# Introduction to Quantum Mechanics of Superconducting Electrical Circuits

- What is superconductivity?
- What is a Josephson junction?
- What is a Cooper Pair Box Qubit?
- Quantum Modes of Superconducting Transmission Lines
  - See R.-S. Huang PhD thesis on Boulder 2004 web page

# What is superconductivity?

See: Boulder Lectures 2000

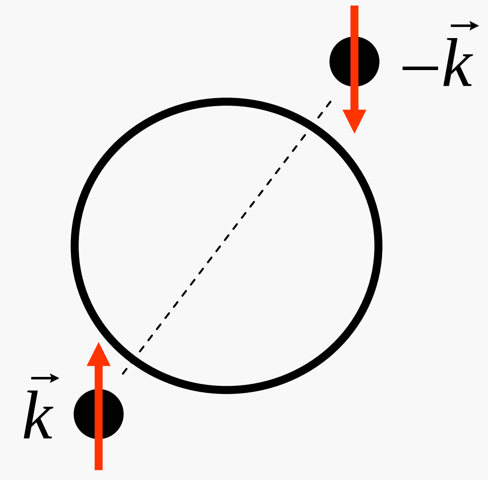
Cooper pairs are like bosons in a BEC -- except --
size of Cooper pair spacing between electrons

$$k_F \xi >> 1$$

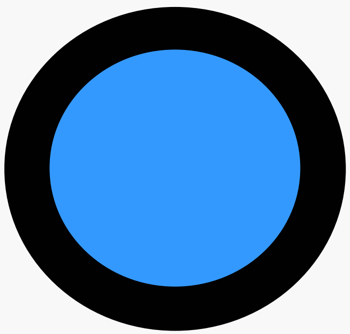
Complex order parameter like BEC

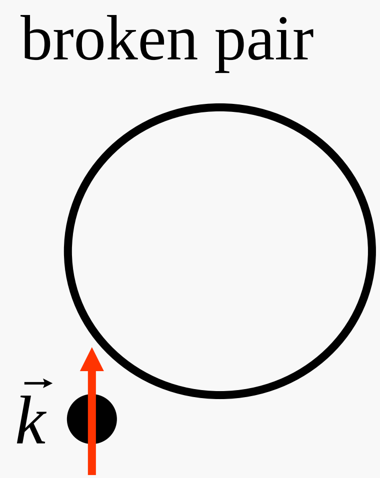
$$\psi(\vec{r}) \sim \left\langle c_{\vec{k}\uparrow} c_{-\vec{k}\downarrow} \right\rangle$$

$$\psi(\vec{r}) \sim |\psi| e^{i\varphi(\vec{r})}$$



# Excitation Gap for Fermions

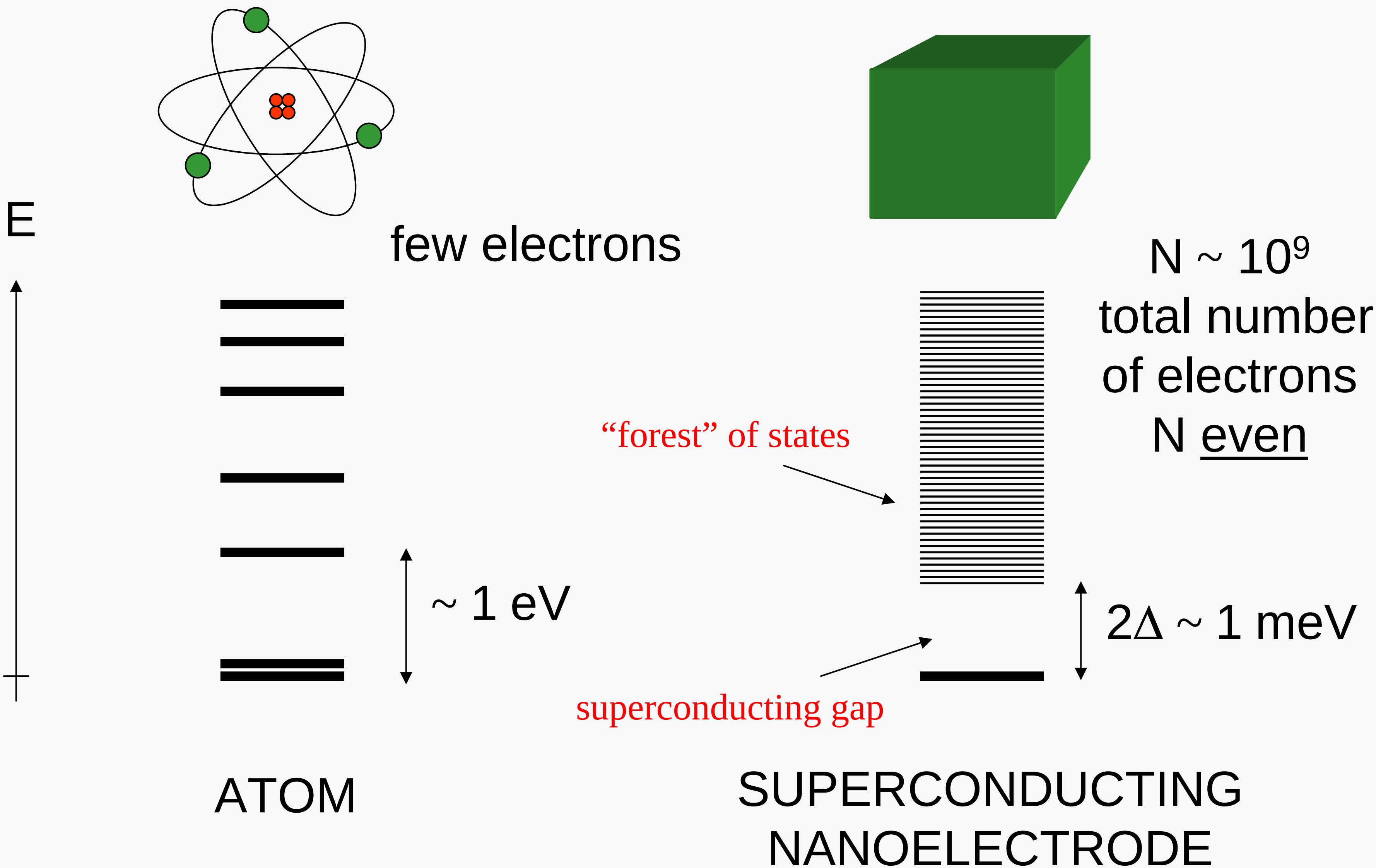




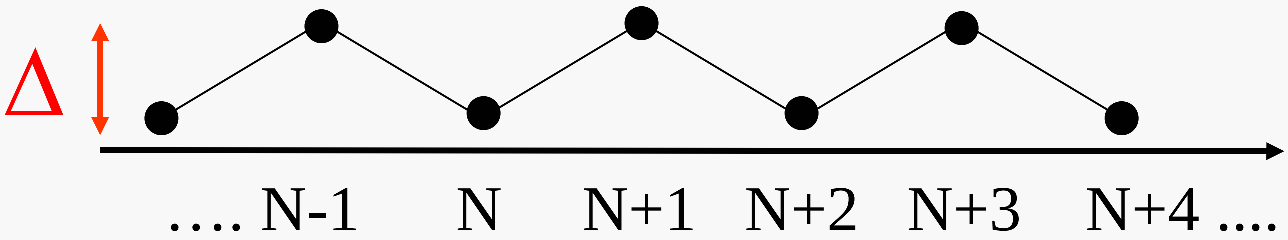

In a semiconductor the gap is tied to the spatial lattice
(only occurs at one special density commensurate with lattice)

In a superconductor, gap is tied to fermi level.  Compressible.

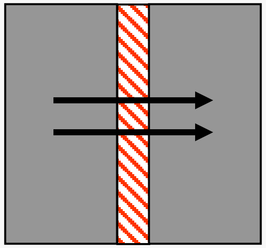
# WHY SUPERCONDUCTIVITY?

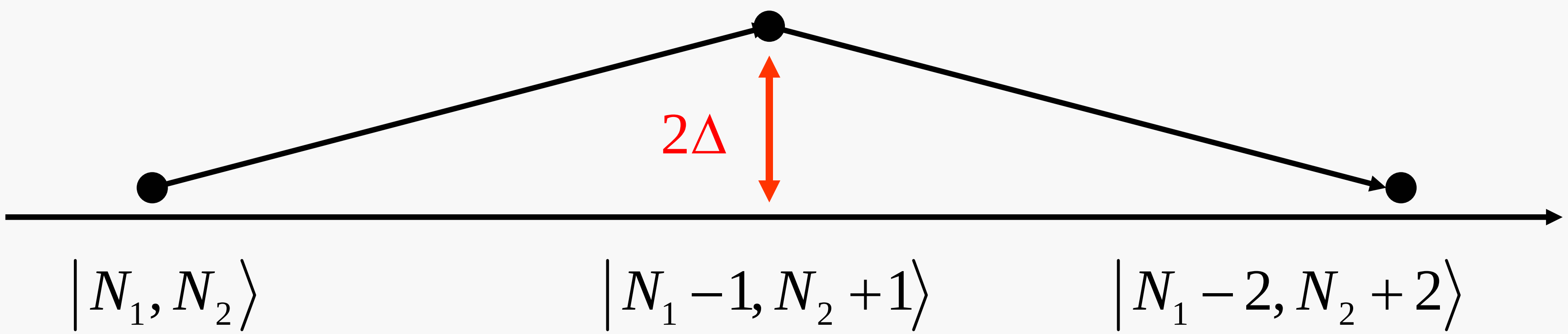


# Energy vs. Particle Number on Grain



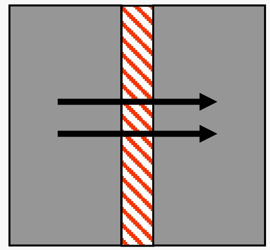

Josephson tunneling between two grains





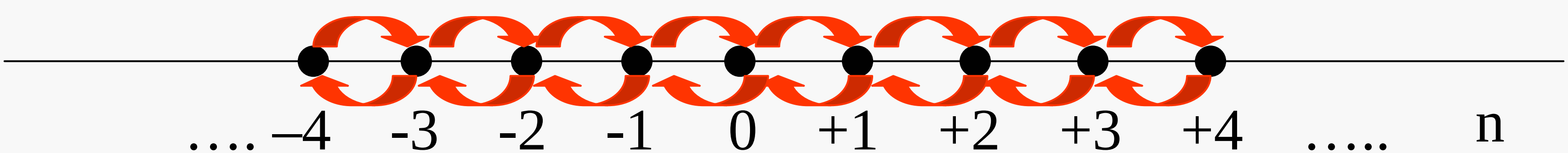
# Josephson Tunneling I

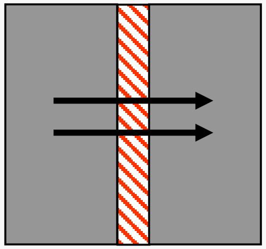
Coherent tunneling of Cooper pairs



Restrict Hilbert space to quantum ground states of the form

$$\left| N_1 - 2n, N_2 + 2n \right\rangle, \qquad \mathrm{n} = \ldots -3, -2, -1, 0, +1, +2, +3 \ldots$$

