

Fractional Statistics and Anyons

for TQFT seminar, Boulder Summer School 2010

Note Title

7/29/2010

§0. Introduction

- Exchange two identical particles
⇒ wave function acquires +1 or -1
 - ↙ bosons
 - ↙ fermions
 - Particles obey statistics other than Bose/Fermi Statistics ⇒ Anyons
 - Ways to describe anyons:
 - Path Integral Formalism (Quantum Mechanics)
 - Chern-Simons gauge field theory
 - Algebraic ways:
 - * Braid Group
 - * Modular Tensor Category
-

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§1. Path Integral formalism of Quantum Statistics

(most material in §1 are from QFT course at the Univ.
of Utah Lectured by Prof. Yong-Shi Wu)

* One particle Path Integral

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}x(t) e^{\frac{i}{\hbar} S[x]}$$

$x(t_i) = x_i$
 $x(t_f) = x_f$

Configuration space ℓ

. is trivial : if any two paths can be continuously deformed into each other.

first

$$\pi_1(\ell) = 0$$

then no complication arises.

homotopy group

. is non-trivial : $\pi_1(\ell) \neq 0$

paths are divided into different classes
(topological sectors). Only paths in the same class can be deformed into each other

e.g. ① One particle on a plane : trivial

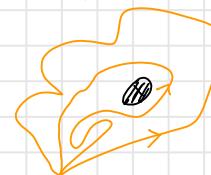
$$\pi_1(\mathbb{R}^2) = 0$$

② With one point excluded on a plane

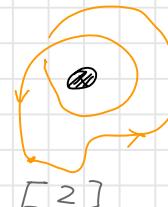
$$\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$$



[0]



[1]



[2]

paths are classified by winding numbers

paths in distinct classes carry different weights!

$$\Rightarrow \sum_{[e] \in \pi_1(\ell)} e^{i\theta([e])} \int D\chi(t) e^{\frac{i}{\hbar} S[\chi]} \text{ paths in } [e]$$

Constraint on the weight $\theta([e])$:

$$\text{Composition rule: } e^{i\theta([\ell_1 \cdot \ell_2])} = e^{i\theta([\ell_1])} e^{i\theta([\ell_2])}$$

Theorem: $e^{i\theta([e])}$ is a 1-dim representation of $\pi_1(\ell)$.

* N identical particles in Euclidean space (in d -dim)

$$\text{Configuration space} = (\underbrace{\mathbb{R}^d \times \dots \times \mathbb{R}^d}_{N \text{ copies}} - \{\text{diag. points}\}) / \mathcal{G}_N$$

exclude any configuration with $\vec{x}_i = \vec{x}_j$ for some $i \neq j$

Topology of the Configuration space:

1) in $d > 2$, $\pi_1(\ell) = S_N$

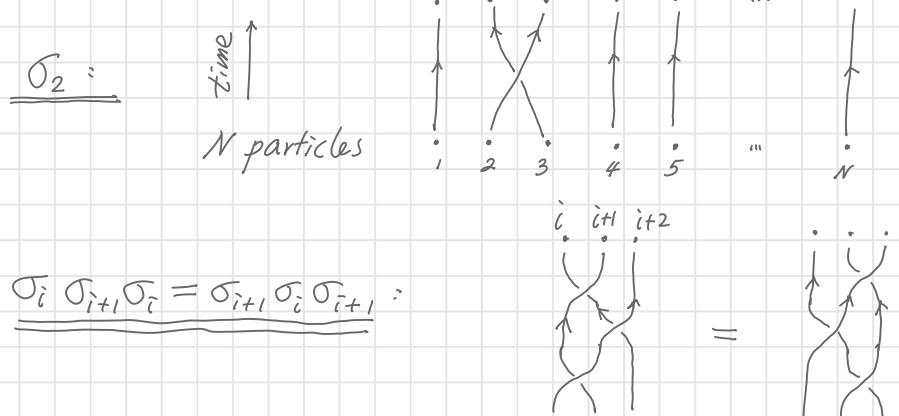
2) in $d = 2$, $\pi_1(\ell) = B_N \Rightarrow \text{Braid Group}$

Review: Braid Group B_N is generated by $\{\sigma_i\}$, $i=1, \dots, N-1$
 ("generated": a group element is a product of σ_i 's)
 satisfying

