Introduction to Circuit QED

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Lecture notes on circuit QED (150 pages)
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https://girvin.sites.yale.edu/lectures

Lecture series on quantum error correction and fault tolerance

arXiv:2111.08894: Introduction to Quantum Error Correction and Fault Tolerance

Videos of above lectures:
https://girvin.sites.yale.edu/video
OUTLINE:

Introduction to Circuit QED

• What is Cavity QED?
• Quantum LC Oscillators
• Josephson Junctions & Transmon Qubits
• Qubits coupled to microwave cavities
QED: Atoms Coupled to Photons

Zero-Point Fluctuations of the Vacuum Affect Atomic Spectra

Irreversible spontaneous decay into the photon continuum:

\[ 2p \rightarrow 1s + \gamma \quad T_1 \sim 1 \text{ ns} \]

Vacuum Fluctuations: electron mass renormalization; Virtual photon emission and reabsorption, Lamb shift lifts 2s-2p degeneracy

Cavity QED: What happens if we trap the photons in engineered discrete modes inside a cavity?

If cavity has no mode at atom’s frequency.

\[ T_{1,cQED} \rightarrow 10^3 T_1 \]
μwave cQED with Rydberg Atoms

cQED at optical frequencies

The state of photons is detected, not atoms.

... measure changes in transmission of optical cavity

(H. J. Kimble, H. Mabuchi)
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Introduction to Circuit QED
Artificial atoms and microwave photons
How to be a quantum electrical engineer

**LC oscillator**

\[ \Phi(t) \]

Define generalized flux

\[ \Phi(t) = \int dt' V(t') \]

\[ \dot{\Phi} = V \]

Faraday induction (up to a minus sign)

Electrostatic energy

\[ \frac{1}{2} C \dot{\Phi}^2 \]

Magnetic energy

\[ \frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2 \]

Lagrangian

\[ L = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \dot{\Phi}^2 \]

[Lumped element LC or single mode of a microwave cavity resonator]
\[ L = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2 \]

velocity coordinate

momentum \( Q = \frac{\delta L}{\delta \dot{\Phi}} = C \dot{\Phi} = CV \)

charge \( Q \) is momentum canonically conjugate to flux.

Hamiltonian
\[ H = Q \dot{\Phi} - L = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \]

harmonic oscillator with "mass" \( m = C \)

"spring constant" \( k = 1/L \)

resonance frequency \( \omega_r = \sqrt{\frac{k}{m}} = \frac{1}{\sqrt{LC}} \)

Hamilton eqn's of motion
\[ \ddot{\Phi} = \frac{\partial H}{\partial \dot{\Phi}} = \frac{Q}{C} = V \quad \checkmark \text{Faraday induction} \]
\[ Q = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I \quad \checkmark \text{charge conservation} \]
\[ \ddot{\Phi} = \frac{\dot{Q}}{C} = -\frac{1}{LC} \Phi \]

\[ I = I_0 \sin(\omega_R t + \theta) \]

\[ V = I_0 Z_R \cos(\omega_R t + \theta) \]

\[ \text{characteristic impedance} \]

\[ I = -\dot{Q} = -C \dot{V} = -\omega_R C Z_R I_0 \sin(\omega_R t + \theta) \]

\[ Z_R = \frac{1}{\omega_R C} = \sqrt{\frac{L}{C}} = \sqrt{1} \]

\[ Z_R \approx 50 - 500 \Omega \quad \text{because impedance of free space} \]

\[ Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377 \Omega \]

quantum of impedance \[ Z_K = \frac{h}{e^2} \approx 25,812 \Omega \]

\[ \alpha_C = \frac{\varepsilon_0}{h c} \left( \frac{1}{4\pi e_0} \right) \approx \frac{1}{137} \]

\[ \frac{Z_0}{Z_K} = 2 \]
Quantizing the oscillator

\[ [\hat{Q}, \hat{\Phi}] = -i\hbar \quad \hat{\Phi} = \frac{\Phi_{2PF}}{2}(a + a^\dagger) \quad \hat{Q} = -i \Phi_{2PF} (a - a^\dagger) \]

\[ [a, a^\dagger] = 1 \quad Q_{2PF} \phi_{2PF} = \frac{\hbar}{2} \]