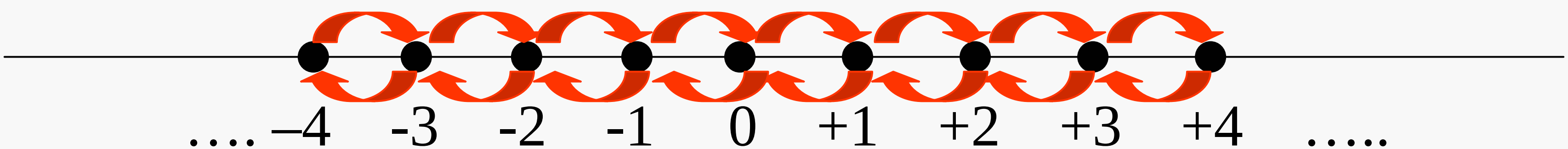




# Josephson Tunneling II

Restrict Hilbert space to quantum ground states of the form

$$\left|N_1 - 2n, N_2 + 2n\right\rangle \Rightarrow \left|n\right\rangle, \qquad \mathrm{n} = \ldots -3, -2, -1, 0, +1, +2, +3\ldots$$




$$T = -\frac{E_J}{2}\sum_{n=-\infty}^{n=+\infty}\Big[\left|n+1\right\rangle\left\langle n\right| + \left|n\right\rangle\left\langle n+1\right|\Big]$$

$$E_J = \frac{G_t \Delta}{8e^2/h}$$

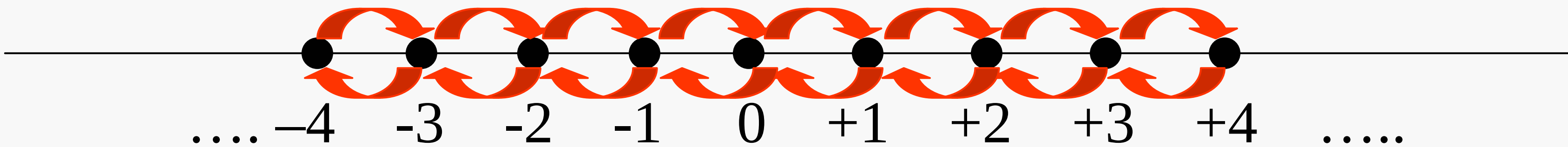
normal state conductance

sc gap

Tight binding model: single ‘particle’ hopping on a 1D lattice

N.B. $E_J \propto \Delta$ not $1/\Delta$

# Josephson Tunneling III




Tight binding model $T = -\frac{E_{\mathrm{J}}}{2} \sum_{n=-\infty}^{n=+\infty} \left[ |n+1\rangle\langle n| + |n\rangle\langle n+1| \right]$

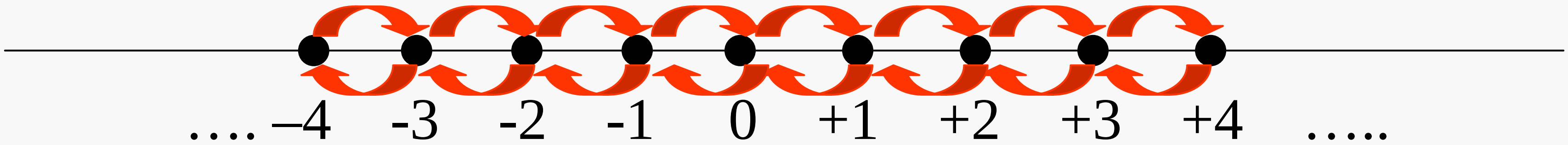
‘position’ $n$

‘wave vector’ $\varphi$ (compact!)

‘plane wave eigenstate’ $|\varphi\rangle = \sum_{n=-\infty}^{n=+\infty} e^{i\varphi n} |n\rangle \quad = \text{"}\, e^{ikx} \,\text{"}$

$$T|\varphi\rangle = -E_{\mathrm{J}} \sum_{n'=-\infty}^{n'=+\infty} \left[ |n'+1\rangle\langle n'| + |n'\rangle\langle n'+1| \right] \sum_{n=-\infty}^{n=+\infty} e^{i\varphi n} |n\rangle$$

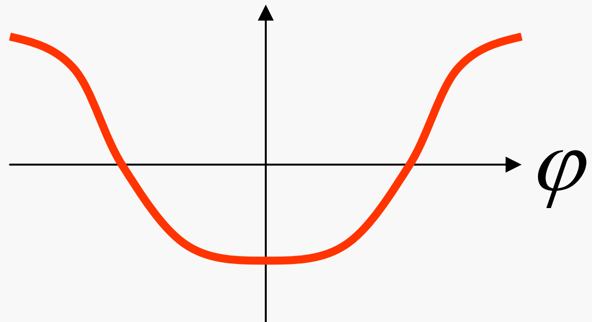
# Josephson Tunneling IV



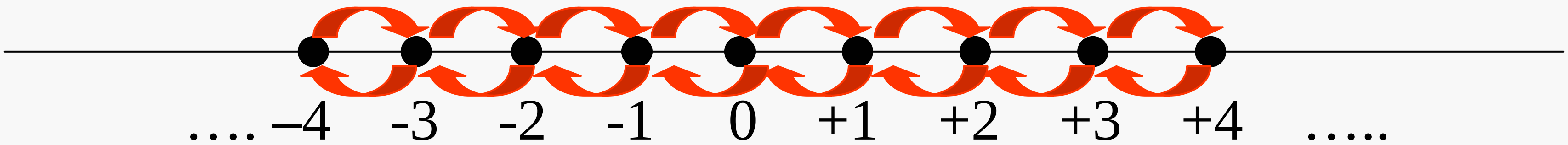

$$T\left|\varphi\right\rangle = -E_{\mathrm{J}} \sum_{n'=-\infty}^{n'=+\infty} \Big[ \left|n'+1\right\rangle\left\langle n'\right| + \left|n'\right\rangle\left\langle n'+1\right| \Big] \sum_{n=-\infty}^{n=+\infty} e^{i\varphi n} \left|n\right\rangle$$

$$T\left|\varphi\right\rangle = -\frac{E_{\mathrm{J}}}{2} \sum_{n=-\infty}^{n=+\infty} e^{i\varphi n} \Big[ \left|n+1\right\rangle + \left|n-1\right\rangle \Big]$$

$$T\left|\varphi\right\rangle = -E_{\mathrm{J}} \cos(\varphi) \left|\varphi\right\rangle$$



# Supercurrent through a JJ




‘position’ $n$

‘momentum’ $\hbar\varphi$

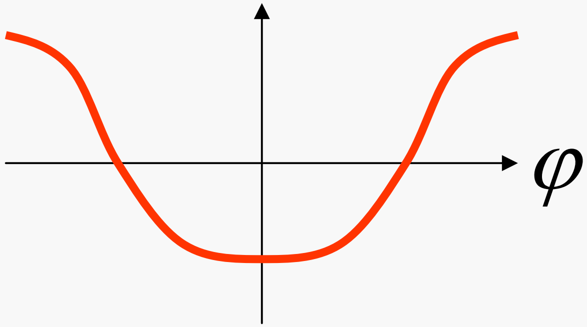
$$T\left|\varphi\right\rangle = -E_{\mathrm{J}}\cos(\varphi)\left|\varphi\right\rangle$$

Wave packet
group velocity

$$\frac{dn}{dt} = \frac{1}{\hbar}\frac{dT}{d\varphi} = \frac{E_J}{\hbar}\sin\varphi$$

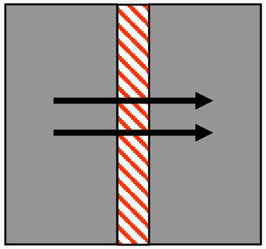
current:

$$I = (2e)\frac{dn}{dt} = \frac{2e}{\hbar}E_J\sin\varphi$$




$$\boxed{I = I_{\mathrm{c}}\sin\varphi}$$

$$I_{\mathrm{c}} \equiv \frac{2e}{\hbar}E_J$$

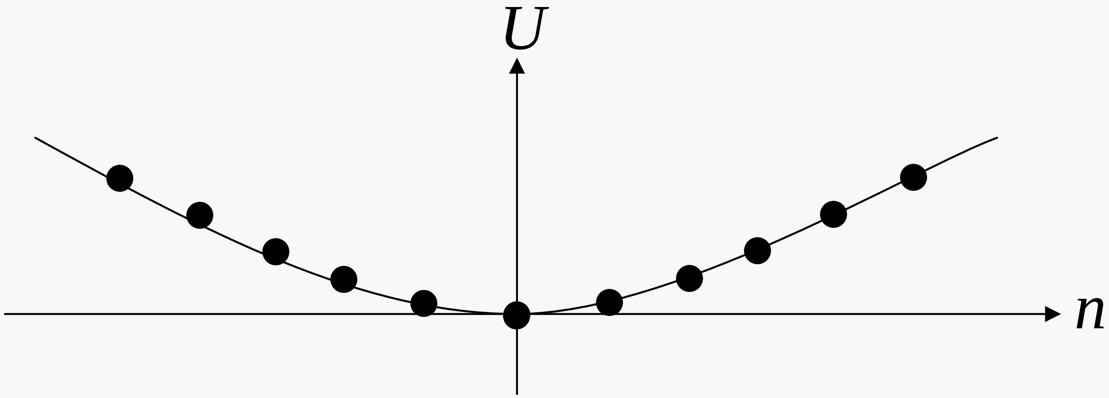


# Charging Energy

$$\left| N_1 - 2n, N_2 + 2n \right\rangle \Rightarrow \left| n \right\rangle, \qquad n = \ldots -3, -2, -1, 0, +1, +2, +3 \ldots$$

$$U = \frac{(2e)^2}{2C} n^2 = 4E_c n^2$$

$$E_c \equiv \frac{e^2}{2C}$$



# ‘Second’ Quantization

$$|\varphi\rangle = \sum_{n=-\infty}^{n=+\infty} e^{i\varphi n} |n\rangle \qquad = " e^{ikx} "$$

