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Introduction to Circuit QED

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Disclosure: SMG is a consultant and equity holder in Quantum Circuits, Inc. and an equity holder in IBM, Inc.

SCIENCE TO SHAPE THE FUTURE OF THE ARMY – ARMY RESEARCH OFFICE 1951
C²QA Co-design Center for Quantum Advantage
U.S. DEPARTMENT OF ENERGY | Office of Science
AFOSR AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Lecture notes on circuit QED (150 pages)
2011 Les Houches Summer School

https://girvin.sites.yale.edu/lectures

Lecture series on quantum error correction and fault tolerance

arXiv:2111.08894: Introduction to Quantum Error Correction and Fault Tolerance

Videos of above lectures:
https://girvin.sites.yale.edu/video

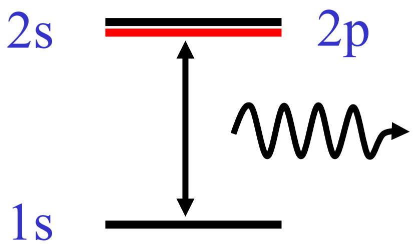
OUTLINE:

Introduction to Circuit QED

- **What is Cavity QED?**
- Quantum LC Oscillators
- Josephson Junctions & Transmon Qubits
- Qubits coupled to microwave cavities

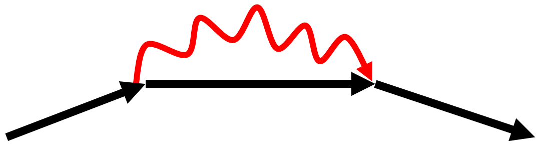
QED: Atoms Coupled to Photons

Zero-Point Fluctuations of the Vacuum Affect Atomic Spectra

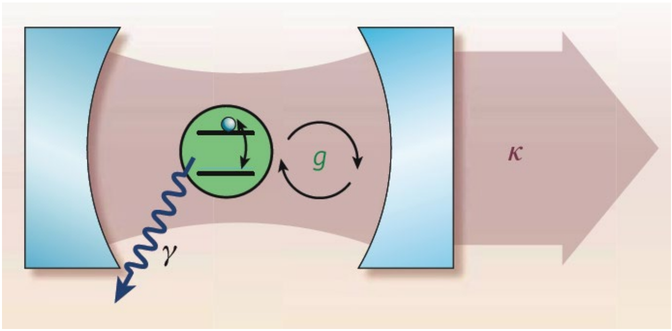


Irreversible spontaneous decay into the photon continuum:

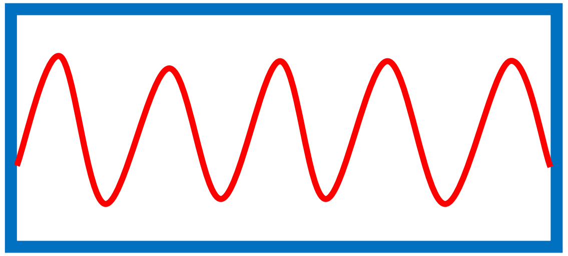
$$2p \rightarrow 1s + \gamma \qquad T_1 \sim 1\ \text{ns}$$



Vacuum Fluctuations: electron mass renormalization;
Virtual photon emission and reabsorption,
Lamb shift lifts 2s-2p degeneracy



Optical cQED



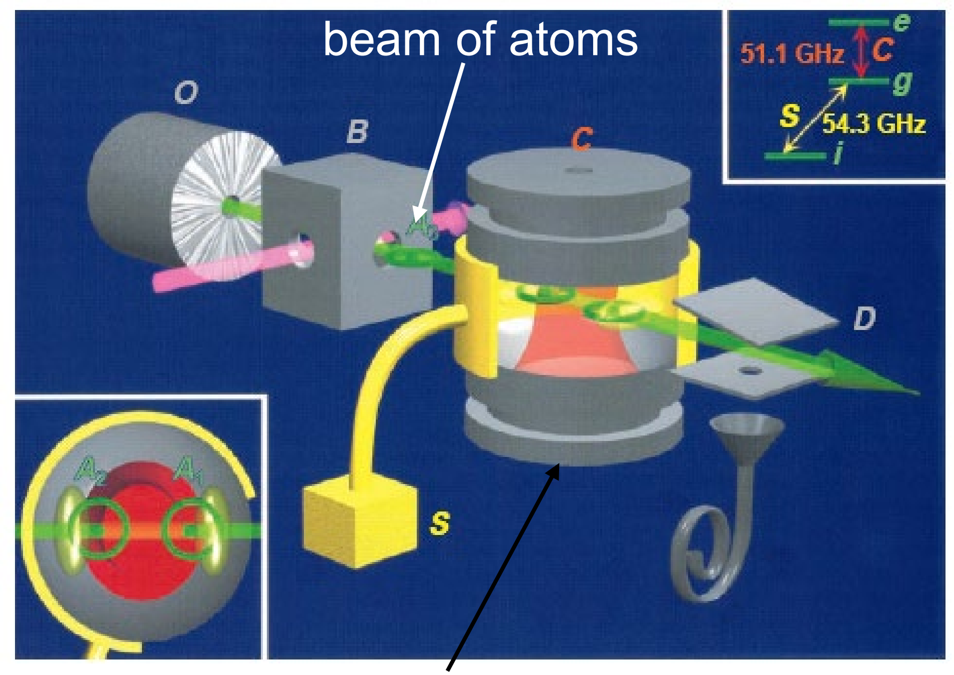
μwave cQED

Cavity QED: What happens if we trap the photons in engineered discrete modes inside a cavity?

$$T_1^{cQED} \rightarrow 10^3 T_1$$

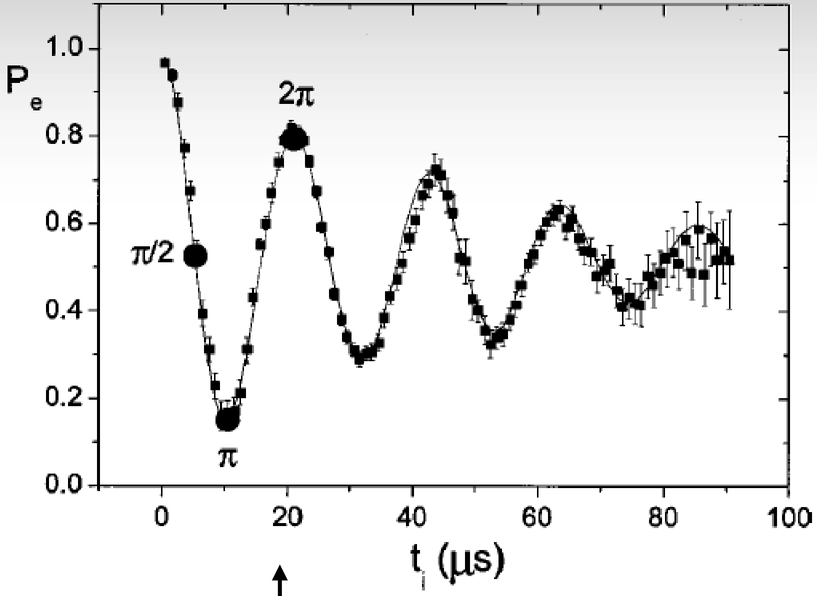
If cavity has no mode at atom's frequency.

μwave cQED with Rydberg Atoms



3-d superconducting cavity
(50 GHz)

vacuum Rabi oscillations

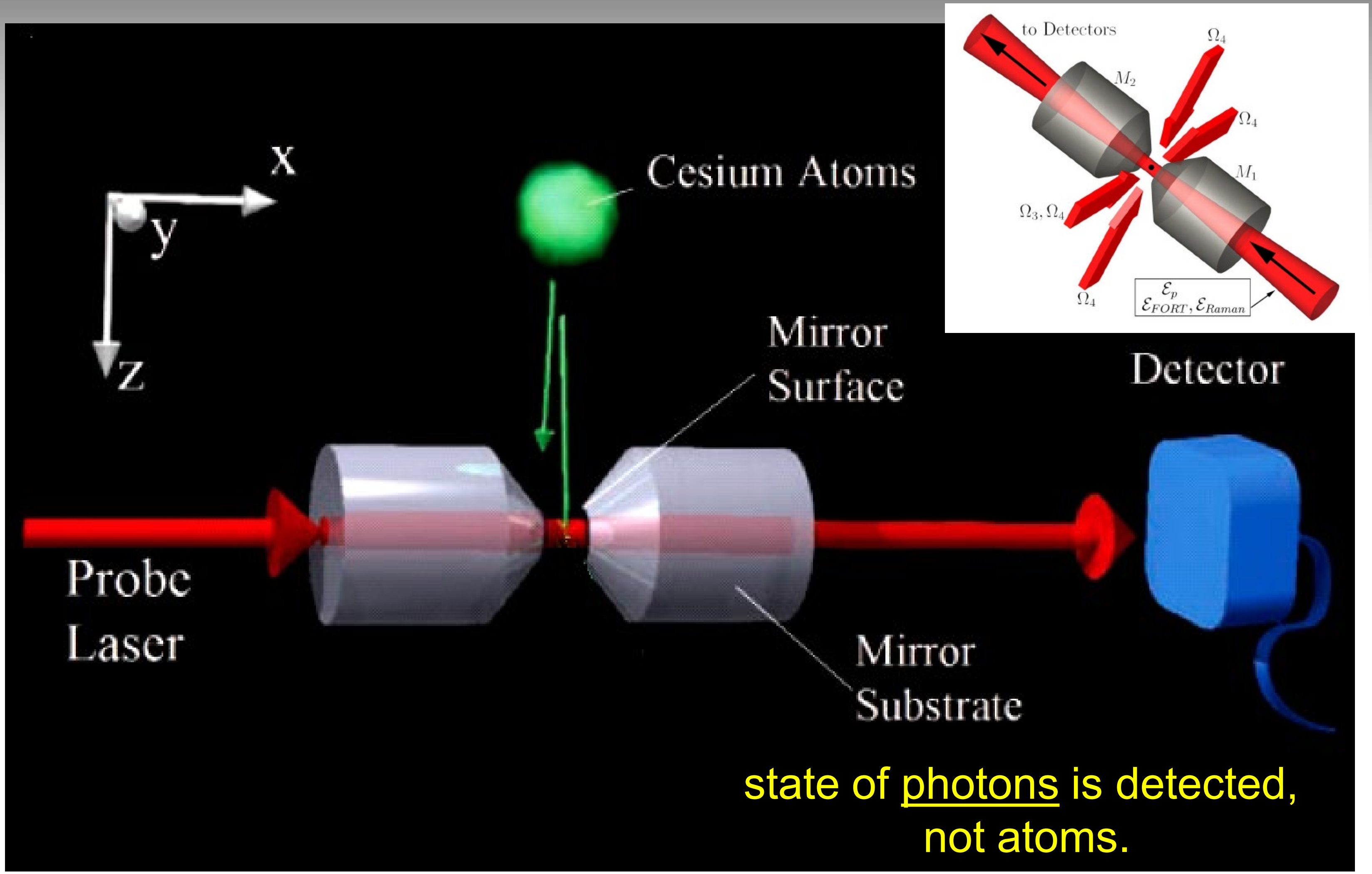


observe dependence of atom final state on time spent in cavity

measure atomic state, or …

Review: S. Haroche Nobel Lecture, Rev. Mod. Phys. 85, 1083 (2013)

cQED at optical frequencies



state of <u>photons</u> is detected,
not atoms.