Virial theorem \( \langle 0 | \hat{Q}^2 | 0 \rangle = \frac{1}{L^2} \left( \frac{1}{2} \hbar \omega R \right) \Rightarrow Q_{2PF} = \sqrt{\frac{\hbar}{2L^2 R}} \)

\[ \langle 0 | \frac{\hat{Q}^2}{2L^2} | 0 \rangle = \frac{1}{L^2} \left( \frac{1}{2} \hbar \omega R \right) \Rightarrow \phi_{2PF} = \sqrt{\frac{\hbar}{2L^2 R}} \]

\[ Q_{2PF} / \phi_{2PF} = \frac{\hbar}{2} \quad \psi(0) = \langle \phi | 0 \rangle \text{ is a minimum uncertainty packet} \]

\[ \frac{Q_{2PF}}{e} = \sqrt{\frac{\hbar}{4 \pi e^2 \frac{1}{Z_R}}} = \sqrt{\frac{Z_K}{4 \pi e^2 Z_R}} \approx \sqrt{\frac{1}{4 \pi}} \approx 3 \]
Quantum Harmonic Oscillators have many important uses but:

Their level spacing is uniform making them impossible to achieve full quantum control with classical signals.

We need anharmonicity to make qubits and auxiliary controllers for oscillators:

\[ H = \hbar \omega \ a \dagger a \]

\[ H = \hbar \left[ \omega \ a \dagger a - \frac{K}{2} a \dagger a a \right] \]

\[ \omega_{12} - \omega_{01} = K \]
Quantum control paradox:

Microwave resonators
- can have very long lifetimes (1ms – 1 s) compared to qubits
- contain a large Hilbert space in a simple empty box
- can replace multiple qubits

But:
- require ancilla non-linear element (e.g. a qubit) to provide universal control

Recent theory papers:


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Joseph tunnel junctions act as non-linear inductors to produce anharmonic oscillators and qubits.

\[ \omega_{01} \neq \omega_{12} \]

\[ |0\rangle = |g\rangle \]

\[ |1\rangle = |e\rangle \]

\[ \omega_{01} \sim 5 - 10 \text{ GHz} \]
‘Circuit QED:’
- microwave photons inside superconducting circuits
- artificial atoms (Josephson junction qubits)

Ultra-strong photon-‘atom’ coupling:
- non-linear quantum optics at the single photon level

Hydrogen atom

\[ f_{1S-2P} \approx 2.46 \times 10^{15} \text{Hz} \]
\[ \tau_{2P} \approx 1.6 \text{ns} \]
\[ Q/2\pi \approx 4 \times 10^6 \]
dipole \( \sim 1 \) Debye

(Not to scale!)

Superconducting oscillator/qubit

\[ f_{01} \approx 7 \times 10^9 \text{Hz} \]
\[ \tau_{2P} \approx 300 \mu\text{s} \]
\[ Q/2\pi \approx 2 \times 10^6 \]
dipole \( \sim 3 \times 10^7 \) Debye
"Transmon" qubit

\[ \hat{Q} = (2e)m \]

Dipole moment \( \sim \hat{Q} \times 1\text{mm} \)

\[ H = \frac{\hat{Q}^2}{2C_\varepsilon} - E_J \cos \left( \frac{2e}{\hbar} \hat{\Phi} \right) \]

\[ C_\varepsilon = C_J + C_{\text{geometric}} \]

\( \Phi = \text{SC order parameter phase} \)

\( \hbar \dot{\varphi} = 2eV = 2e \dot{\hat{Q}} \)

Josephson relation

Subtlety: \( \hat{Q} = (2e) \hat{N} \) is discrete not continuous.

For \( E_J \gg E_C \equiv \frac{e^2}{2C_\varepsilon} \), \( < \varphi > \ll 2\pi \) so can expand the cosine and safely ignore the subtleties.