Number Operator: $\hat{n} \equiv -i \dfrac{\partial}{\partial \varphi}$

$\varphi$ Now viewed as coordinate

$$H = 4E_{\mathrm{c}}\hat{n}^2 - E_{\mathrm{J}} \cos(\varphi)$$

$$H = -4E_{\mathrm{c}} \frac{\partial^2}{\partial \varphi^2} - E_{\mathrm{J}} \cos(\varphi)$$

$$H\Psi_j(\varphi) = E_j(\varphi)\Psi_j(\varphi)$$

$\Psi_j(\varphi)$ Describes quantum amplitude for (quantum) phase

# Weak Charging Limit: Josephson Plasma Oscillations

$$E_{\mathrm{c}} \ll E_{\mathrm{J}}$$

Simple harmonic oscillator

$$H = 4E_{\mathrm{c}}\hat{n}^2 - E_{\mathrm{J}}\cos(\varphi)$$

$$H = -4E_{\mathrm{c}}\frac{\partial^2}{\partial\varphi^2} - E_{\mathrm{J}}\cos(\varphi)$$

$$H \approx -4E_{\mathrm{c}}\frac{\partial^2}{\partial\varphi^2} + E_{\mathrm{J}}\left[-1 + \frac{\varphi^2}{2}\right]$$

spring constant $K = E_{\mathrm{J}}$

mass $M = \frac{1}{\hbar^2 8E_{\mathrm{c}}}$

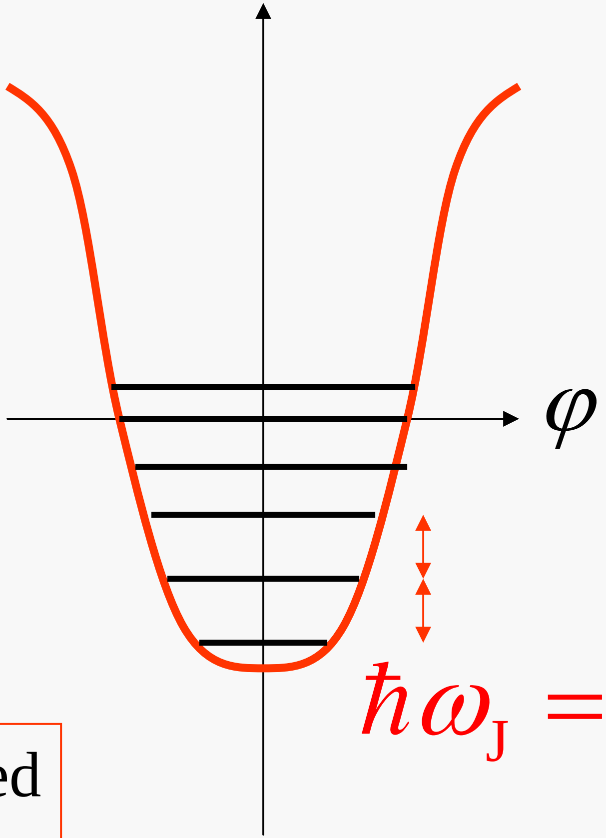
$$\hbar\omega_{\mathrm{J}} = \sqrt{8E_{\mathrm{c}}E_{\mathrm{J}}}$$

# Weak Charging Limit: Josephson Plasma Oscillations

$$E_c \ll E_J$$ (heavy 'mass', small quantum fluctuations)

spring constant $K = E_J$

mass $M = \dfrac{1}{\hbar^2 8E_c}$




$$\hbar\omega_J = \sqrt{8E_c E_J}$$

Weak anharmonicity can be used to form qubit and readout levels

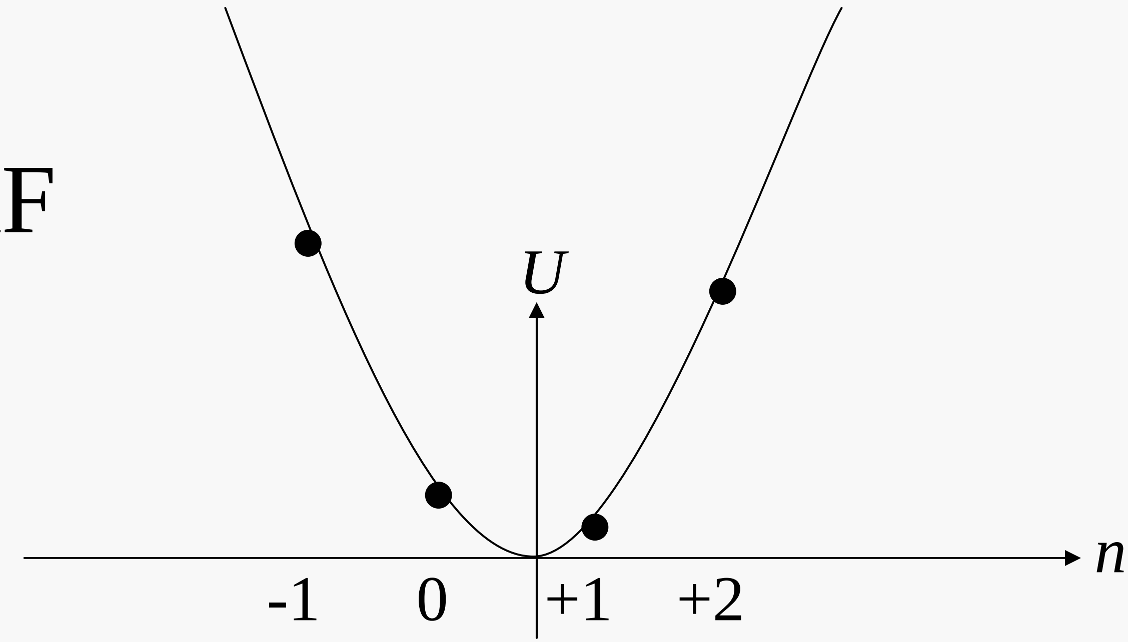
# Strong Charging Limit: Cooper Pair Box

$$4E_c \geq E_J$$

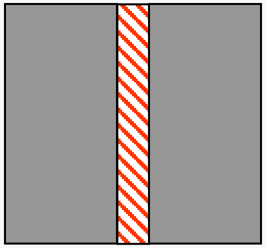
$\varphi$ Fluctuates too wildly;
better to use number (position) representation

$$C \sim 500\text{aF}$$

$$E_C \sim 2\text{K}$$

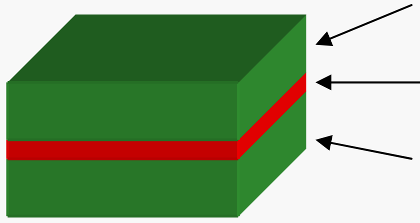



Random offset charge due to built in asymmetry



# THE BARE JOSEPHSON JUNCTION: SIMPLEST SOLID STATE ATOM

Josephson junction with no wires to rest of world:




Hamiltonian:

$$H = 4E_C\left(\hat{n} - Q_r/2e\right)^2 - E_J\cos\varphi + \ldots..$$

negligible terms at small T

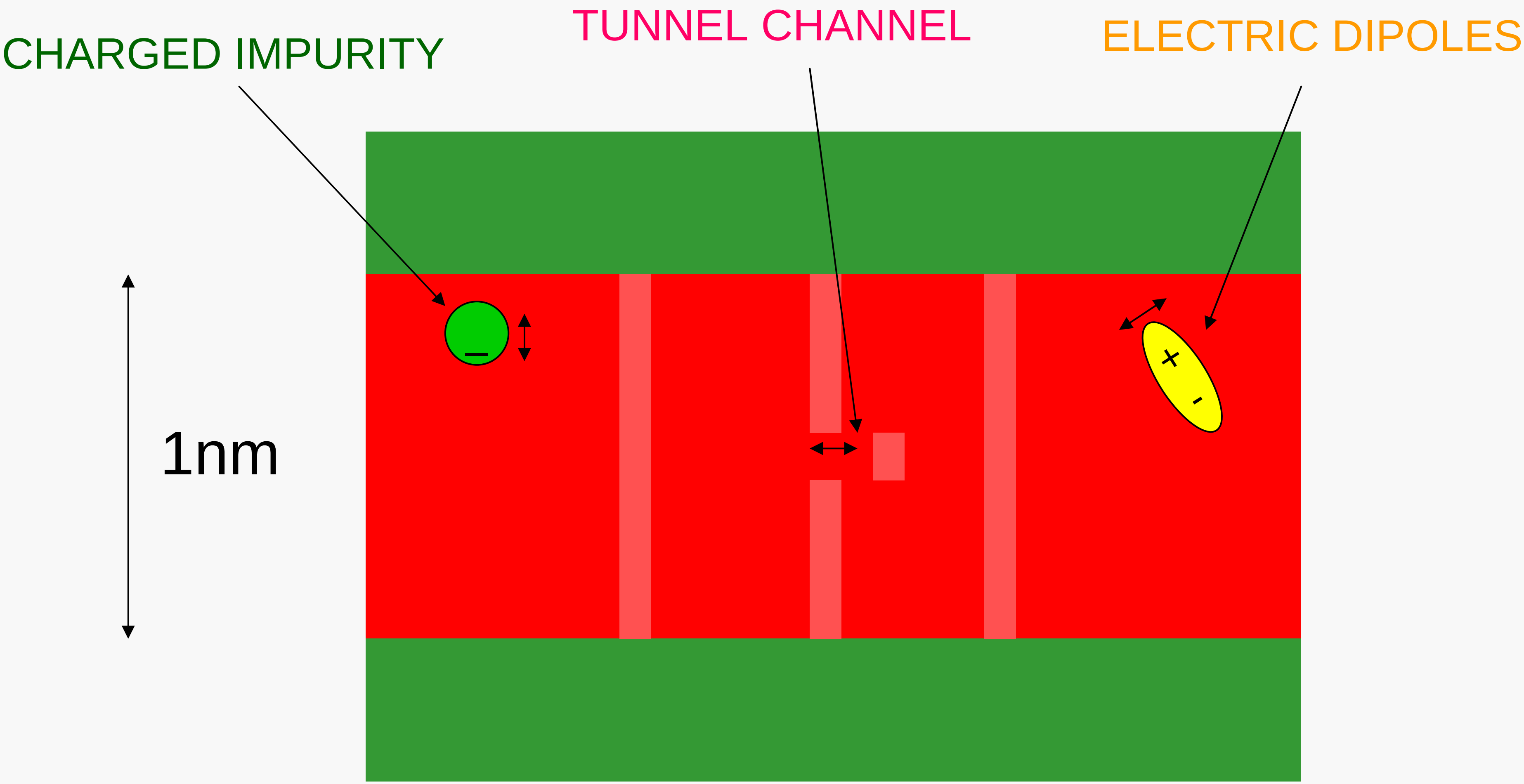
3 parameters:

$Q_r$ residual offset charge

$$E_J = \frac{G_t\Delta}{8e^2/h}$$ Josephson energy

$$E_C = \frac{e^2}{2C_J}$$ Coulomb energy

# IMPERFECTIONS OF JUNCTION PARAMETERS




CAN IN PRINCIPLE MAKE JUNCTION PERFECT, BUT MEANWHILE.....