… measure changes in transmission of optical cavity

(H. J. Kimble, H. Mabuchi)

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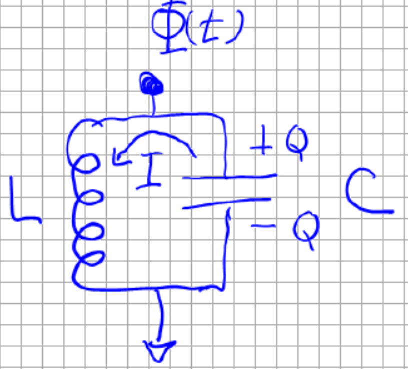
Introduction to Circuit QED

Artificial atoms and microwave photons

How to be a quantum electrical engineer

LC oscillator [Lumped element LC or single mode of a microwave cavity resonator]

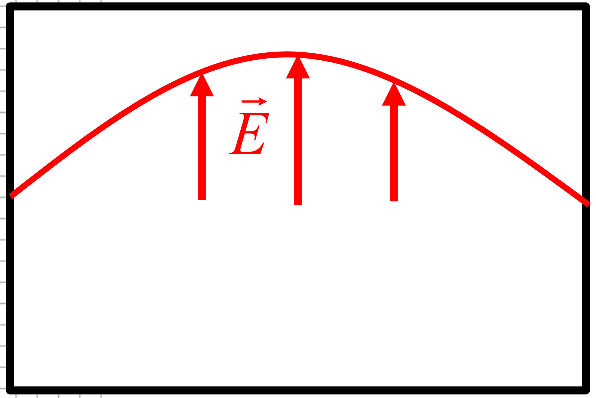


Define generalized flux

$$\Phi(t) \equiv \int^{t} d\tau\, V(\tau)$$

$$\dot{\Phi} = V$$

Faraday induction (up to a minus sign)



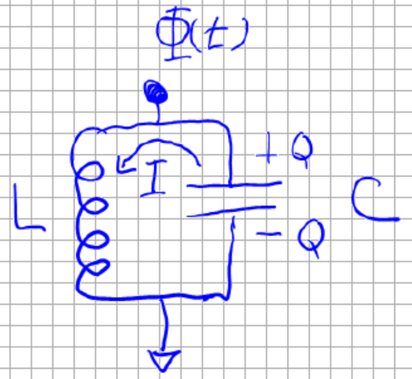
electrostatic energy $\frac{1}{2} C \dot{\Phi}^2$

magnetic energy $\frac{1}{2} L I^2 = \frac{1}{2L} \Phi^2$ $(\Phi = IL)$

Lagrangian $\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$

$$\mathcal{L} = \frac{1}{2} C \dot{\Phi}^2 - \frac{1}{2L} \Phi^2$$

velocity → $\dot{\Phi}$ coordinate → Φ



momentum $Q \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}} = C \dot{\Phi} = CV$

charge Q is momentum canonically conjugate to flux.

Hamiltonian $H = Q\dot{\Phi} - \mathcal{L} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$

harmonic oscillator with "mass" $m = C$

"spring constant" $k = 1/L$

resonance frequency $\omega_R = \sqrt{\frac{k}{m}} = \frac{1}{\sqrt{LC}}$

Hamilton eqn's of motion

$$\dot{\Phi} = \frac{\partial H}{\partial Q} = \frac{Q}{C} = V$$ ✓ Faraday induction

$$\dot{Q} = -\frac{\partial H}{\partial \Phi} = -\frac{\Phi}{L} = -I$$ ✓ charge conservation

$$\ddot{\Phi} = \frac{\dot{Q}}{C} = -\frac{1}{LC}\Phi$$

$$I = I_0 \sin(\omega_R t + \theta)$$

$$V = I_0 Z_R \cos(\omega_R t + \theta)$$

← characteristic impedance

[Nothing to do with dissipation since current and voltage are 90 degrees out of phase.]

$$I = -\dot{Q} = -C\dot{V} = +\underbrace{\omega_R C Z_R}_{=1} I_0 \sin(\omega_R t + \theta)$$

$$\boxed{Z_R = \frac{1}{\omega_R C} = \sqrt{\frac{L}{C}}}$$

$Z_R \sim 50 - 500\,\Omega$ because impedance of free space

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$$

quantum of impedance $Z_K = \frac{h}{e^2} \approx 25{,}812\ \Omega$

$$\alpha = \frac{e^2}{\hbar c}\frac{1}{[4\pi\epsilon_0]} \approx \frac{1}{137}$$

$$\boxed{Z_0 = 2\alpha Z_K}$$

Quantizing the oscillator

$$[\hat{Q}, \hat{\Phi}] = -i\hbar \qquad \hat{\Phi} = \Phi_{ZPF}(a + a^\dagger) \qquad \hat{Q} = -i\, Q_{ZPF}(a - a^\dagger)$$

$$[a, a^\dagger] = 1 \qquad Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$$

virial thm $\langle 0| \frac{\hat{Q}^2}{2C} |0\rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Rightarrow Q_{ZPF} = \sqrt{\frac{\hbar}{2 Z_R}}$

$$\langle 0| \frac{\hat{\Phi}^2}{2L} |0\rangle = \frac{1}{2}\left(\frac{1}{2}\hbar\omega_R\right) \Longrightarrow \Phi_{ZPF} = \sqrt{\frac{\hbar}{2} Z_R}$$

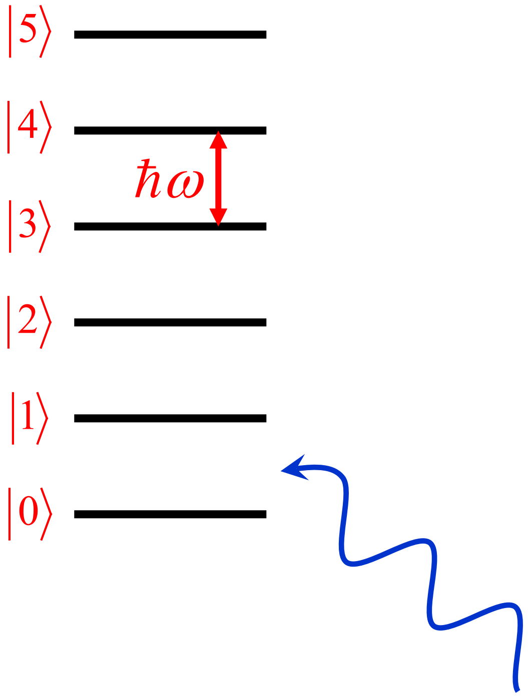
$Q_{ZPF}\,\Phi_{ZPF} = \frac{\hbar}{2}$ ✓ $\quad \psi(\Phi) = \langle \Phi | 0 \rangle$ is a minimum uncertainty packet

$$\frac{Q_{ZPF}}{e} = \sqrt{\frac{h}{4\pi e^2} \frac{1}{Z_R}} = \sqrt{\frac{Z_K}{4\pi Z_R}} \sim \sqrt{\frac{137}{4\pi}} \sim 3$$

Quantum Harmonic Oscillators have many important uses but:

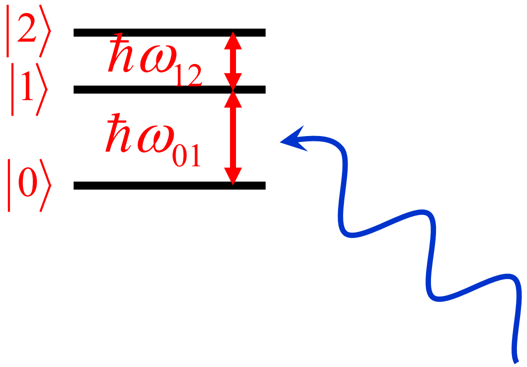
Their level spacing is uniform making them impossible to achieve full *quantum* control with *classical* signals.

$$H = \hbar\omega\, a^{\dagger} a$$



We need anharmonicity to make *qubits* and *auxiliary controllers* for oscillators:

$$H = \hbar\left[\omega\, a^{\dagger} a - \frac{K}{2} a^{\dagger} a^{\dagger} a a\right]$$



$$\omega_{12} - \omega_{01} = K$$

Quantum control paradox:

Microwave resonators

- can have very long lifetimes (1ms – 1 s) compared to qubits
- contain a large Hilbert space in a simple empty box
- can replace multiple qubits

But:

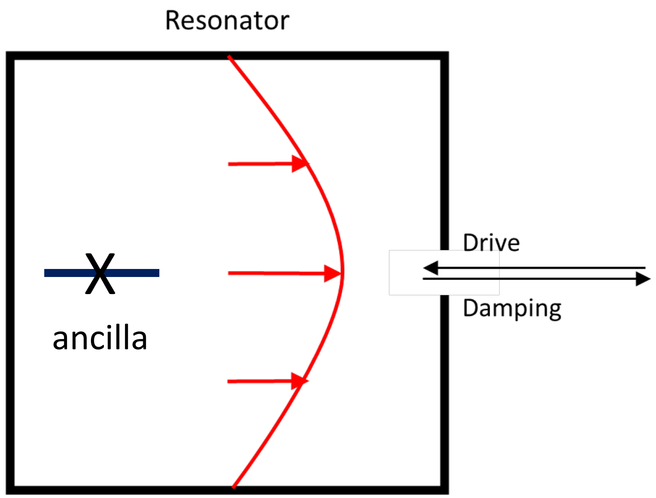
- require ancilla non-linear element (e.g. a qubit) to provide universal control

Recent theory papers:

‘Quantum control of bosonic modes with superconducting circuits,’ Wen-Long Ma et al., *Science Bulletin* **66**, 1789 (2021)

‘Photon-Number-Dependent Hamiltonian Engineering for Cavities,’ Chiao-Hsuan Wang et al. *Phys. Rev. Applied* **15**, 044026 (2021)

‘Constructing Qudits from Infinite Dimensional Oscillators by Coupling to Qubits,’ Yuan Liu et al., *Phys. Rev. A* **104**, 032605 (2021)