Typically \( \frac{E_J}{E_C} \sim 10^2 \)

\[ \hat{\Phi} \text{ is angular momentum} \]

\[ \hat{N} \text{ conjugate to angle } \varphi \]
The Josephson relation and Hamiltonian

\[ H = 4E_c \hat{n}^2 - E_J \cos \phi \]

\[ \hat{n} = -i \frac{\partial}{\partial \phi} \]
Josephson Tunnel Junctions

Normal tunnel junction

Superconducting tunnel junction

Unique ground state for $N$ pairs on an island

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.
Josephson Tunnel Junctions

\[ |\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m |m\rangle \] 

Exactly the same Hilbert space as a 1D tight-binding model (integer position)

\[ |\varphi\rangle = \sum_m e^{i\varphi_m} |m\rangle \]

plane waves in 1st BZ (only)

linear momentum \(-\pi < \varphi < +\pi\)

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

integer \(m \leftrightarrow \varphi\) compact
Josephson Tunnel Junctions

\[ |\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m |m\rangle \]

Exactly the same Hilbert space as a 1D tight-binding model (integer position)

Or:
a quantum rotor (integer angular momentum)

Or:
position basis \(|m\rangle\)
plane waves in 1st BZ (only) \(|\varphi\rangle = \sum_{m} e^{i\varphi m} |m\rangle\)
linear momentum \(-\pi < \varphi < +\pi\)

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

angular momentum basis \(|m\rangle\)
position basis \(|\varphi\rangle = \sum_{m} e^{i\varphi m} |m\rangle\)
angular position \(-\pi < \varphi < +\pi\)

integer \(m \Leftrightarrow \varphi\) compact
Josephson Tunnel Junction as a capacitor
(N.B. ignoring offset charge)

\[ Q = (2e)m \]

\[ U = \frac{Q^2}{2C} = 4 \frac{e^2}{2C} m^2 = 4E_c m^2 \]

Quantum Rotor

\[ T = \frac{L^2}{2I} = -\frac{1}{2I} \frac{d^2}{d\phi^2} \]

\[ T \left| m \right> = \frac{m^2}{2I} \left| m \right> \]

Unique ground state for N pairs on an island

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Superconducting tunnel junction

\[ +Q \quad \begin{array}{c} \text{Unique ground state for N pairs on an island} \\ \text{Total number of Cooper pairs that have tunneled} \\ \text{uniquely determines the low-energy quantum state} \\ \text{of a pair of islands.} \end{array} \]
Cooper Pair Tunneling (Josephson Effect)

\[ H_j = -\frac{E_j}{2} \sum_m \left\{ |m+1\rangle \langle m| + |m\rangle \langle m+1| \right\} \]

[tight-binding hopping matrix element that changes position by \( \pm 1 \)]

\[ H_j = -E_j \cos \varphi \]
[gravitational potential producing a torque that changes the angular momentum by \( \pm 1 \)]

Unique ground state for \( N \) pairs on an island

Quantum Rotor

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.
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Transmon Qubit in 3D Cavity

$g = \frac{\vec{d} \cdot \vec{E}_{\text{rms}}}{h}$

Huge dipole moment: strong coupling

$|\vec{d}| = 2e \times 1 \text{ mm} \approx 10^7 \text{Debye!!}$

$V_{\text{dipole}} = g \sigma^x (a + a^\dagger)$

$g \sim 100 \text{ MHz}$

$Q_{\text{ZPF}} \sim \frac{1}{2e \sqrt{16\pi\alpha}} \sim 1 - 3$
$H = \omega_r a^\dagger a + \frac{\omega_q}{2} \sigma^z + g\sigma^x [a + a^\dagger] + H_{\text{damping}}$  

[Rabi]

$H = \omega_r a^\dagger a + \frac{\omega_q}{2} \sigma^z + g[a\sigma^+ + a^\dagger\sigma^-] + H_{\text{damping}}$  

[Jaynes-Cummings]

$H = \omega_r a^\dagger a + \frac{\omega_q}{2} \sigma^z + \chi \sigma^z a^\dagger a + H_{\text{damping}}$  

[Dispersive]

**Strong Dispersive Limit**
Strong Dispersive Hamiltonian

\[ H = \omega_r a^\dagger a + \frac{\omega_q}{2} \sigma^z + \chi \sigma^z a^\dagger a + H_{\text{damping}} \]

resonator  qubit  dispersive coupling

\[ \chi >> \kappa, \Gamma \]