# JUNCTION PARAMETER FLUCTUATIONS

$$Q_r = Q_r^{stat} + \Delta Q_r(t)$$

$$E_J = E_J^{stat} + \Delta E_J(t)$$

$$E_C = E_C^{stat} + \Delta E_C(t)$$

| param. | dispers. |
|---|---|
| $Q_r^{stat}$ | **random!** |
| $E_J^{stat}$ | 10% |
| $E_C^{stat}$ | 10% |

| noise | sd@1Hz |
|---|---|
| $\Delta Q_r/2e$ | $\sim 10^{-3}\text{Hz}^{-1/2}$ |
| $\Delta E_J/E_J$ | $10^{-5}$-$10^{-6}\text{Hz}^{-1/2}$ |
| $\Delta E_C/E_C$ | $<10^{-6}\text{Hz}^{-1/2}$? |

**$0^{th}$ order problem: get rid of randomness of static offset charge!**

2 solutions: - control offset charge with a gate ➡ Cooper pair box (charge qubit)

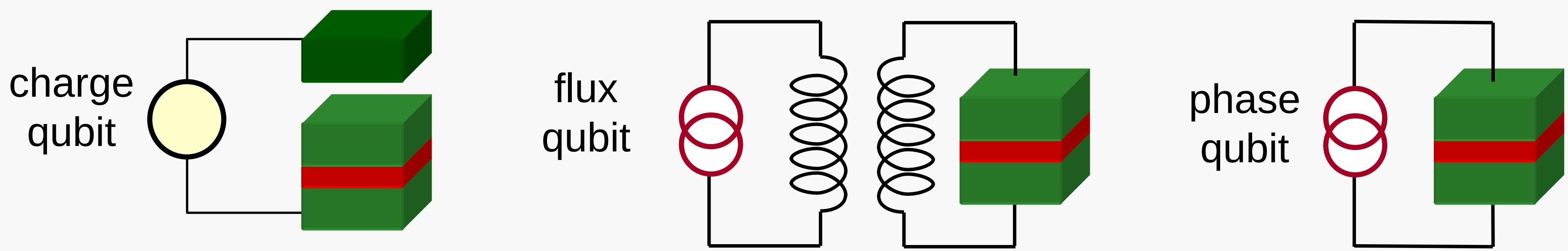
or

- make $E_J/E_C$ very large ➡ RF-SQUID (flux qubit)

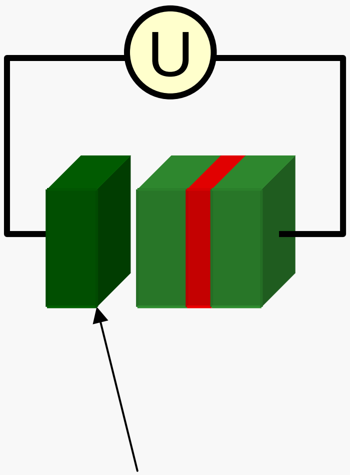
Cur. Biased J. (phase qubit)

# SENSITIVITY TO NOISE OF VARIOUS QUBITS

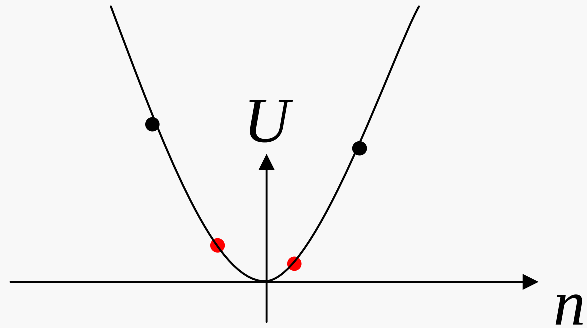
| | junction noise | | | port noise | | |
|---|---|---|---|---|---|---|
| qubits | $\Delta Q_r$ | $\Delta E_J$ | $\Delta E_C$ | technical | quantum | back-action |
| <u>charge</u> | ⚡ | ⚡ | | ⚡ | ⚡ | ⚡ |
| flux | | ⚡⚡⚡ | ⚡⚡⚡ | ⚡⚡⚡ | ⚡⚡⚡ | ⚡⚡⚡ |
| phase | | ⚡⚡ | ⚡ | ⚡⚡ | ⚡⚡ | ⚡ |



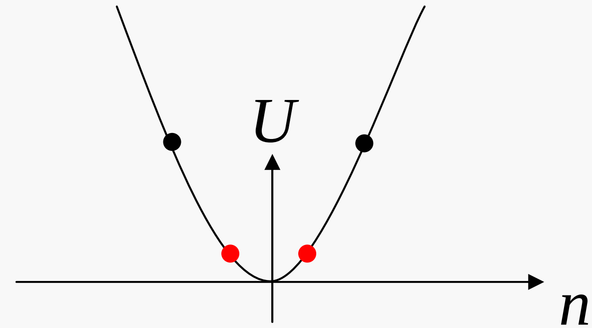
# COOPER BOX STRATEGY




compensate
static
offset charge
with gate




charge deg. point

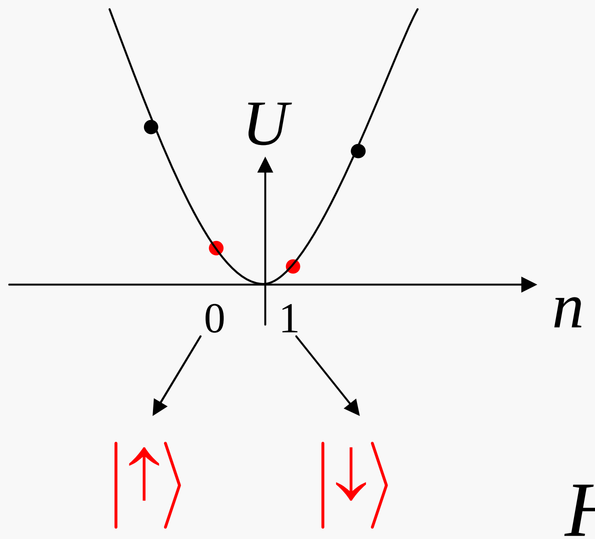



$$H = 4E_C\left(\hat{n} - N_g\right)^2 - E_J \cos\varphi + \ldots..$$

gate charge can be controlled
$N_g=C_gU/2e$

$$T = -E_J \cos\varphi = -\frac{E_J}{2}\sum_{n=-\infty}^{n=+\infty}\left[\left|n+1\right\rangle\left\langle n\right| + \left|n\right\rangle\left\langle n+1\right|\right]$$

# Mapping to two level system

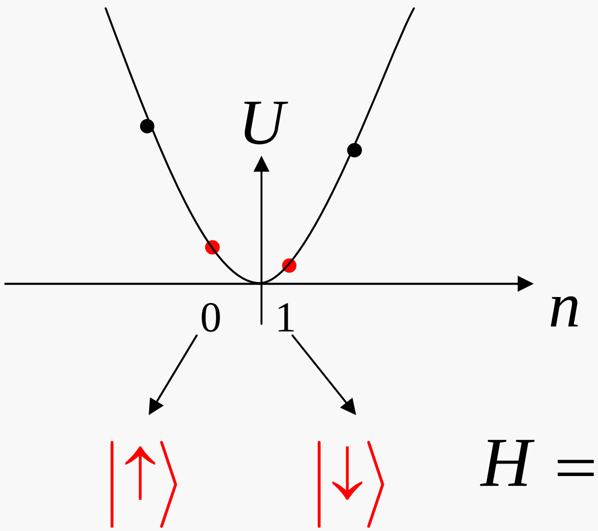



$$H = 4E_C\left(\hat{n} - N_g\right)^2 - E_J\cos\varphi + \ldots..$$

$$\hat{n} = \frac{1-\sigma^z}{2}$$

$$T = -E_J\cos\varphi = -\frac{E_J}{2}\sum_{n=-\infty}^{n=+\infty}\left[|n+1\rangle\langle n| + |n\rangle\langle n+1|\right] \Rightarrow -\frac{E_J}{2}\sigma^x$$

# Mapping to two level system



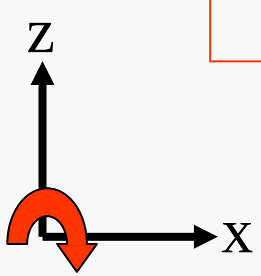

$$\hat{n} = \frac{1-\sigma^z}{2} \qquad [\hat{n}^2 = \hat{n}]$$

$$H = -2E_C\left(1-2N_g\right)\sigma^z - \frac{E_J}{2}\sigma^x + \text{constant.....}$$

Rotate:

$$x \to -z$$

$$z \to +x$$



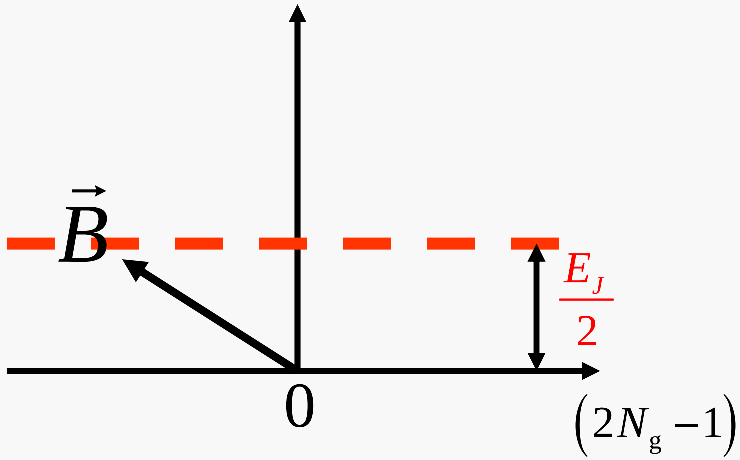

$$H = -2E_C\left(1-2N_g\right)\sigma^x + \frac{E_J}{2}\sigma^z + \text{constant.....}$$