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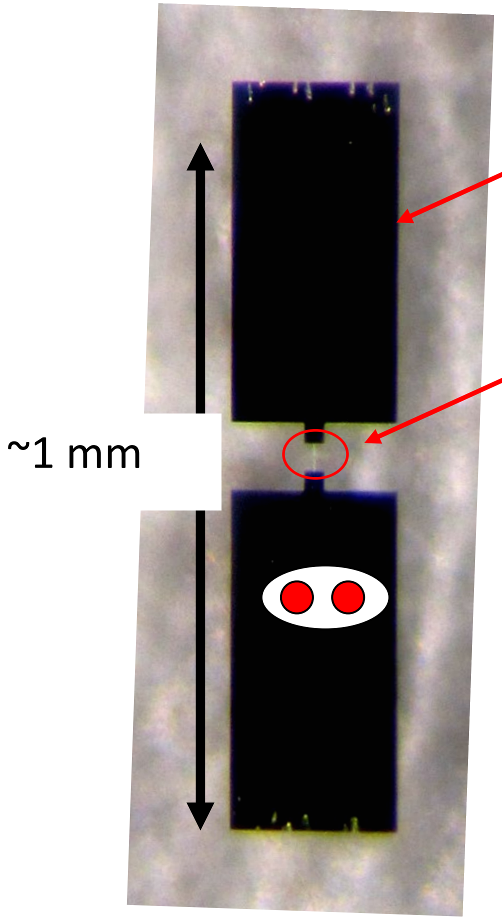
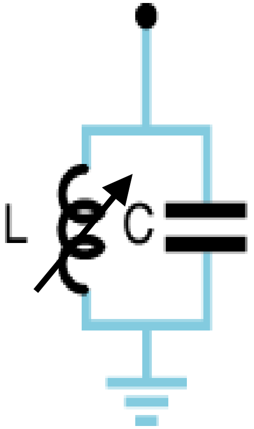
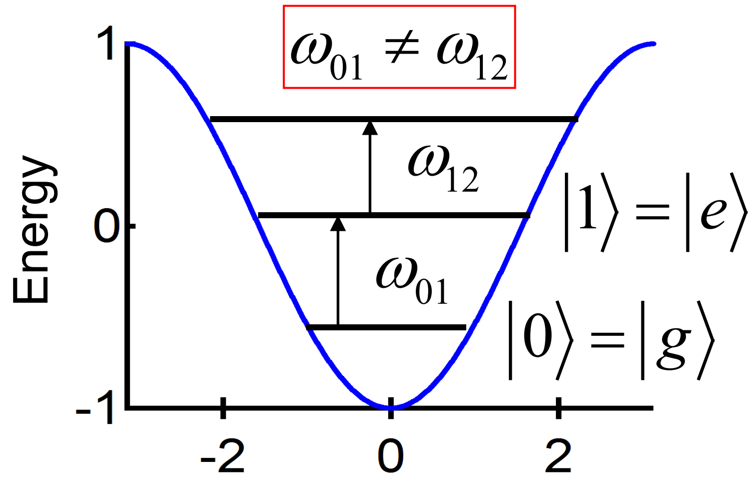
Joseph tunnel junctions act as non-linear inductors to produce anharmonic oscillators and qubits



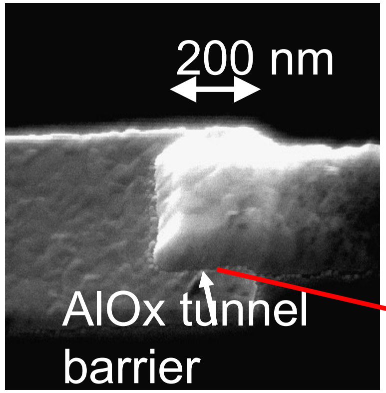
Antenna pads are capacitor plates

Josephson tunnel junction

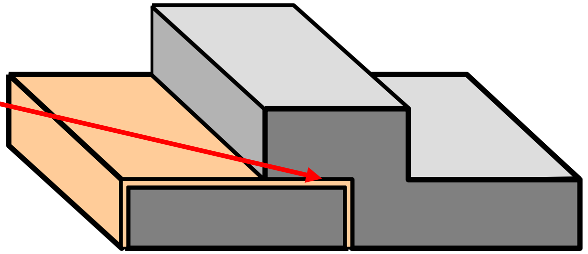




$$\omega_{01} \sim 5-10\ \mathbf{GHz}$$



‘transistor of quantum computing’



‘Circuit QED:’

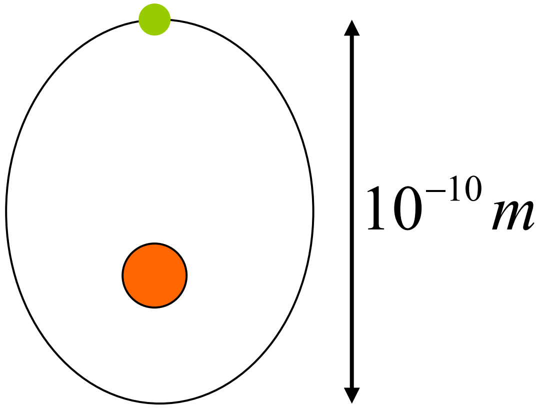
-microwave photons inside superconducting circuits

-artificial atoms (Josephson junction qubits)

Ultra-strong photon-‘atom’ coupling:

-non-linear quantum optics at the single photon level

Hydrogen atom



$f_{1S-2P} \approx 2.46\times10^{15}\,\mathrm{Hz}$

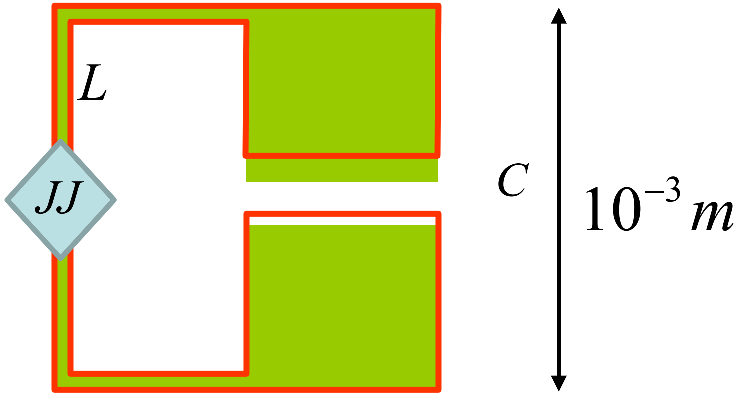
$\tau_{2P} \approx 1.6\,\mathrm{ns}$

$\mathrm{Q}/2\pi \approx 4\times10^{6}$

$\text{dipole} \sim 1\ \text{Debye}$

(Not to scale!)

Superconducting oscillator/qubit



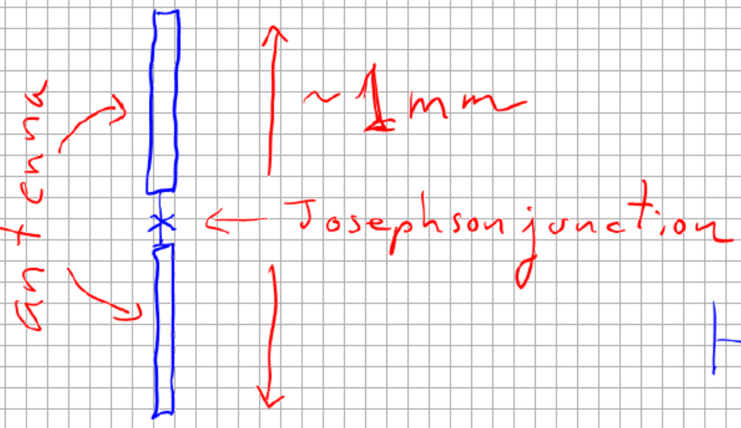
$f_{01} \approx 7\times10^{9}\,\mathrm{Hz}$

$\tau_{2P} \approx 300\,\mu\mathrm{s}$

$\mathrm{Q}/2\pi \approx 2\times10^{6}$

$\text{dipole} \sim 3\times10^{7}\ \text{Debye}$

"Transmon" qubit



dipole moment

$\sim Q_{ZPF} \times 1\,mm$

$\hat{Q} = (2e)m$

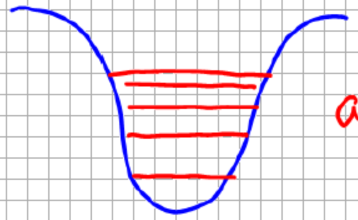
$$H = \frac{\hat{Q}^2}{2C_\Sigma} - E_J \cos\left(\frac{2e}{\hbar}\hat{\Phi}\right)$$

$C_\Sigma \equiv C_J + C_{geometric}$

$\varphi \equiv$ SC order parameter phase

$\hbar\dot{\varphi} = 2eV = 2e\dot{\Phi}$

Josephson relation



anharmonic oscillator

Subtlety: $\hat{Q} = (2e)\hat{n}$ is discrete not continuous.

For $E_J \gg E_c \equiv \frac{e^2}{2C_\Sigma}$ $\quad \langle\varphi^2\rangle \ll 2\pi$ so can expand the cosine and safely ignore the subtleties.

Typically $\frac{E_J}{E_c} \sim 10^2$

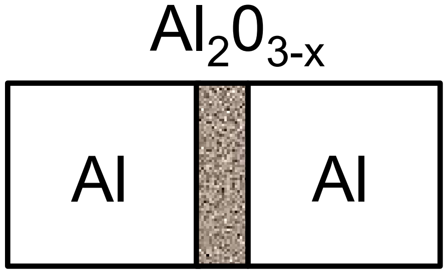
[$\hat{n}$ is angular momentum conjugate to angle φ.]

The Josephson relation and Hamiltonian

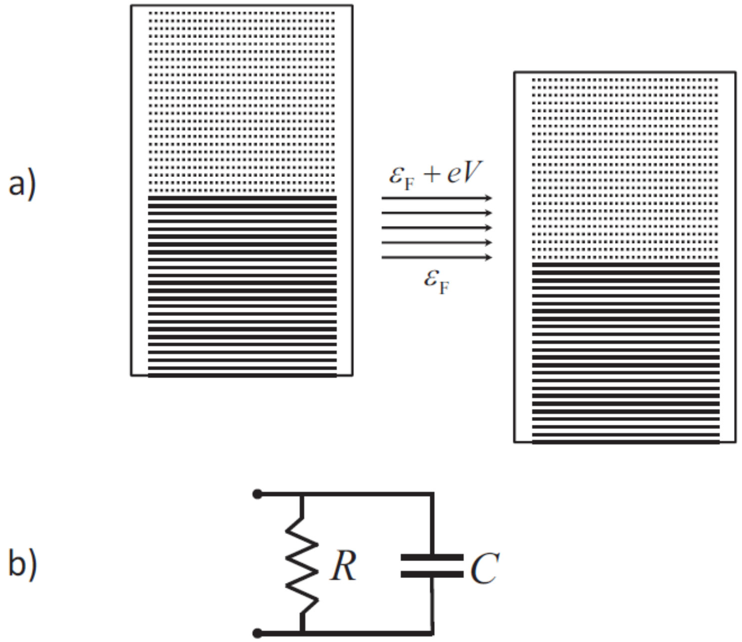
$$H = 4E_c\hat{n}^2 - E_J\cos\varphi$$

$$\hat{n} = -i\frac{\partial}{\partial\varphi}$$

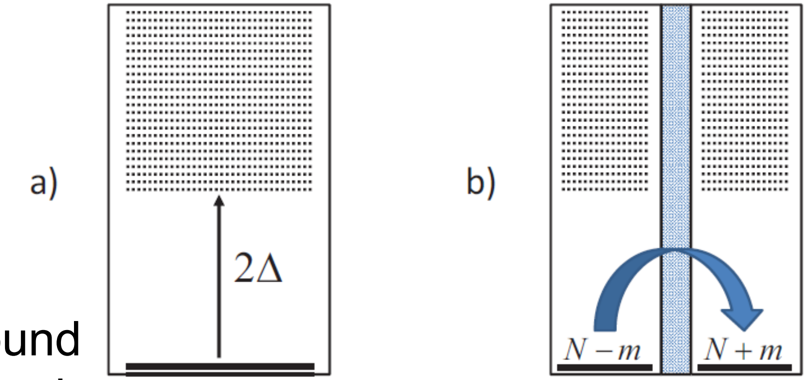
Josephson Tunnel Junctions



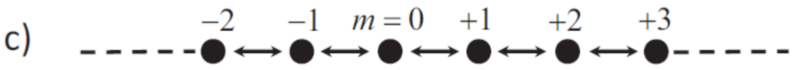
Normal tunnel junction



Superconducting tunnel junction



Unique ground state for N pairs on an island



Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Josephson Tunnel Junctions

$$|\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m \,|m\rangle$$

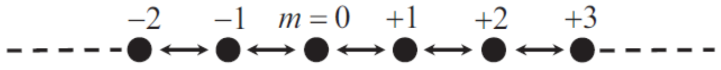
Exactly the same Hilbert space as a 1D tight-binding model (integer *position*)