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

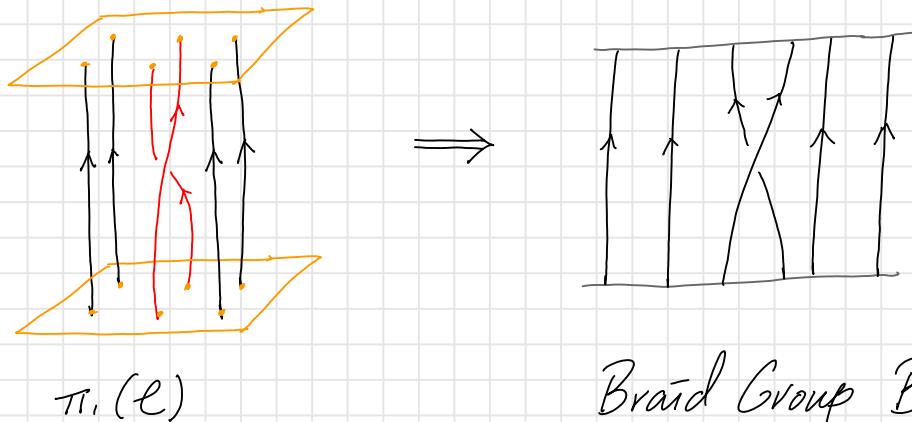
Graphical presentation:

σ_i : to exchange particle i and $i+1$



if add condition $\sigma_i^{-1} = \sigma_i$, B_N becomes S_N

To recognize $\pi_1(\mathcal{C})$ as B_N :



* Quantum Statistics

$e^{i\theta(\mathcal{C})}$: 1-dim representation of B_N
 \Rightarrow abelian anyons

higher-dim representation of B_N
 \Rightarrow non-abelian anyons

Examples:

- Quasiparticles in Fractional Quantum Hall system at $v = \frac{1}{3}, \frac{1}{5}, \dots$
 \Rightarrow abelian anyons

- Quasiparticles in Fractional Quantum Hall system at $v = \frac{5}{2}$
 \Rightarrow non-abelian anyons

* Topological Term

The question: Can we absorb $\theta(\mathcal{C})$ in the action $S[x]$?

The trick: find a total derivative term.

Integral of total derivative \Rightarrow topological term.

$$S_{\text{top}}[x] = -\frac{\theta}{\pi} \int dt \frac{d}{dt} \sum_{i < j} \arg(\vec{x}_i - \vec{x}_j)$$

- Topological Lagrangian does not affect Equation of Motion.

- but contributes in quantum theory.

Wilczek's Anyon model

$$L_{\text{top}}[\vec{x}] = -\frac{\theta}{2\pi} \frac{d}{dt} \sum_{i \neq j} \arg(\vec{x}_i - \vec{x}_j)$$

$$= -\frac{\theta}{\pi} \sum_{i \neq j} \frac{(\frac{d}{dt} \vec{x}_i) \cdot (\hat{z} \times (\vec{x}_i - \vec{x}_j))}{|\vec{x}_i - \vec{x}_j|^2}$$

$$\arg(\vec{r}) = \tan^{-1} \frac{r_y}{r_x}$$

Legendre transformation gives us an extra term called "topological term".

For free particles,

$$H = \sum_i \frac{1}{2m} (\hat{p}_i - \vec{A}_i(\vec{x}_i))^2$$

$$\text{with } \vec{A}_i = \frac{\theta}{\pi} \sum_{i \neq j} \frac{(\vec{x}_i - \vec{x}_j) \times \hat{z}}{|\vec{x}_i - \vec{x}_j|^2}$$

creating perpendicular magnetic field of strength

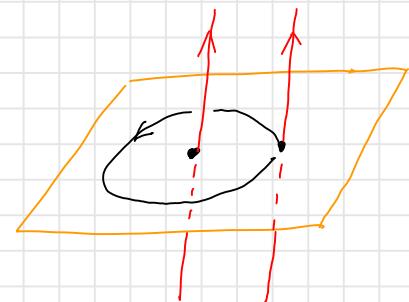
$$\vec{\nabla}_i \times \vec{A}_i = 2\theta \sum_{i \neq j} S(\vec{x}_i - \vec{x}_j)$$

physical interpretation: flux tubes centered at the coordinates of particles penetrating the plane!