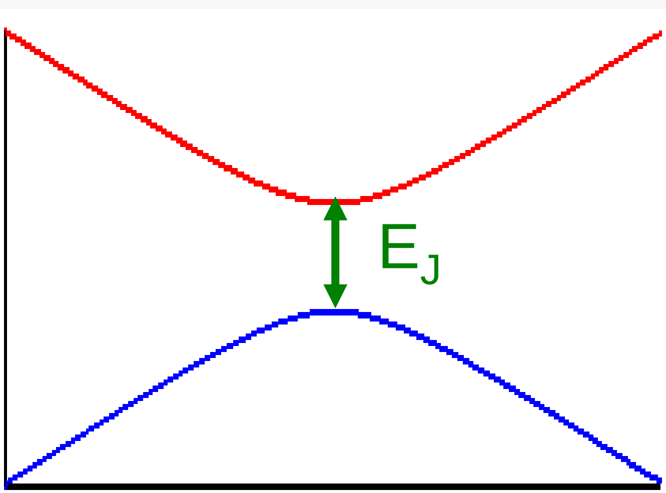
# Mapping to two level system

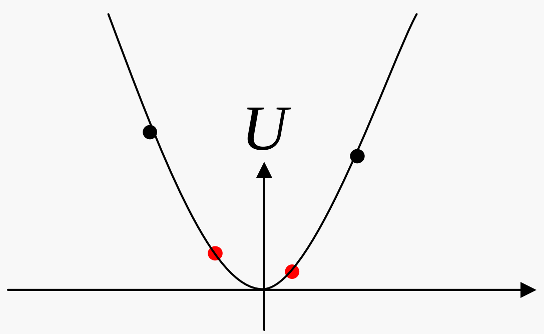
$$H = -2E_C\left(1-2N_g\right)\sigma^x + \frac{E_J}{2}\sigma^z + \text{constant.....}$$

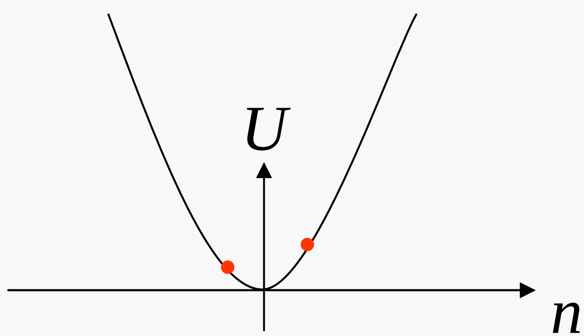
$$H = \vec{B}\bullet\vec{\sigma}$$

$$\varepsilon_\pm = \pm\sqrt{B_x^2 + B_z^2}$$

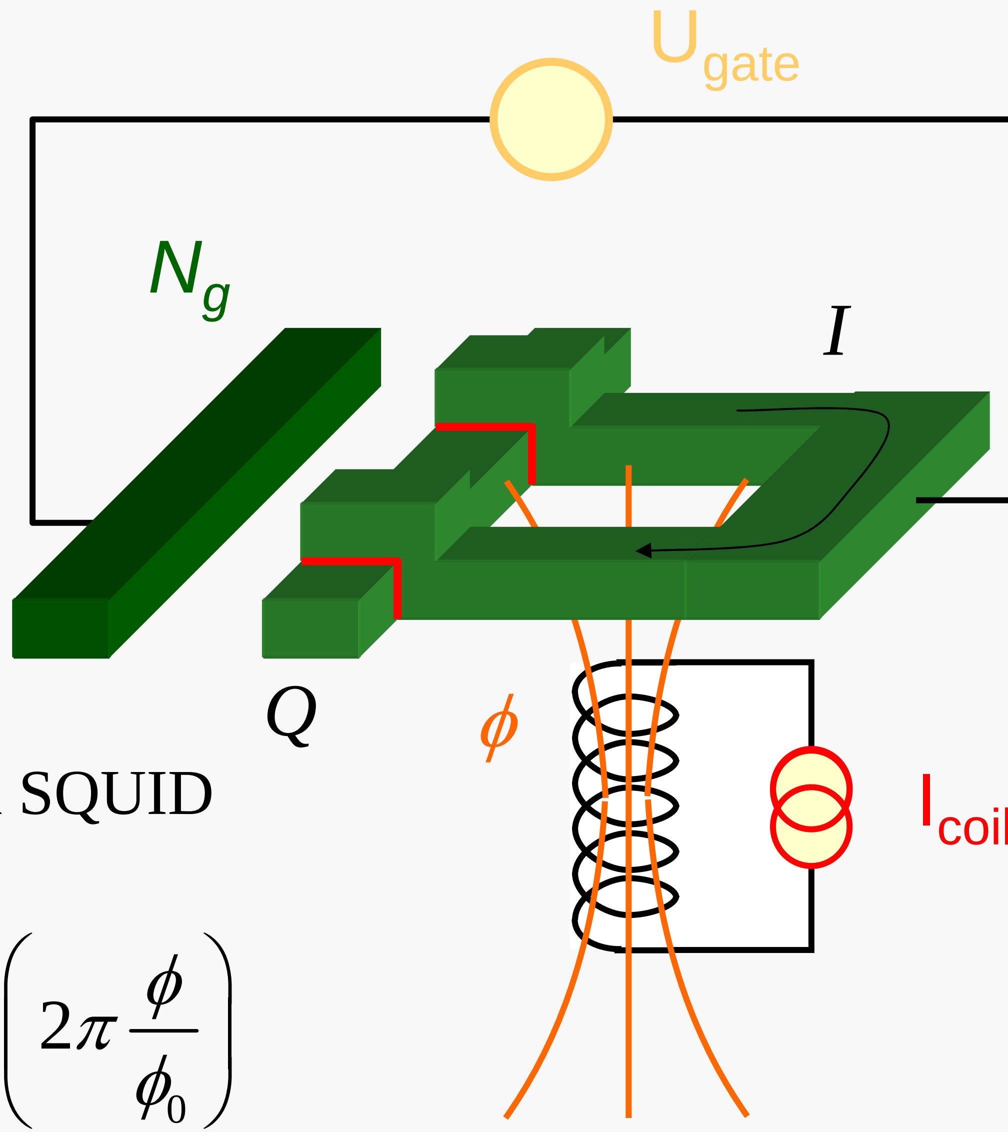


# SPLIT COOPER PAIR BOX

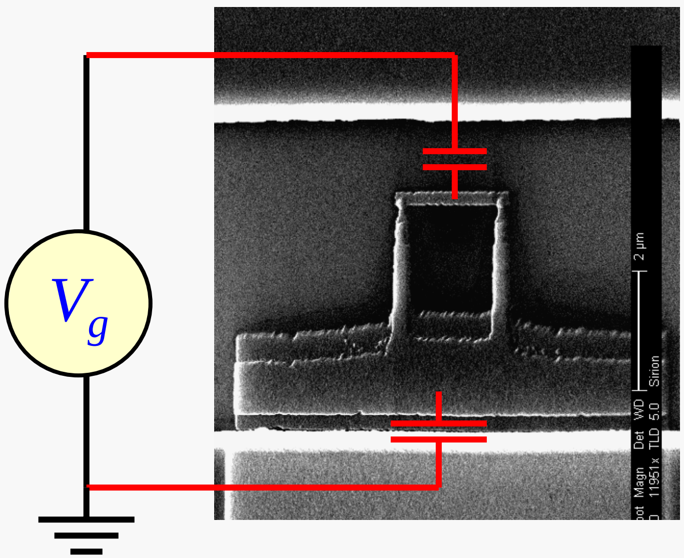



JJ is actually a SQUID

$$E_J = E_J^{\max} \cos\left(2\pi \frac{\phi}{\phi_0}\right)$$

# Split Cooper Pair Box

$$H = -2E_C\left(1-2N_g\right)\sigma^x + \frac{E_J}{2}\sigma^z + \text{constant.....}$$



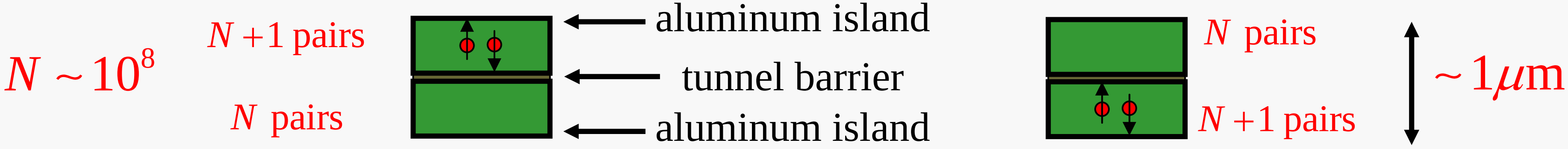

JJ is actually a SQUID

$$E_J = E_J^{\max}\cos\left(2\pi\frac{\phi}{\phi_0}\right)$$

(Buttiker ’87; Bouchiat et al., 98)

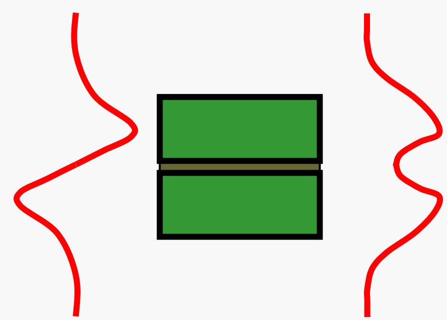
# Superconducting Tunnel Junction as a
# Covalently Bonded Diatomic 'Molecule'

(simplified view at charge deg. point)



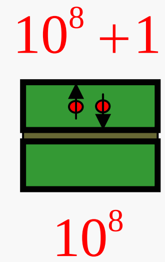

## Cooper Pair Josephson Tunneling Splits the Bonding and Anti-bonding 'Molecular Orbitals'

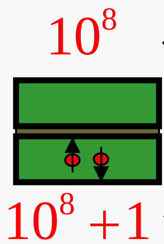
anti-bonding



bonding

# Bonding Anti-bonding Splitting

$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\Big( |$ 

$\rangle \pm |$ 

$\rangle \Big)$

$$E_{\text{anti-bonding}} - E_{\text{bonding}} = E_{\text{J}} \sim 7\text{ GHz} \sim 0.3\text{ K}$$