position basis $|m\rangle$

plane waves in 1st BZ (only) $\quad |\varphi\rangle = \sum_m e^{i\varphi m}|m\rangle$

linear momentum $-\pi < \varphi < +\pi$

c)



Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

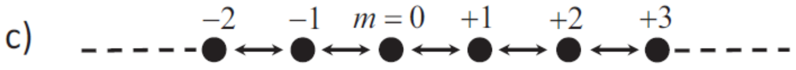
integer $m \Leftrightarrow \varphi$ compact

Josephson Tunnel Junctions

$$|\psi\rangle = \sum_{m=-\infty}^{+\infty} \psi_m \,|m\rangle$$

Exactly the same Hilbert space as a 1D tight-binding model (integer *position*)

Or:
a quantum rotor
(integer *angular momentum*)

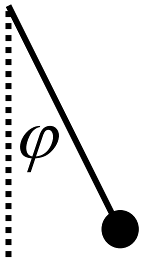


Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state

position basis $|m\rangle$

plane waves in 1st BZ (only) $|\varphi\rangle = \sum_m e^{i\varphi m}|m\rangle$

linear momentum $-\pi < \varphi < +\pi$



angular momentum basis $|m\rangle$

position basis $|\varphi\rangle = \sum_m e^{i\varphi m}|m\rangle$

angular position $-\pi < \varphi < +\pi$

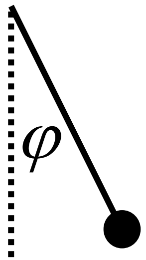
integer $m \Leftrightarrow \varphi$ compact

Josephson Tunnel Junction as a capacitor
(N.B. ignoring offset charge)

$$Q = (2e)m$$

$$U = \frac{Q^2}{2C} = 4\frac{e^2}{2C}m^2 = 4E_c\, m^2$$

Quantum Rotor

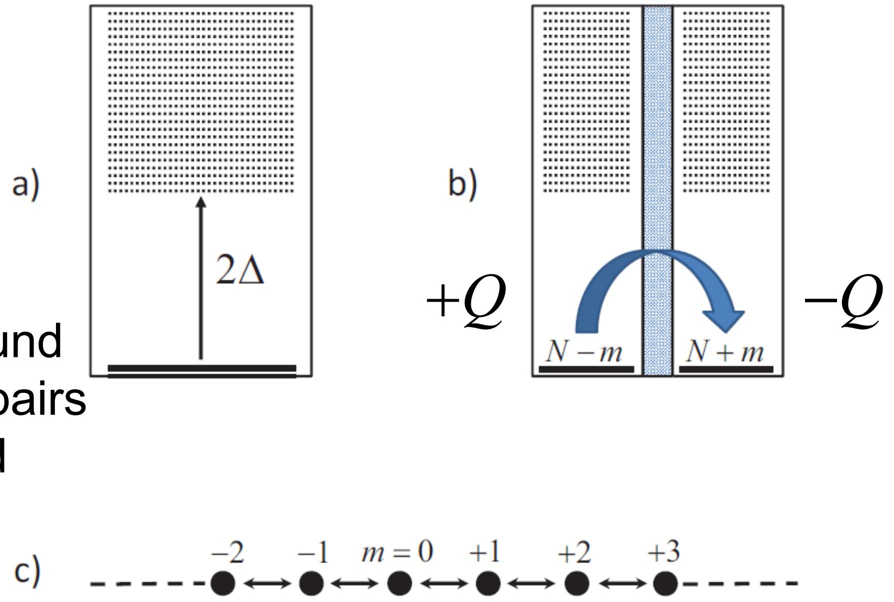


$$T = \frac{L^2}{2I} = -\frac{1}{2I}\frac{d^2}{d\varphi^2}$$

$$T|m\rangle = \frac{m^2}{2I}|m\rangle$$

Superconducting tunnel junction

Unique ground state for N pairs on an island



Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

Cooper Pair Tunneling (Josephson Effect)

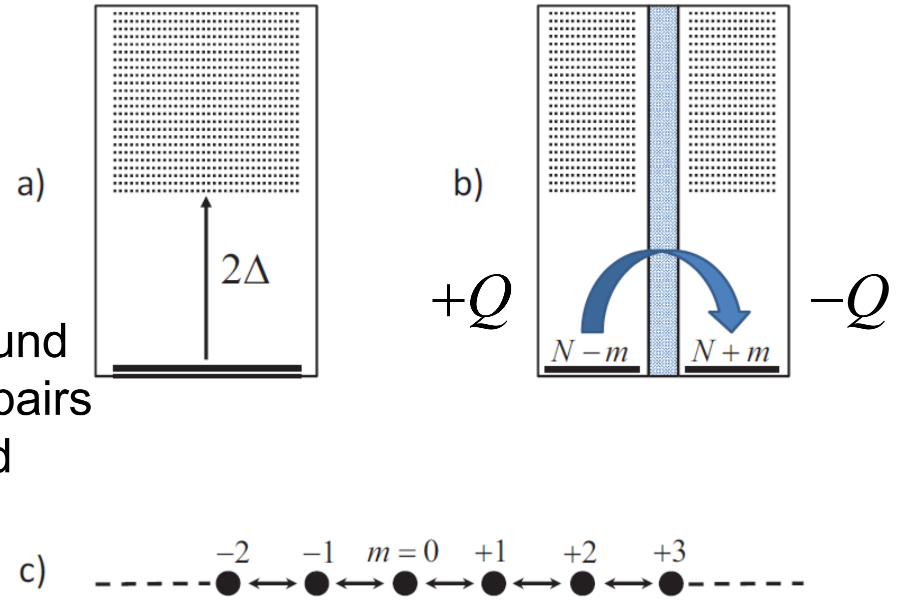
$$H_J = -\frac{E_J}{2}\sum_m \left\{ |m+1\rangle\langle m| + |m\rangle\langle m+1| \right\}$$

[tight-binding hopping matrix element that changes position by ± 1]

$$H_J = -E_J \cos\varphi$$

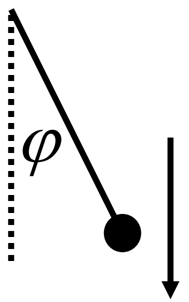
[gravitational potential producing a torque that changes the angular momentum by ± 1]

Superconducting tunnel junction



Unique ground state for N pairs on an island

Quantum Rotor



'gravity'

Total number of Cooper pairs that have tunneled uniquely determines the low-energy quantum state of a pair of islands.

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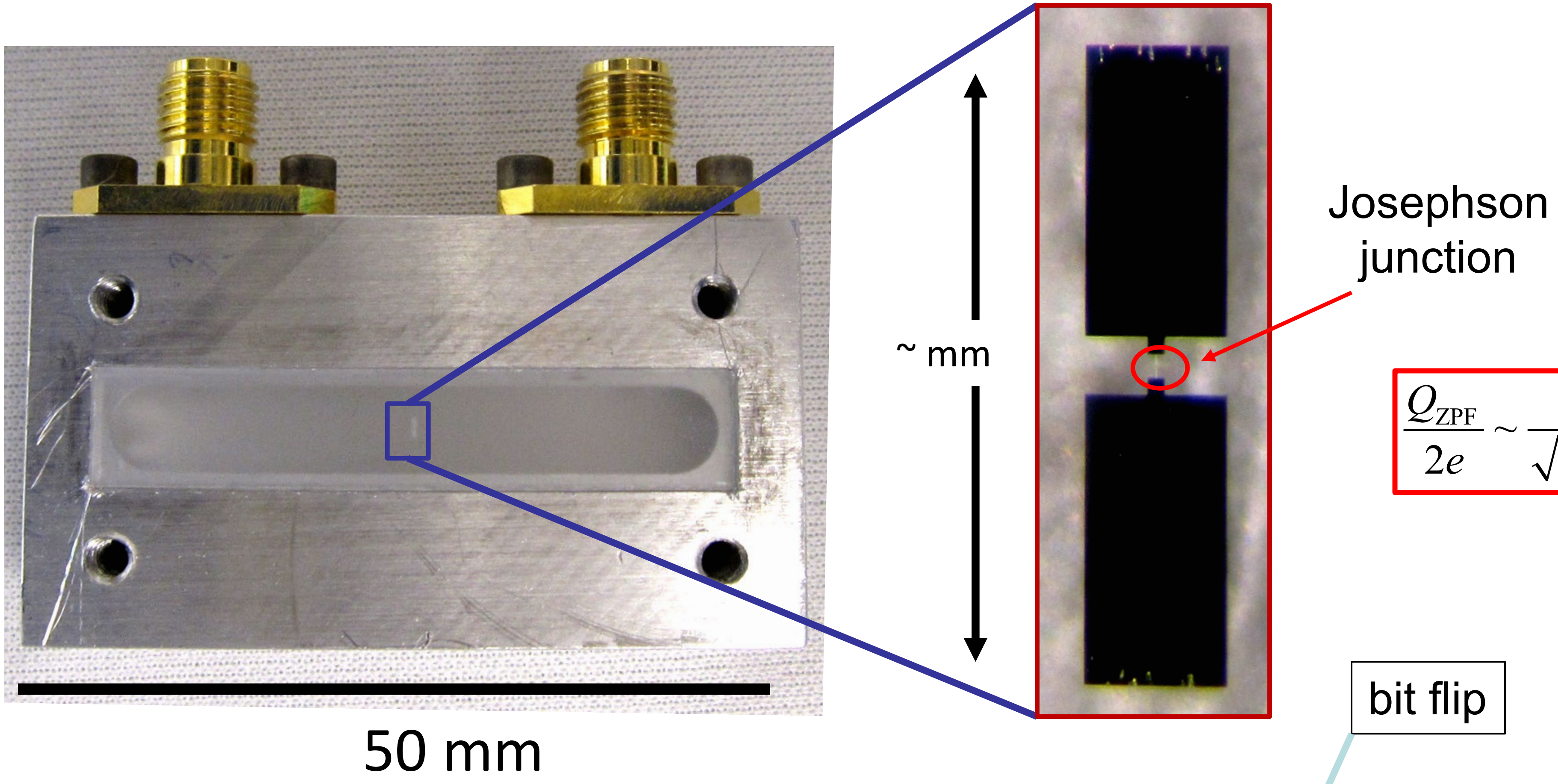
Transmon Qubit in 3D Cavity



$$\frac{Q_{\mathrm{ZPF}}}{2e} \sim \frac{1}{\sqrt{16\pi\alpha}} \sim 1-3$$

bit flip

$$V_{\mathrm{dipole}} = g\sigma^x (a + a^\dagger)$$

$$g = \frac{\vec{d} \bullet \vec{E}_{rms}}{h}$$

$$\left|\vec{d}\right| = 2e \times 1 \text{ mm} \approx 10^7 \text{Debye!!}$$

Huge dipole moment: strong coupling

g ~ 100 MHz

Microwave resonator

Transmon qubit

Dipole Coupling

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + g\sigma^x[a + a^\dagger] + H_{\text{damping}} \qquad \text{[Rabi]}$$