⇒ Wilczek's anyon model

charge-flux composite:

$$\theta = \frac{1}{2} q \phi$$



move one particle around another one
↔ exchange two particles twice

Chern-Simons Theory

$$\mathcal{L} = \mathcal{L}_0 + \frac{k}{4\pi} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma + a_\mu \bar{j}^\mu$$

Equation of Motion

$$\frac{k}{2\pi} \epsilon^{\mu\nu\sigma} \partial_\nu a_\sigma = -\bar{j}^\mu$$

$$\int d^2x (\partial_1 a_2 - \partial_2 a_1) = -\frac{2\pi}{k} j^0$$

Effect of Chern-Simons term:

to endow the charged particles with
the flux $\frac{2\pi}{k}$

$$\Rightarrow \text{Anyon statistics } \frac{\Theta}{\pi} = \frac{1}{k}$$

§2. Quantum Double model

§2.1 topological quantum field theory,

Review: Topological Quantum field theory. (TQFT)

① Def: a quantum field theory whose partition function is topologically invariant.

(under any $x^\mu \mapsto x'^\mu = \lambda^\mu_\nu x^\nu$)

that preserve the topology of spacetime)

② TQFT has a constant Hamiltonian

$$H = 0$$

Obtained by:

In most cases
are trivial

Restrict any Hamiltonian to its ground states

by Integrating higher energy degrees of freedom

③ two kinds of well-studied TQFT

- Levin-Wen model

(Other names: String-net, Toric code,
Quantum Doubles...)

- Chern-Simons theory



Theoretically
well-understood



no Hamiltonian formulation
on lattice



Physical Relevance:
Not Clear!



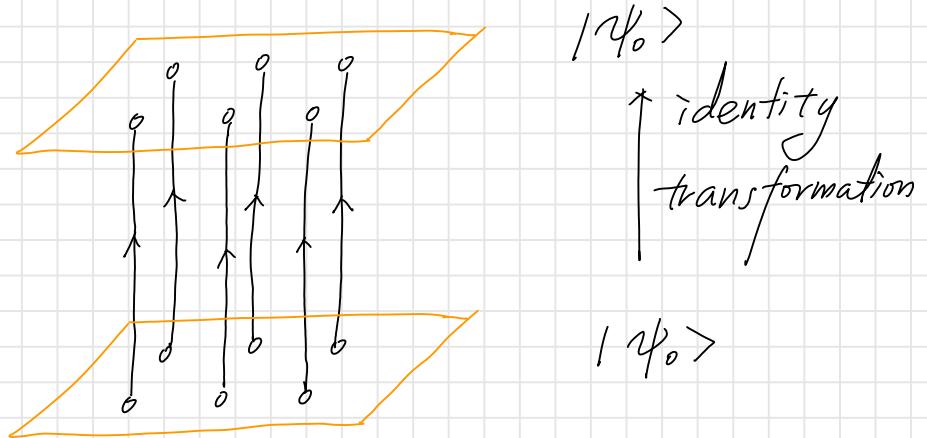
Fractional Quantum
Hall liquids

(Do Levin-Wen type TQFT in this lecture)

* Anyons from TQFT

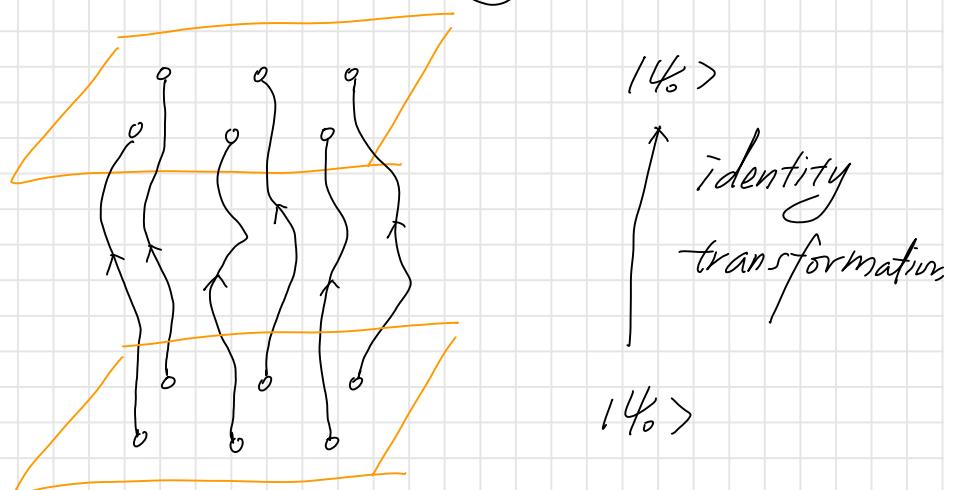
Consider a sphere with N "punctures"