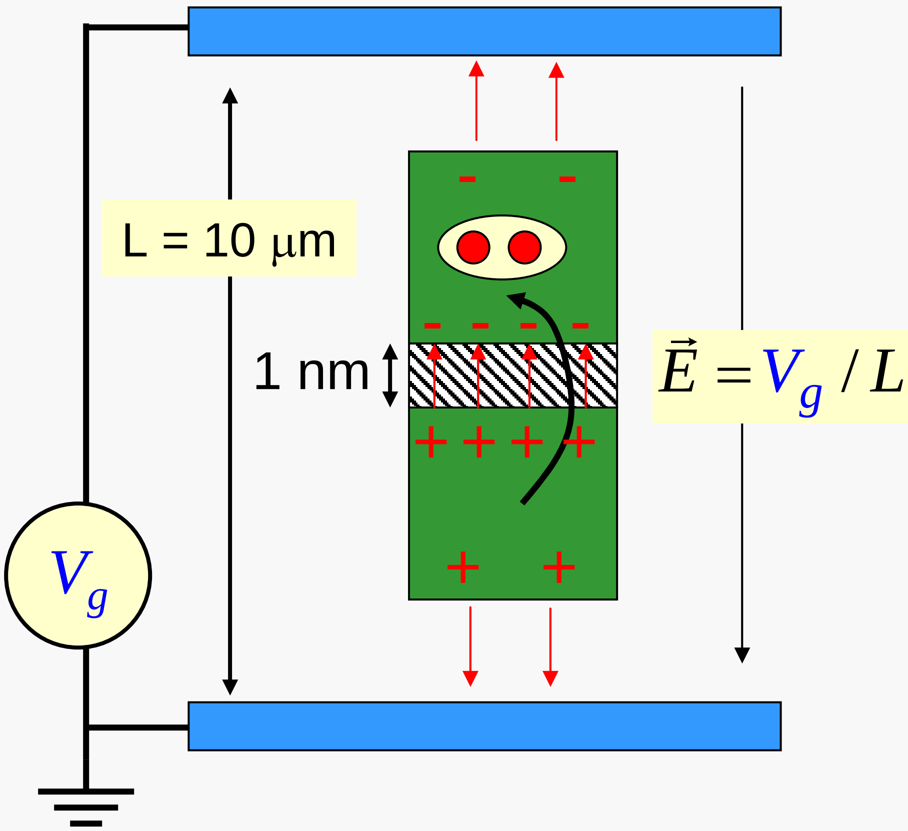
↑ Josephson coupling

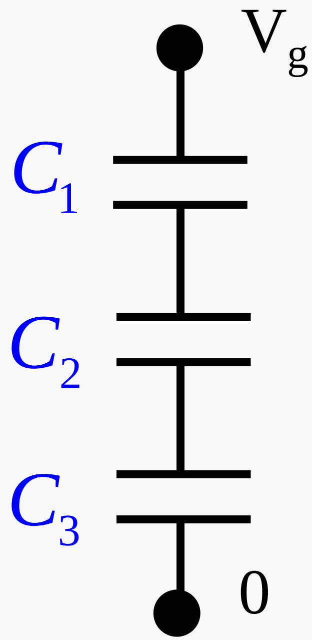
$|\uparrow\rangle =$ bonding

$|\downarrow\rangle =$ anti-bonding

$$H = -\frac{E_{\text{J}}}{2}\sigma^z$$

# Dipole Moment of the Cooper-Pair Box (determines polarizability)




$$d = (2e)L\frac{1/C_2}{1/C_1 + 1/C_2 + 1/C_3}$$

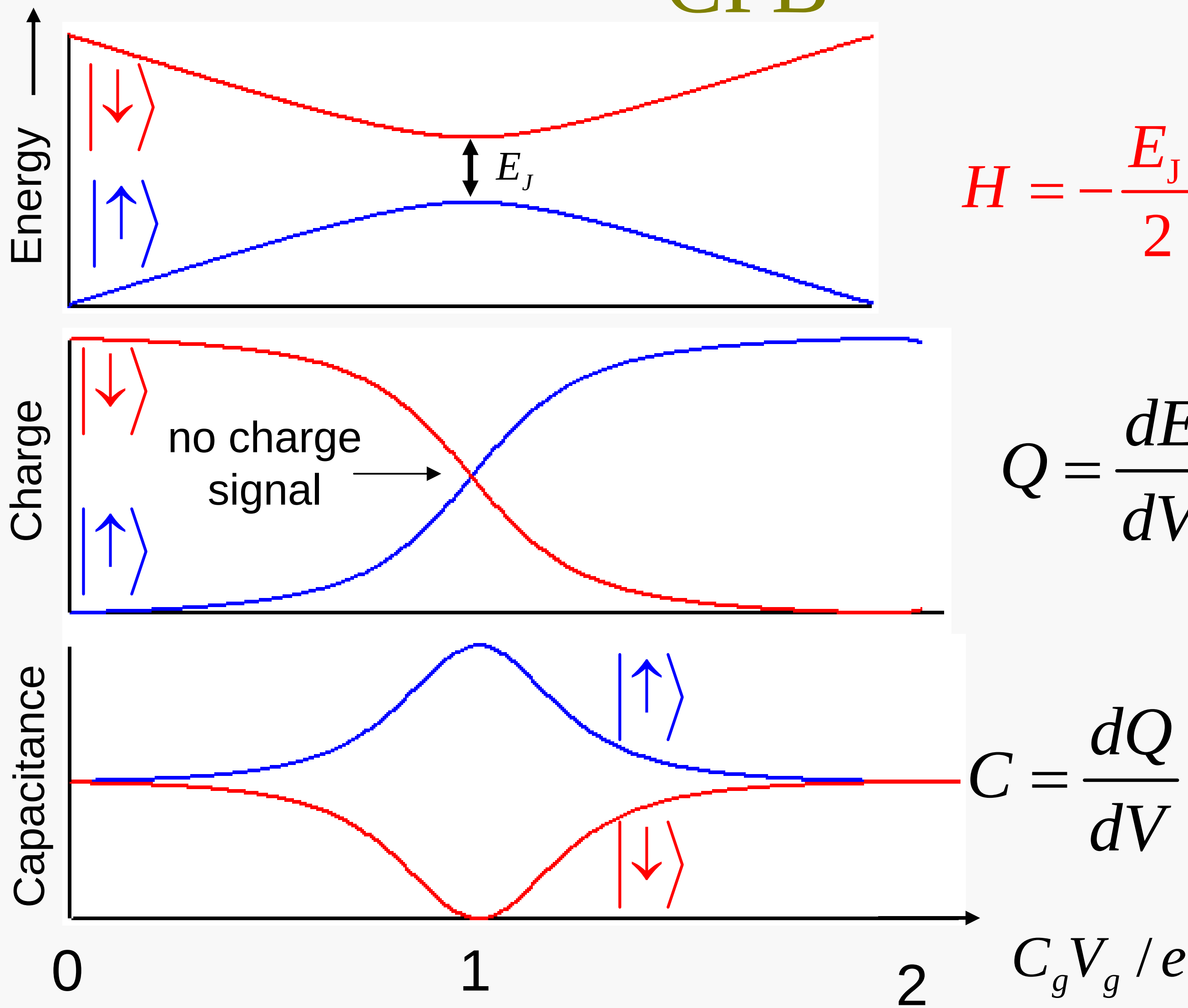
$|\uparrow\rangle =$ bonding

$|\downarrow\rangle =$ anti-bonding

$$H = -\frac{E_J}{2}\sigma^z - \frac{d}{L}V_g\sigma^x$$

$$|\vec{d}| \sim 2e\text{-}\mu\text{m}$$

# Energy, Charge, and Capacitance of the CPB



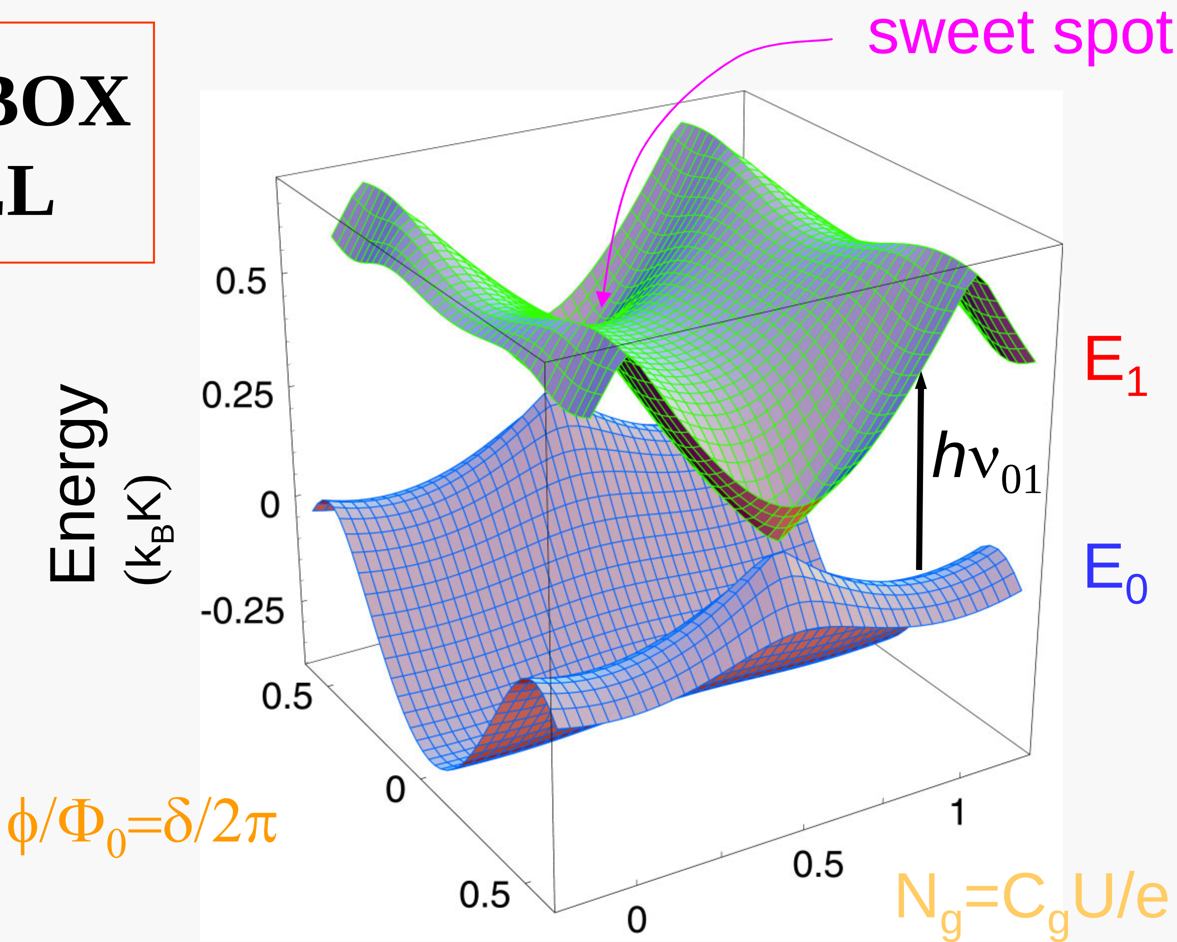

$$H = -\frac{E_J}{2}\sigma^z - \frac{d}{L}V_g\sigma^x$$

$$Q = \frac{dE}{dV}$$ charge

$$C = \frac{dQ}{dV}$$ polarizability is state dependent

deg. pt. = coherence sweet spot

# RESPONSES OF BOX IN EACH LEVEL




loop current $$I_k = \frac{\partial E_k}{\partial \phi}$$