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + g[a\sigma^+ + a^\dagger\sigma^-] + H_{\text{damping}} \qquad \text{[Jaynes-Cummings]}$$

$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + \chi\sigma^z a^\dagger a + H_{\text{damping}} \qquad \text{[Dispersive]}$$

Strong Dispersive Limit

Strong Dispersive Hamiltonian

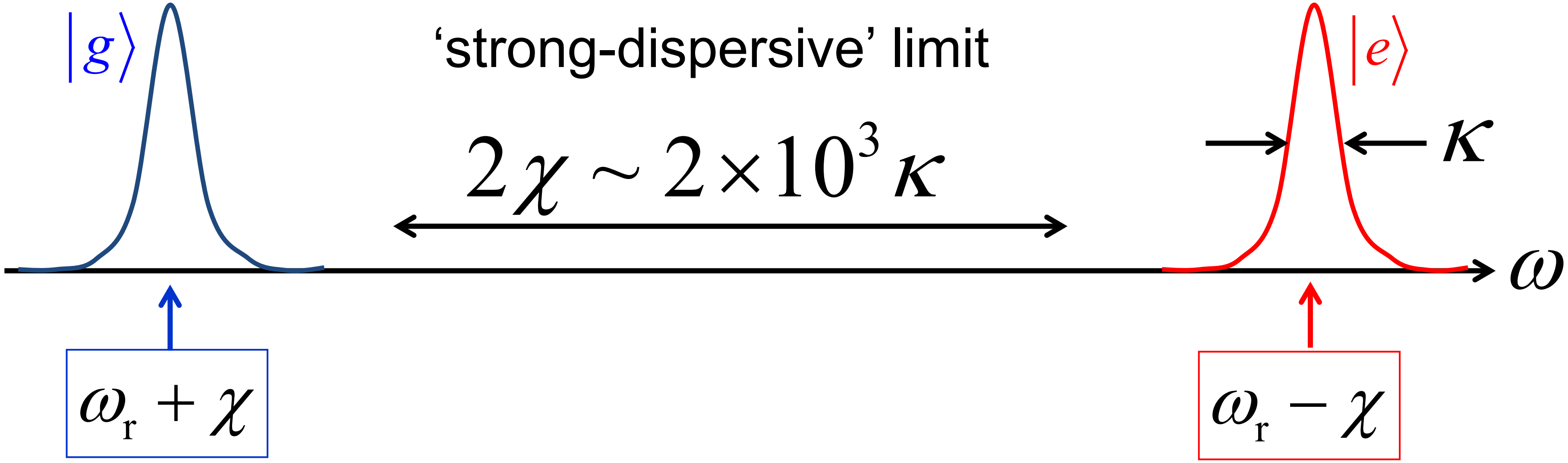
$$H = \omega_r a^\dagger a + \frac{\omega_q}{2}\sigma^z + \chi\sigma^z a^\dagger a + H_{\text{damping}}$$

$$\chi >> \kappa, \Gamma$$

resonator qubit dispersive coupling

$$\text{cavity frequency} = \omega_r + \chi\sigma^z$$

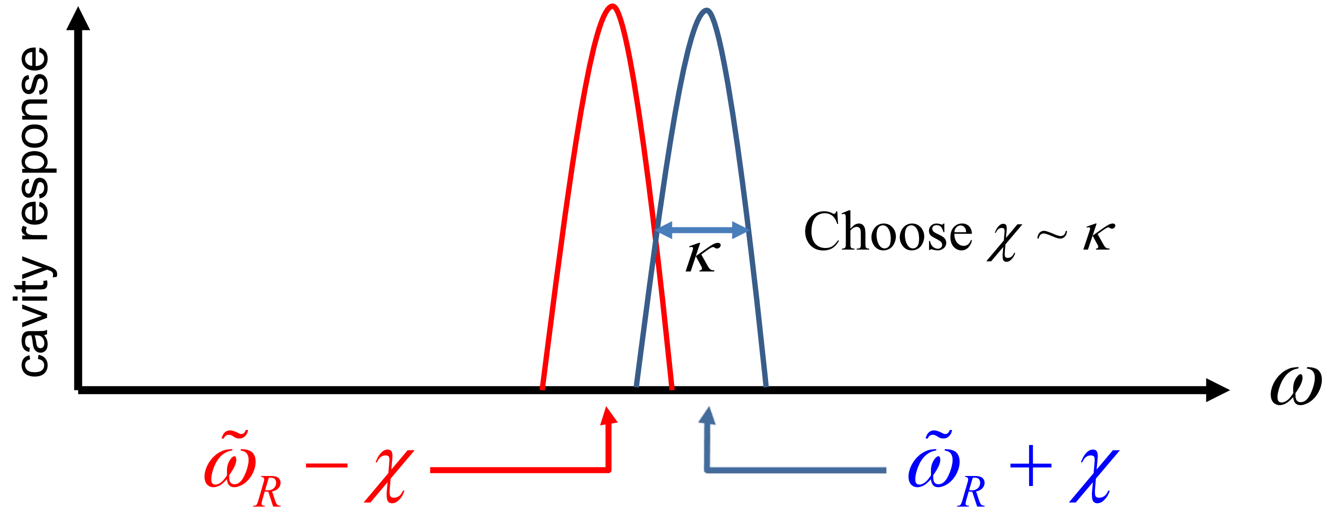
[Cavity frequency can be used to readout state of qubit]

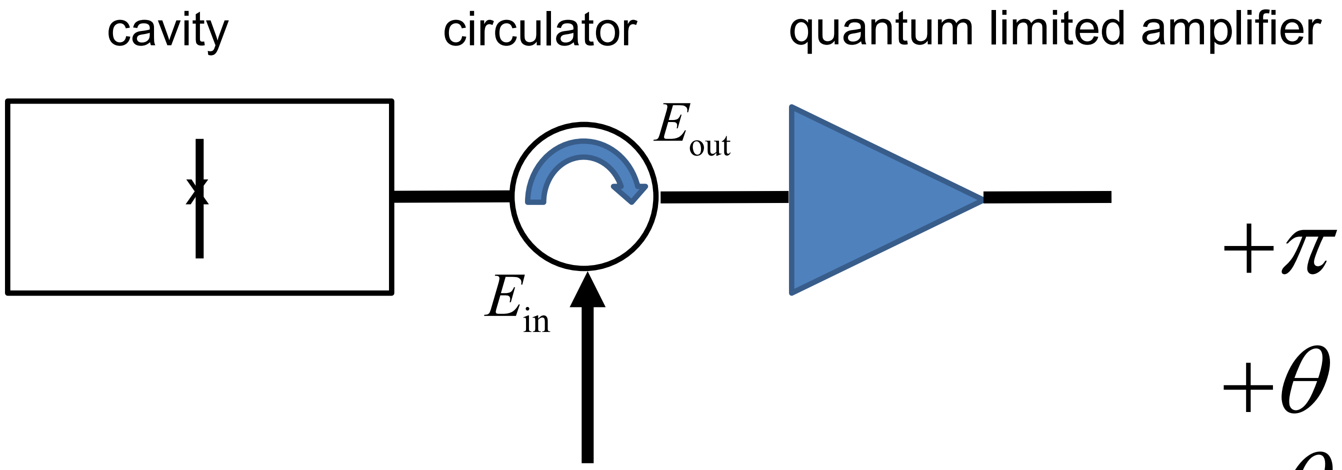


Using (not so) strong dispersive coupling to measure the state of the qubit

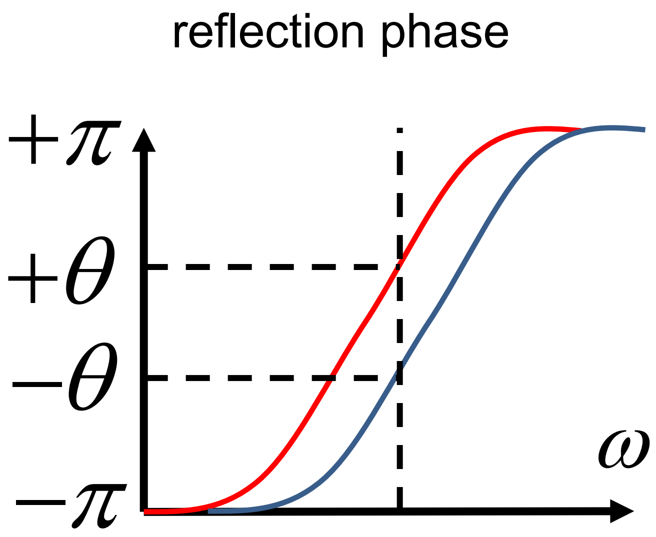
Additional notes:
The S matrix for reflection of microwave photon from a resonator is derived in the separate PDF document ‘Reflection from a resonator’

