrow$ will come back to q_{EA}

$\Rightarrow F(q(x)) = \int_{\text{along the path}} d^dx\,(\_\_ + V(q_{EA})) > \int d^dx\, V(q_{EA}) = \frac{4}{3}\pi R^3 V(q_{EA})$

③ $*$ start from q_{EA} with $E_c^{init} = V(q_E) \Rightarrow$ will go to $q = 0$



$E^{total} = \frac{c}{2}\left(\frac{dq}{dr}\right)^2 - V(q)$



$F(q(x)) = F(a) + F(b) + F(c) \to \int_{R+\ln V_{qEA}}^{R} 4\pi r^2 dr\, V(q_{EA}) \underset{\text{to leading order}}{\simeq} \underbrace{V(q_{EA}) 4\pi R^2 (-\ln V_{qEA})}_{\underset{T \to T_K}{\to} 0} -$

$F(a) = 0$

$F(b) = Y 4\pi R^2$

$Y = \int_0^{q_{EA}} \sqrt{2c V(q)}\, dq \sim O(1)$

G. Tarjus HTC.

leading order for $T \to T_K$ ($V(q_{EA}) \to 0$) $F(q(x)) = Y 4\pi R^2$

There is a cross over between ① and ③ solution:

$\frac{4}{3}\pi R^3 V(q_{EA}) = Y 4\pi R^2 \to R_c = cste\,\dfrac{Y}{V(q_{EA})} \sim \dfrac{Y}{S_c(T)}$.

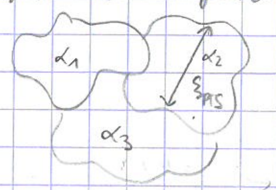
<u>Conclusion:</u>

$R < R_c \to$ ① dominates $\to q(r=0) = q_{EA}$.

The system is amorphously ordered on the scale $R_c = \xi_{PTS} \sim \frac{Y}{TS_c} \xrightarrow{T \to T_K} \infty$

point to set

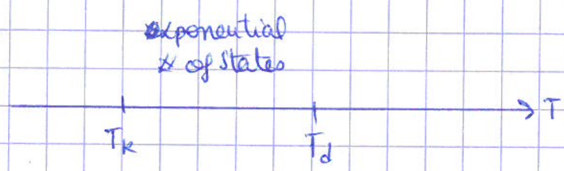
$T > T_K$ order to finite scale

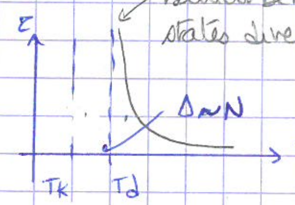


→ "mosaic state" introduced by KIRKPATRICK, THIRUMALAI, WOLYNES
↳ pictorial, a priori non-correct, caricature

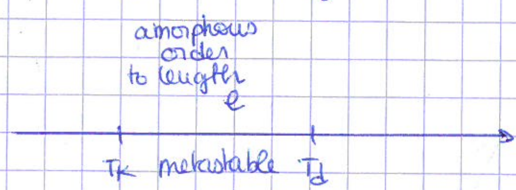
$T = T_K$ → infinitely ordered, true glass transition

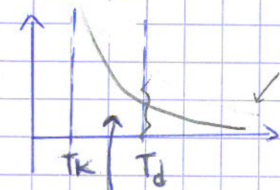
mean field theory





Finite dimension





barrier $\Delta \sim (\xi_{PTS})^{\psi}$ → $\begin{cases} \xi_{PTS} \sim \frac{1}{S_c} \sim \frac{1}{T - T_K} \\ \tau \sim e^{\Delta/T} \sim e^{(1/(T-T_K))^{\psi}} \end{cases}$

3.2 With the fluctuations

$$F(q(x)) = \int d^d x \, \frac{(\nabla q)^2}{2} + V(q) + \text{random term}$$

↳ $\int d^d x \, h(x) q(x)$ $\quad \overline{h(x)} = 0$, $\quad \overline{h(x) h(x')} = \delta(x - x') \Delta$

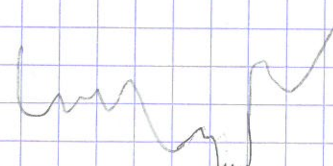
1. permeation length $\xi_p \sim 1/\sqrt{S_c}$
2. Mode coupling theory
3. How big is Δ? → will it destroy the order? (always too strong to have a glass transition in 2d)

⇒ WHAT WE COULD HAVE TALKED ABOUT:

→ There has been method developped to prove that there is a glass transition like for the p-spin for interacting particles systems in $d \to +\infty$.

→ Spin glasses behave differently

→ Gardner transition and its relation to jamming



→ Scaling argument arguments / Simulations

⇒ FINAL WORD:

Mean field theory does provide new ideas to study rough energy landscape new-