Cavity frequency = \( \omega_r + \chi \sigma^z \)

'Cstrong-dispersive' limit

\[ 2\chi \sim 2 \times 10^3 \kappa \]

[Cavity frequency can be used to readout state of qubit]
Using (not so) strong dispersive coupling to measure the state of the qubit

Additional notes:
The S matrix for reflection of microwave photon from a resonator is derived in the separate PDF document ‘Reflection from a resonator’
Can read out qubit state by measuring cavity resonance frequency

Choose $\chi \sim \kappa$

$E_{\text{out}}(\omega) = e^{i\theta(\omega)}E_{\text{in}}(\omega)$
State of qubit is entangled with the ‘meter’ (microwave phase).
Then ‘meter’ is read with amplifier.

\[
|\psi_{\text{in}}\rangle = \{a |\uparrow\rangle + b |\downarrow\rangle\}|\alpha\rangle
\]
\[
|\psi_{\text{out}}\rangle = a |e^{+i\theta \alpha}\rangle |\uparrow\rangle + b |e^{-i\theta \alpha}\rangle |\downarrow\rangle
\]

\[
E_{\text{out}}(\omega) = e^{i\theta(\omega)} E_{\text{in}}(\omega)
\]
Quadrature amplitudes $X, Y$ are canonically conjugate, leading to quantum vacuum noise.
Using (not so) strong dispersive coupling to measure the state of the qubit

Quantum limited amplifiers developed...

Data from: M. Hatridge et al., Science 339, 178 (2013)

Readout fidelity > 99.5% in ~ 300 nsec
Using strong-dispersive coupling to measure the photon number distribution in a cavity

**Strong Dispersive Hamiltonian**

\[
H = \omega_r a^\dagger a + \frac{\omega_q}{2} \sigma^z + \chi \sigma^z a^\dagger a + H_{\text{damping}}
\]

\( \chi \gg \kappa, \Gamma \)

resonator  qubit  dispersive coupling

**Reinterpretation of same Hamiltonian:**

Quantized Light Shift of Qubit Transition Frequency

\[
H = \omega_r a^\dagger a + \frac{1}{2} \sigma^z \left[ \omega_q + 2 \chi a^\dagger a \right] + H_{\text{damping}}
\]

Spectrum of qubit depends on cavity photon number
Using strong-dispersive coupling to measure the photon number distribution in a cavity

Measure photon number in high Q storage cavity via dispersive coupling to transmon.

Readout transmon state via dispersive coupling to low Q readout resonator.
quantized light shift of qubit frequency
(coherent microwave state)

\[ \frac{\omega_q + 2 \chi a^\dagger a}{2} \sigma^z \]

N.B. power broadened
100X

Readout Voltage (mV)

spectroscopy frequency (GHz)
- quantized light shift of qubit frequency (coherent microwave state)

\[ \frac{\omega_q + 2\chi a^+ a}{2} \sigma_z \]

N.B. power broadened 100X

Microwaves are particles!
- quantized light shift of qubit frequency  
(coherent microwave state)  
\[
\frac{\omega_q + 2 \chi a^\dagger a}{2} \sigma^z
\]

New low-noise way to do axion dark matter detection by QND photon counting  
 Photon number parity

\[ \hat{P} = (-1)^{a^\dagger a} = \sum_{n=0}^{\infty} \left| n \right>(-1)^n \left< n \right| \]

Remarkably easy to measure using our quantum engineering toolbox

and

Measurement is 99.8% QND
Measuring Photon Number Parity

- use quantized light shift of qubit frequency

\[ \frac{\omega_q + 2\chi a^\dagger a}{2} \sigma^z \]

\[ e^{-i2\chi \hat{n}_t \frac{\sigma^z}{2}} = e^{-i\pi \hat{n} \frac{\sigma^z}{2}} \]

\[ \hat{n} = 1, 3, 5, \ldots \]
\[ \hat{n} = 0, 2, 4, \ldots \]
400 consecutive parity measurements (99.8% QND)
Summary of Lecture I:

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• What is Cavity QED?
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  • Control and measurement of both qubit and cavity