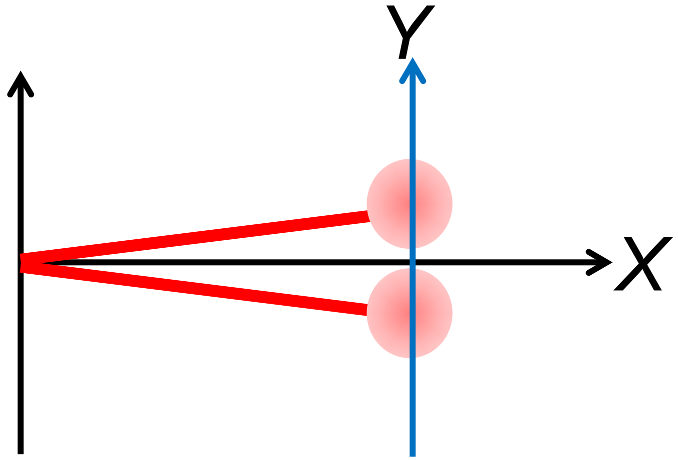
Can read out qubit state by measuring cavity resonance frequency





$$E_{\text{out}}(\omega) = e^{i\theta(\omega)} E_{\text{in}}(\omega)$$

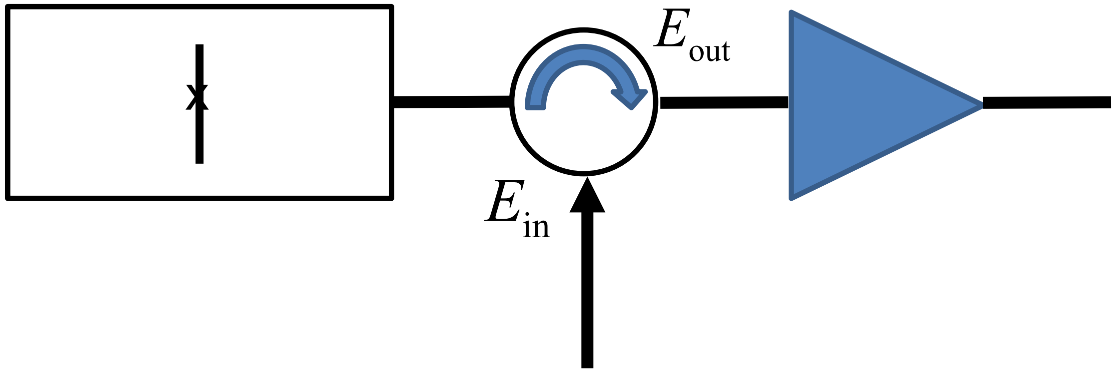




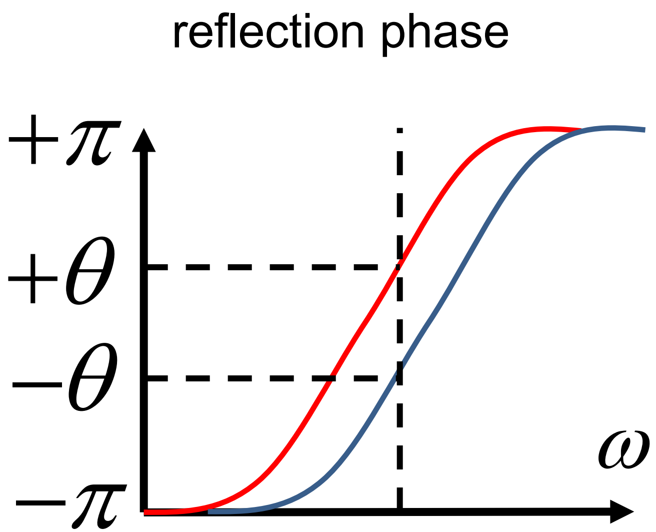
$$|\psi_{\text{in}}\rangle = \{a|\uparrow\rangle + b|\downarrow\rangle\}|\alpha\rangle$$

$$|\psi_{\text{out}}\rangle = a|e^{+i\theta}\alpha\rangle|\uparrow\rangle + b|e^{-i\theta}\alpha\rangle|\downarrow\rangle$$

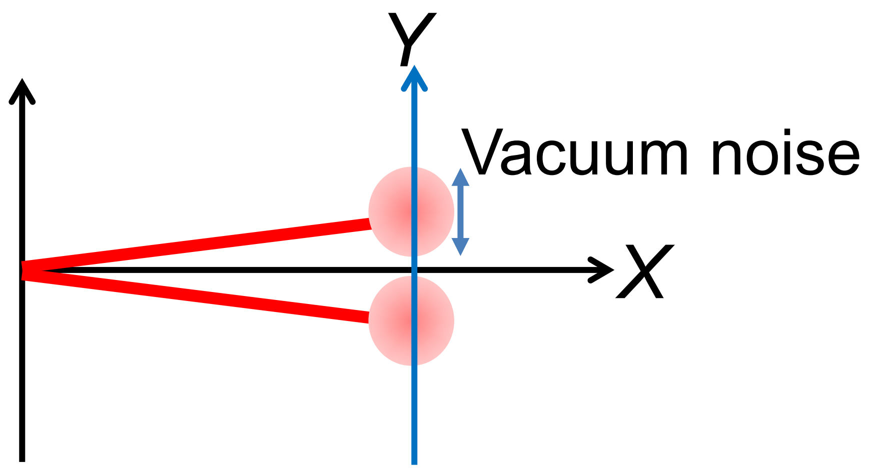
State of qubit is entangled with the 'meter' (microwave phase)
Then 'meter' is read with amplifier.



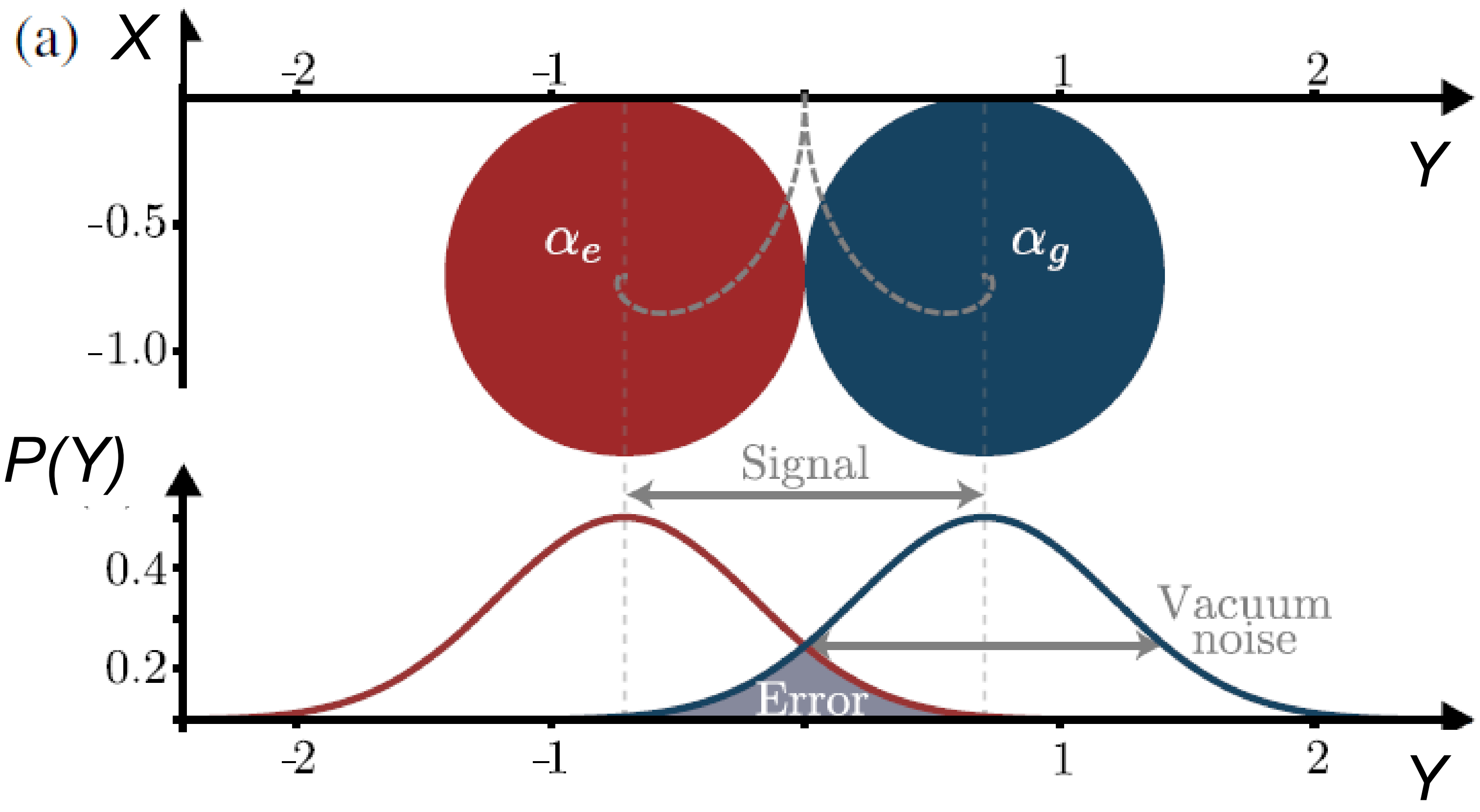
$$E_{\text{out}}(\omega) = e^{i\theta(\omega)}E_{\text{in}}(\omega)$$



Readout fidelity
vacuum noise → shot noise

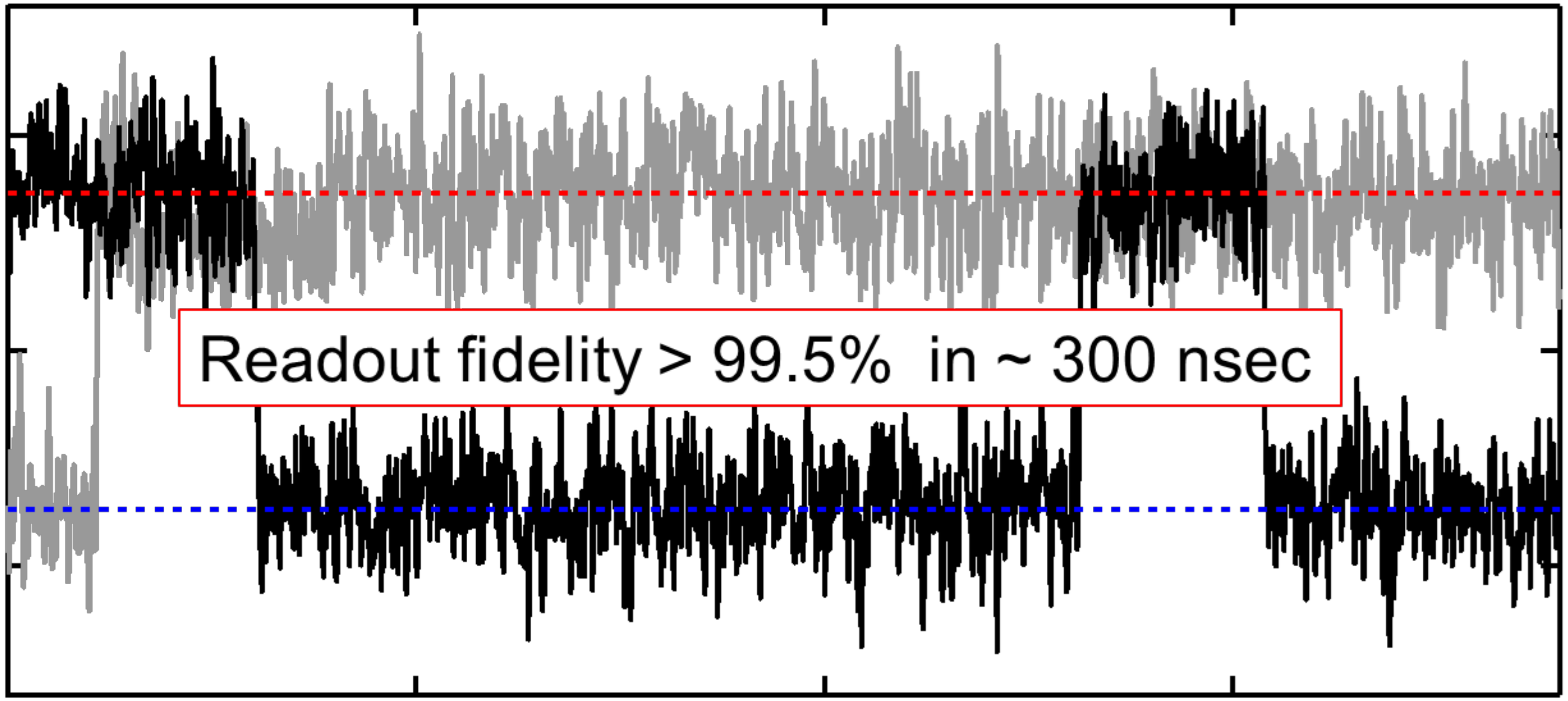


Quadrature amplitudes X, Y are canonically conjugate, leading to quantum vacuum noise