island charge $$Q_k = \frac{\partial E_k}{\partial U}$$

box inductance $$L_k = \left( \frac{\partial^2 E_k}{\partial \phi^2} \right)^{-1}$$

box capacitance $$C_k = \frac{\partial^2 E_k}{\partial U^2}$$

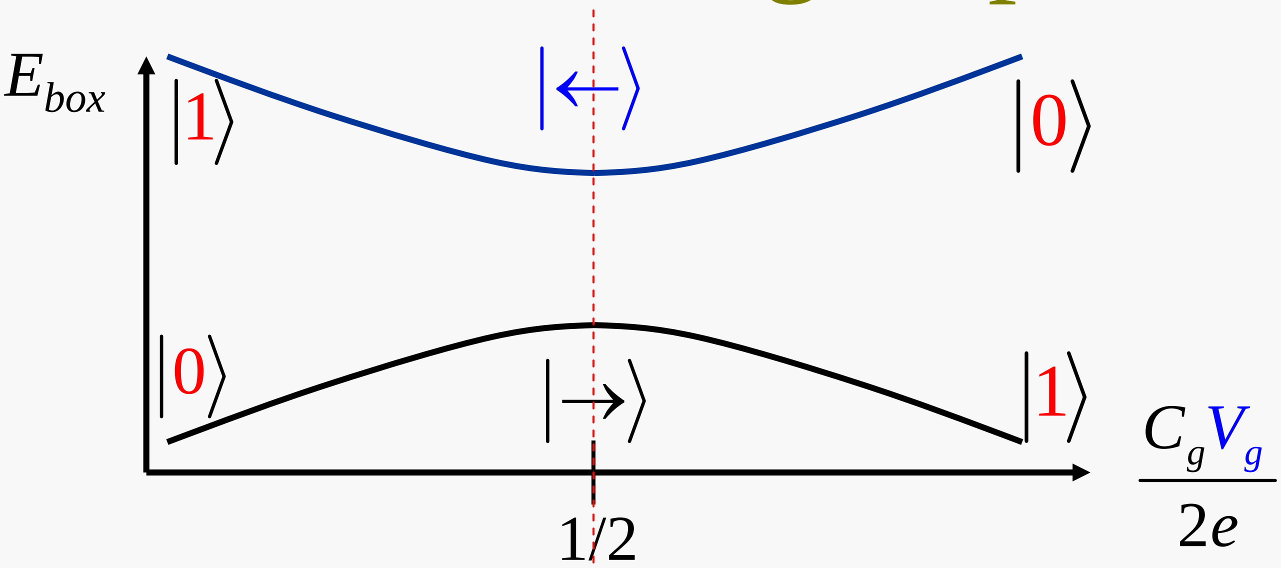
# Dephasing, decoherence, decay

$$\hbar \approx 0.658 \times 10^{-15}\,\text{eV-s}$$

$1\,\text{nV}$ acting on charge $e$ for $1\ \mu\text{s}\ = 10^{-15}\ \text{eV-s}$

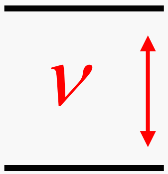
Spins very coherent but hard to couple and *very* hard to measure.

# Avoiding Dephasing




'sweet spot' for $T_2$

Vion et al. Science 2002

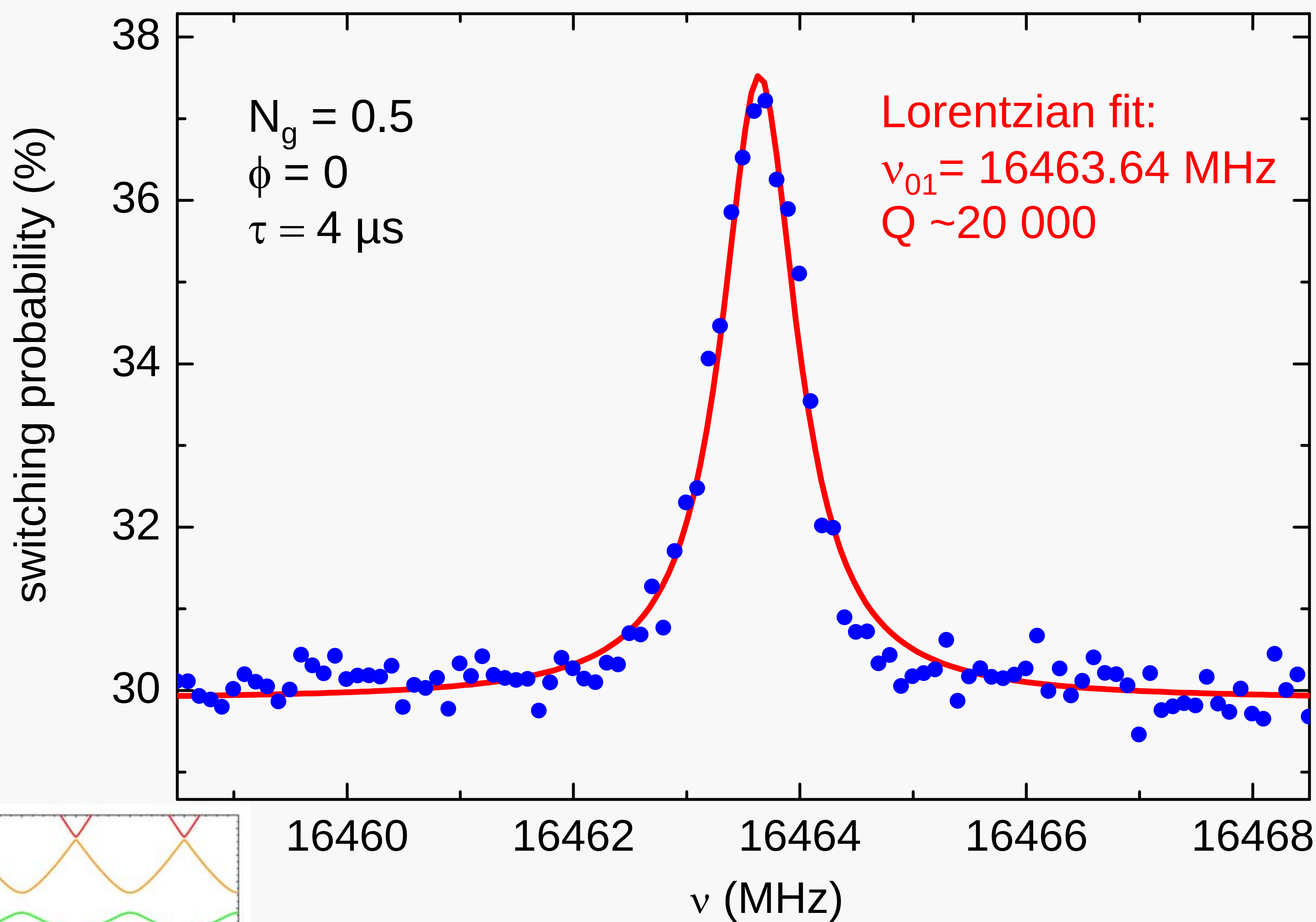


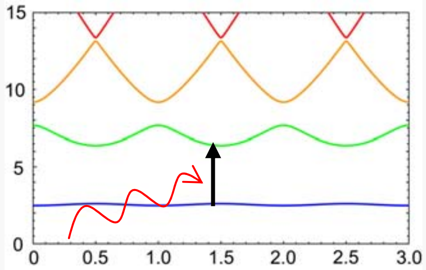

$$\frac{\partial \nu}{\partial V_g} = \langle \rightarrow | Q | \rightarrow \rangle - \langle \leftarrow | Q | \leftarrow \rangle = 0$$

$$\varphi(t) = -i \int_0^t dt' \, \nu(t')$$

$$|\psi(t)\rangle = \cos(\theta/2)|1\rangle + e^{i\varphi(t)} \sin(\theta/2)|0\rangle$$