Using (not so) strong dispersive coupling to measure the state of the qubit

Dispersive readout proposed in: Blais et al., Phys. Rev. A 69, 062320 (2004)
First experiment: Wallraff et al., Nature 431, 162 (2004)
Quantum limited amplifiers developed…
First single-shot quantum jumps observed: R. Vijay et al., Phys. Rev. Lett. 106, 110502 (2011)



Data from: M. Hatridge et al., Science 339, 178 (2013)

Using strong-dispersive coupling to measure the photon number distribution in a cavity

Strong Dispersive Hamiltonian

$$H = \omega_\mathrm{r} a^\dagger a + \frac{\omega_\mathrm{q}}{2}\sigma^z + \chi \sigma^z a^\dagger a + H_\mathrm{damping}$$

$$\chi >> \kappa, \Gamma$$

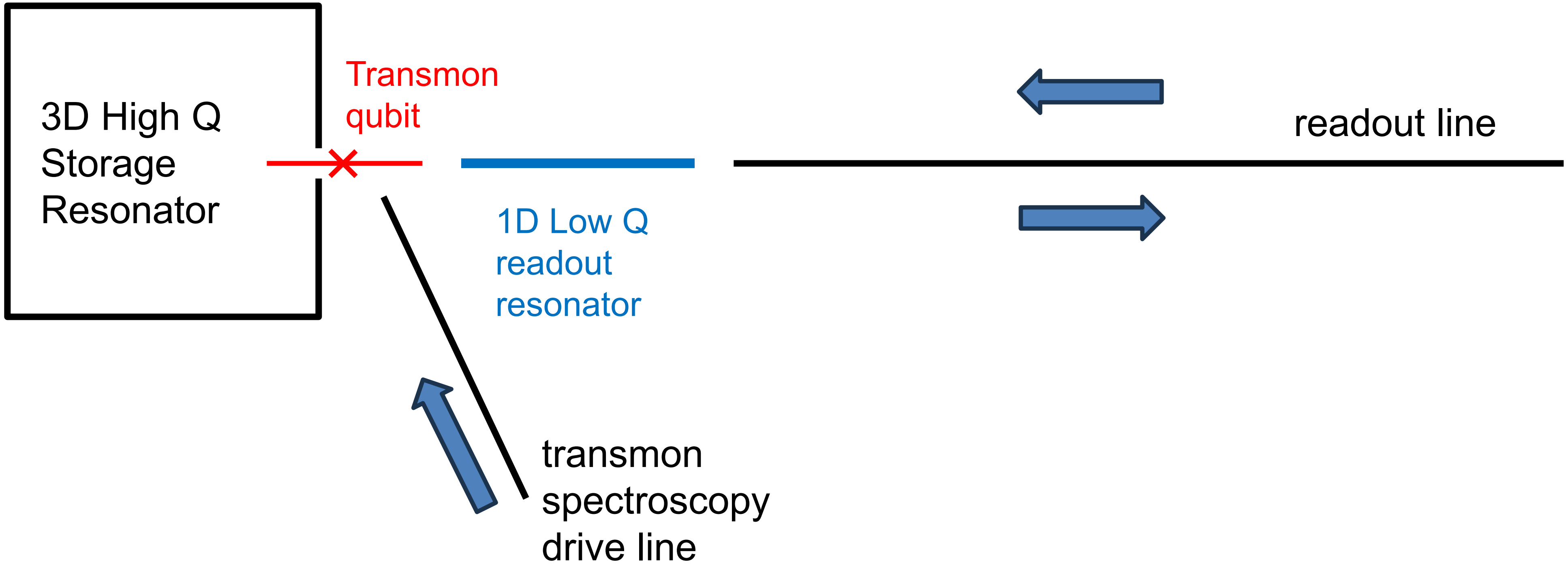
resonator qubit dispersive coupling

Reinterpretation of same Hamiltonian: Quantized Light Shift of Qubit Transition Frequency

$$H = \omega_\mathrm{r} a^\dagger a + \frac{1}{2}\sigma^z \left[\omega_\mathrm{q} + 2\chi a^\dagger a\right] + H_\mathrm{damping}$$

Spectrum of qubit depends on cavity photon number

Using strong-dispersive coupling to measure the photon number distribution in a cavity

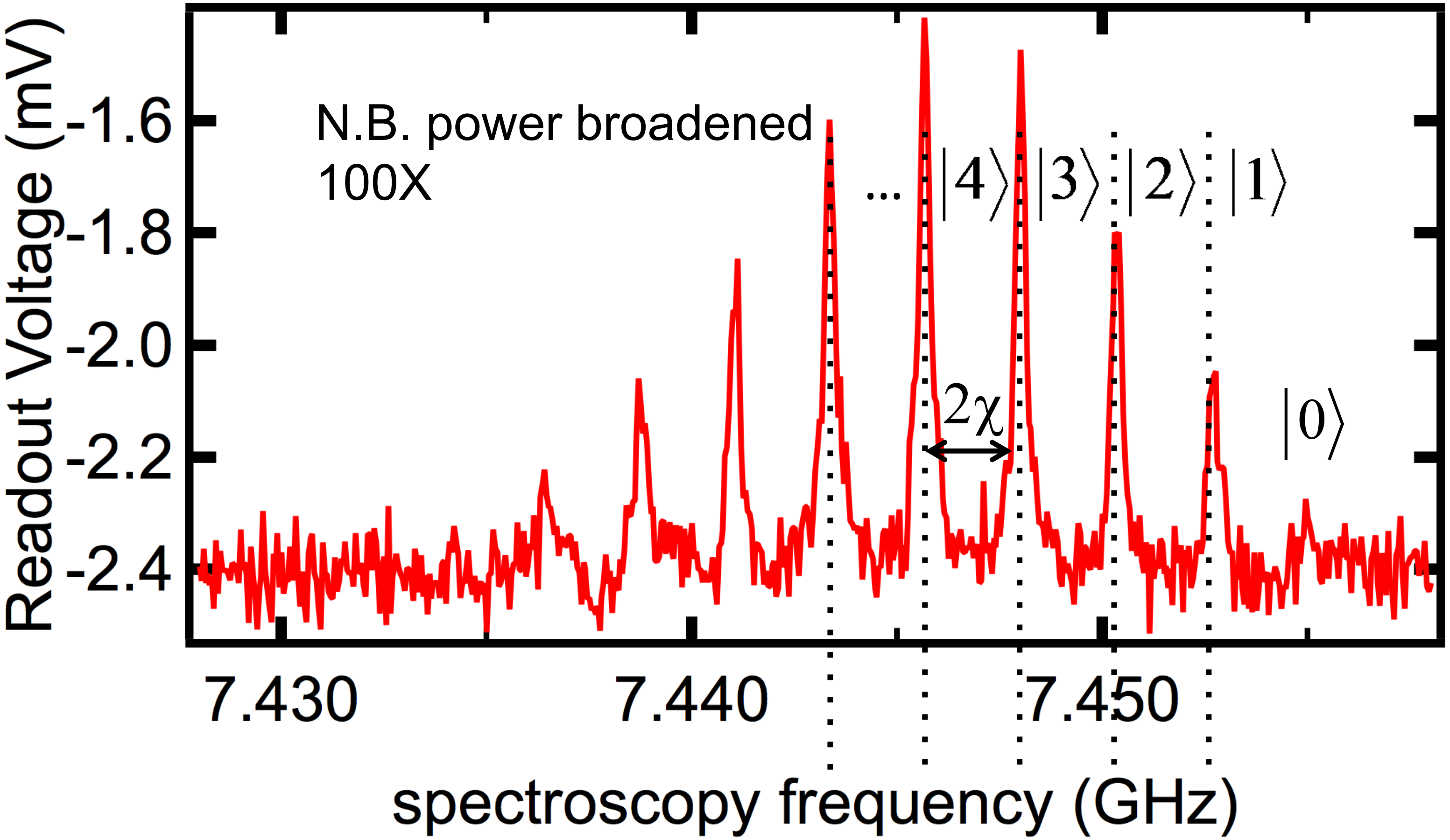


Measure photon number in high Q storage cavity via dispersive coupling to transmon.

Readout transmon state via dispersive coupling to low Q readout resonator.

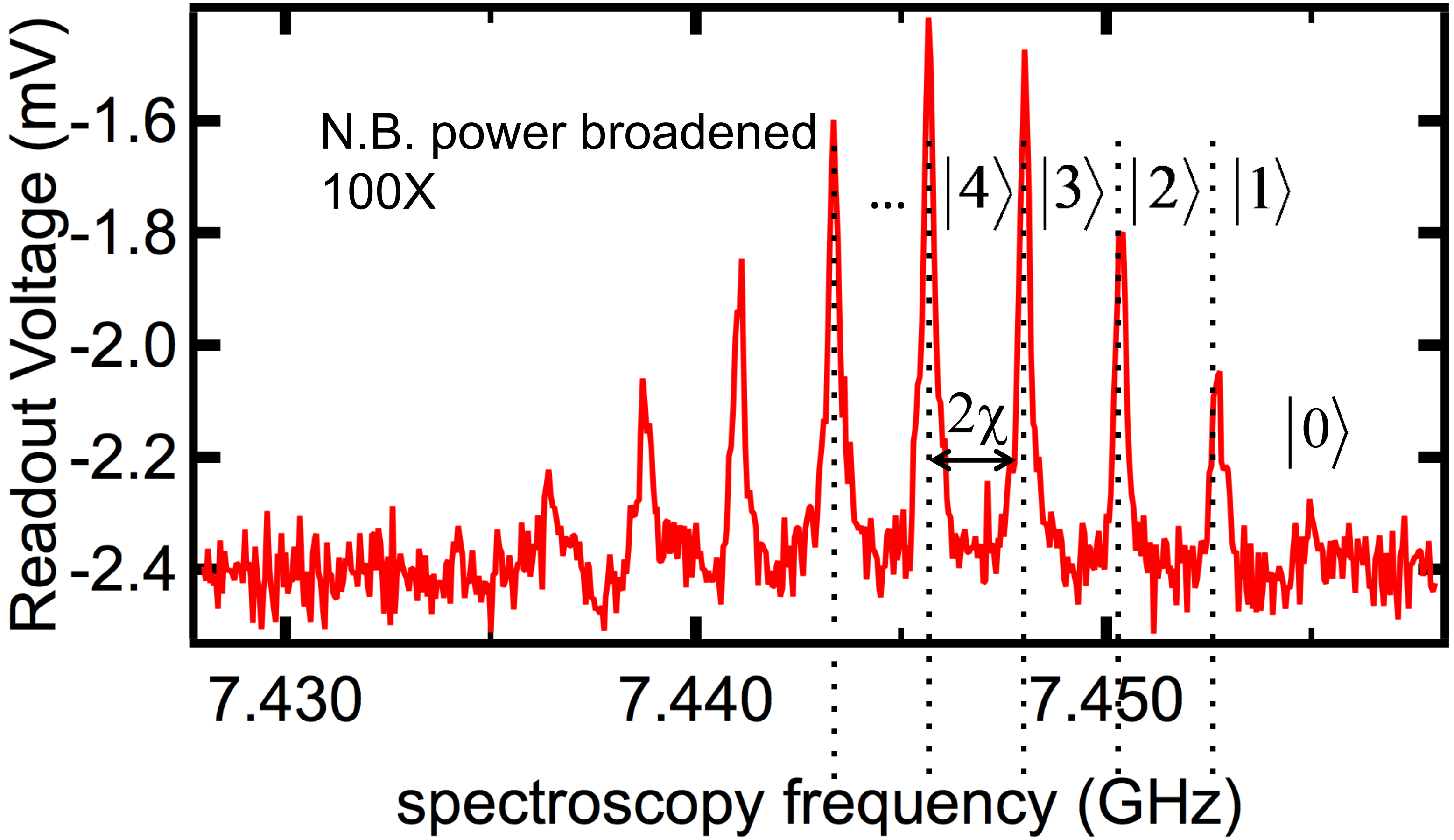
- quantized light shift of qubit frequency
  (coherent microwave state)

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



- quantized light shift of qubit frequency (coherent microwave state)

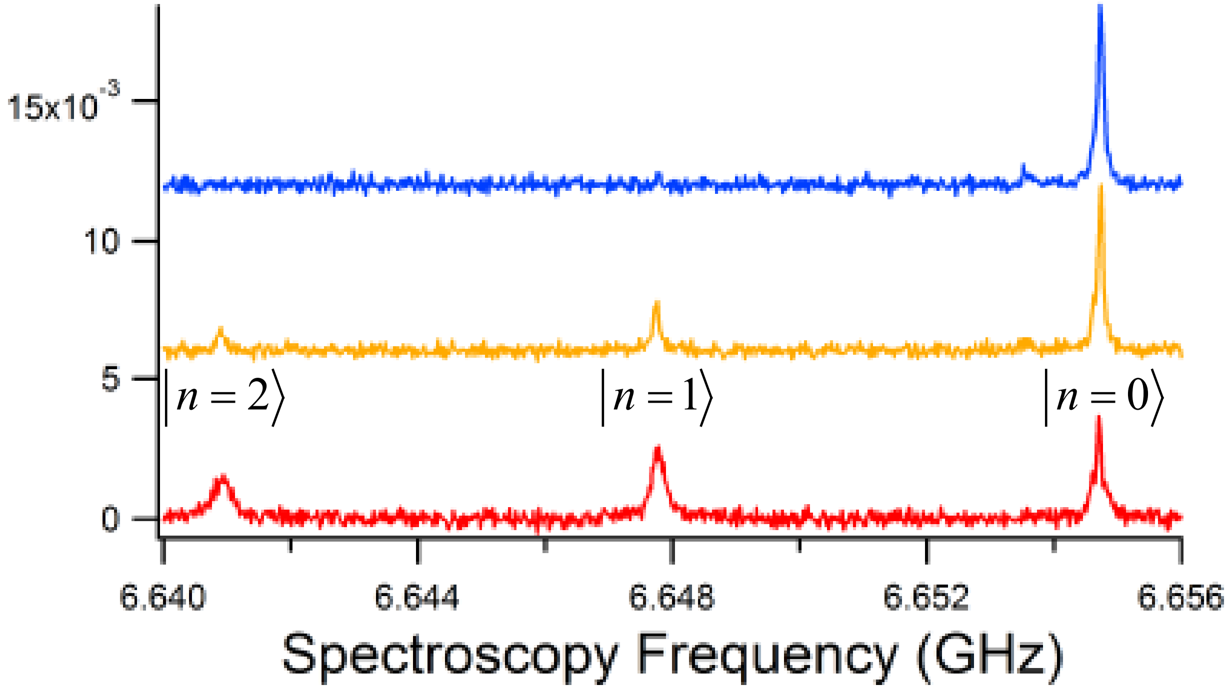
$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



Microwaves are particles!

- quantized light shift of qubit frequency
  (coherent microwave state)

$$\frac{\omega_q + 2\chi a^\dagger a}{2}\sigma^z$$



New low-noise way to do axion dark matter detection by QND photon counting
Zheng et al. arXiv:1607.02529 → A. Chou: PRL **126**, 141302 (2021)

Photon number parity

$$\hat{P} = (-1)^{a^\dagger a} = \sum_{n=0}^{\infty} |n\rangle (-1)^n \langle n|$$

Remarkably <u>easy</u> to measure using our quantum engineering toolbox

and

Measurement is 99.8% QND

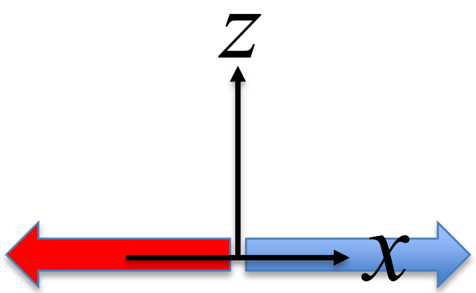
Measuring Photon Number Parity

- use quantized light shift of qubit frequency

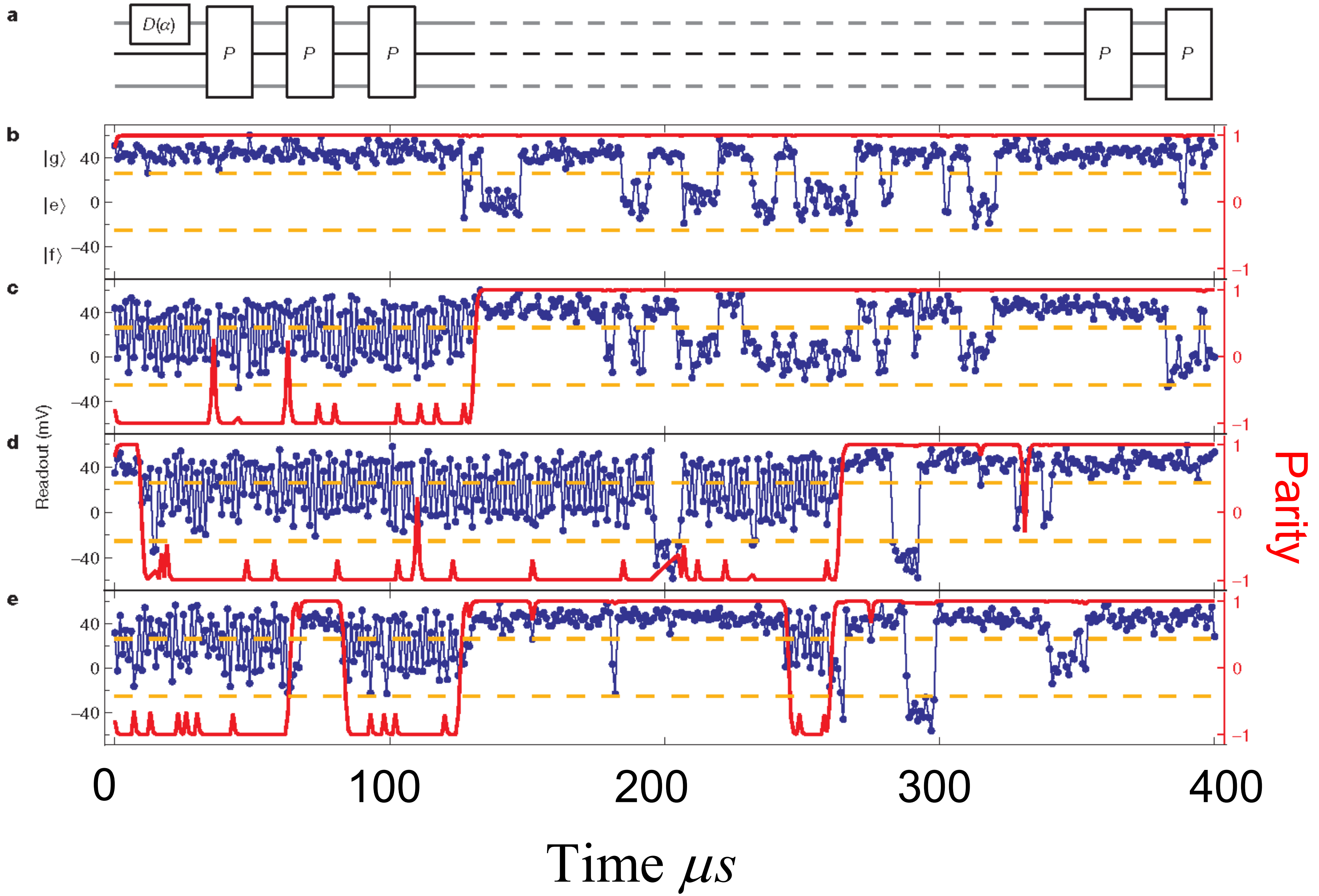
$$\frac{\omega_{\mathrm{q}} + 2\chi a^{\dagger} a}{2} \sigma^{z}$$

$$e^{-i 2\chi \hat{n} t \frac{\sigma^{z}}{2}} = \mathrm{e}^{-i\pi \hat{n} \frac{\sigma^{z}}{2}}$$

$\hat{n} = 1, 3, 5, \ldots$



$\hat{n} = 0, 2, 4, \ldots$



Nature **511**, 444 (2014)

400 consecutive parity measurements (99.8% QND)

Summary of Lecture I:

Introduction to Circuit QED

- What is Cavity QED?
- Quantum LC Oscillators
- Josephson Junctions & Transmon Qubits
- Qubits coupled to microwave cavities
  - Control and measurement of both qubit and cavity