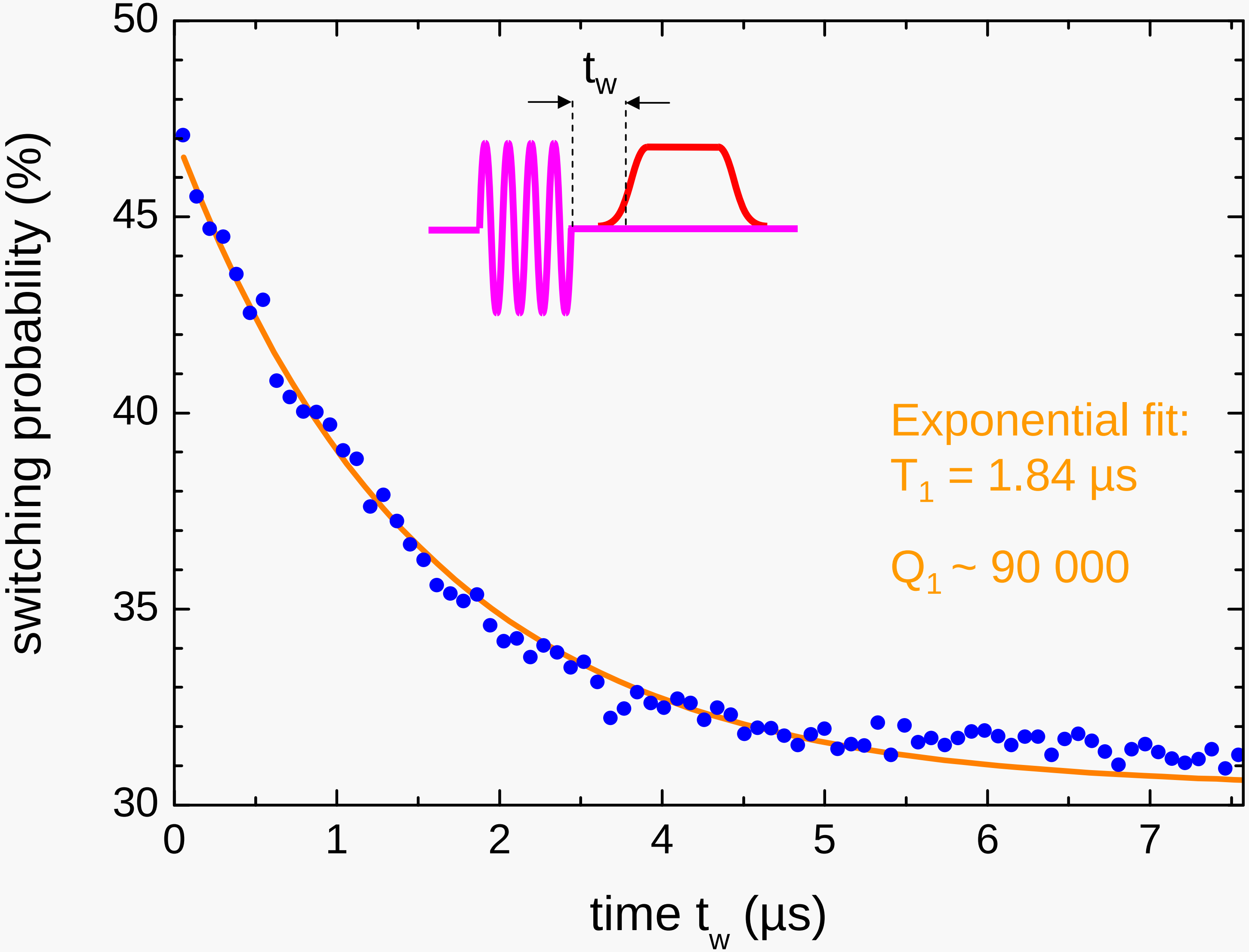
# ‘Atomic spectral line’



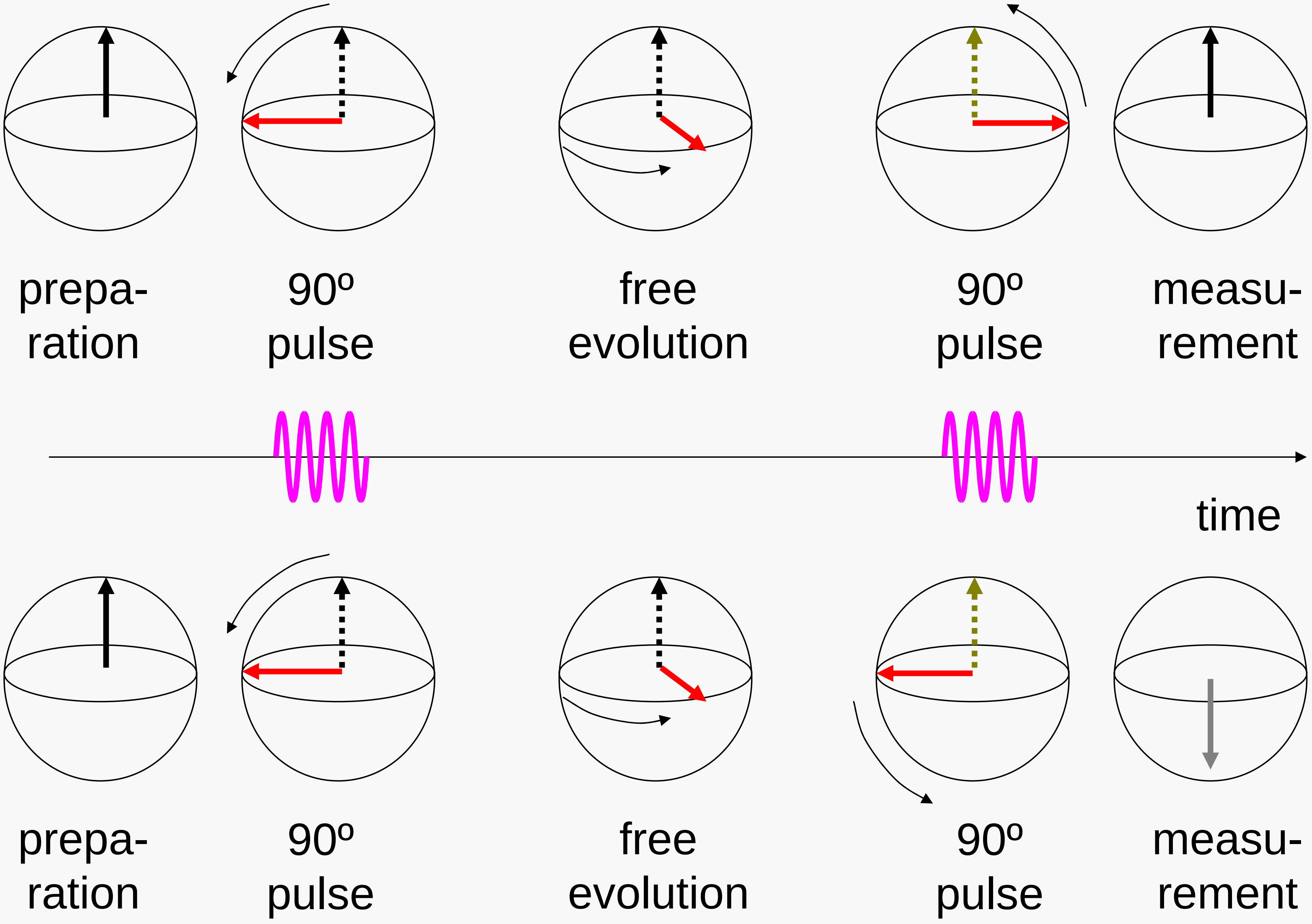




Courtesy M. Devoret

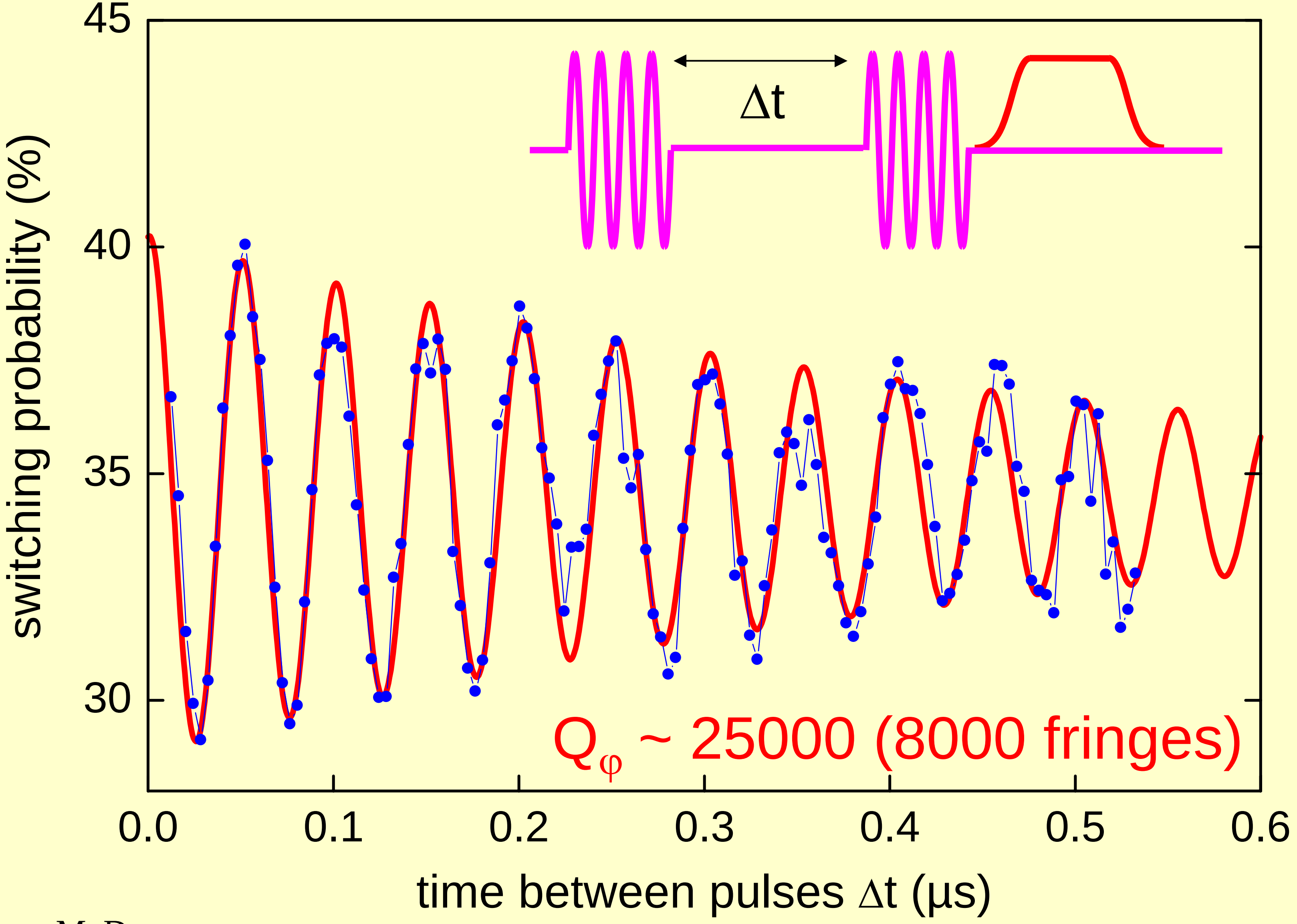
# RELAXATION TIME AT OPTIMAL POINT



# PRINCIPLE OF RAMSEY EXPERIMENT



# RAMSEY FRINGES

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