Time evolution
of TQFT



but, TQFT does not "know" the change of the metric:

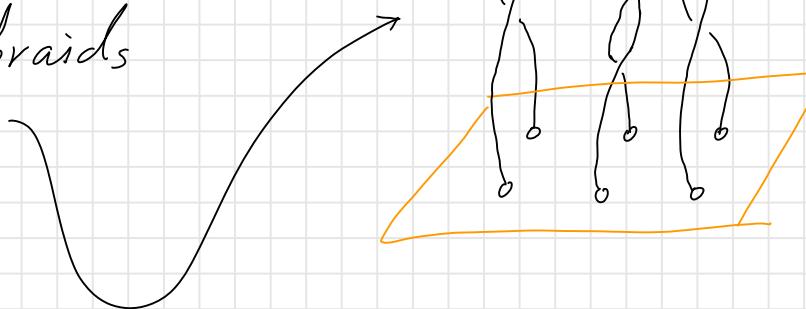
move these
punctures,
finally go back
to the same
configuration



If Ground States are degenerate, and move punctures around each other:

worldlines of punctures

are braids



$U_{ab} |\psi_0^b \rangle$

unitary transformation

$|\psi_0^a \rangle$

Time evolution of TQFT is given by
Braid Group Representations

"punctures" called anyons.

Example : Chern-Simons theory

$$\mathcal{L} = \mathcal{L}_0 + \frac{k}{4\pi} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma + a_\mu j^\mu$$

let j^μ , $\begin{cases} j^0 = \sum_{i=1}^N \delta(\vec{x} - \vec{x}_i) \\ \vec{j} = 0 \end{cases}$

creates N punctures

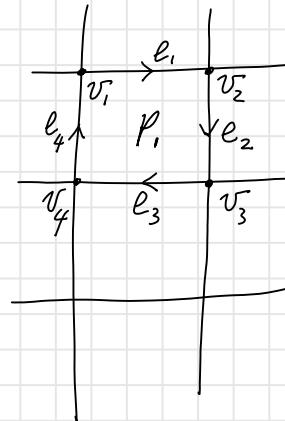
§2.2 Quantum Double model

- * is a Levin-Wen type TQFT
- * Generalization of Toric code
- * Lattice gauge theory

Lattice Gauge theory could be any random graph

Given a graph Γ consisting

- vertices labelled by $\{v_1, v_2, \dots\} = V$
- (oriented) edges " " $\{e_1, e_2, \dots\} = E$
- plaquette " " $\{P_1, P_2, \dots\} = P$



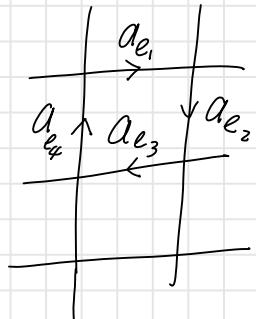
Given a finite group G .

Hilbert space spanned by

$$\left\{ |a_{e_1}, a_{e_2}, a_{e_3}, \dots, a_{e_E}\rangle \right\}$$

$$\text{dimension} = |G|^{\#E}$$

assign a group element to each edge

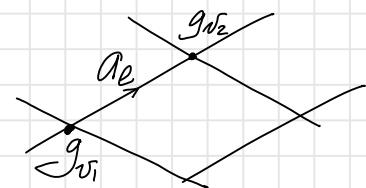


. local Gauge transformation

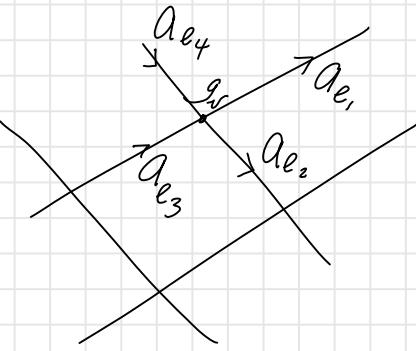
assign a group element $\{g_v\}$ to vertices $\{v\}$

action on the state

$$|\dots a_e \dots\rangle \mapsto |\dots, g_{v_2} a_e g_{v_1}^{-1}, \dots\rangle$$



$$g_v | \alpha_{e_1}, \alpha_{e_2}, \alpha_{e_3}, \alpha_{e_4}, \dots \rangle \\ = | \alpha_{e_1} g_v^{-1}, \alpha_{e_2} g_v^{-1}, g_v \alpha_{e_3}, g_v \alpha_{e_4} \dots \rangle$$

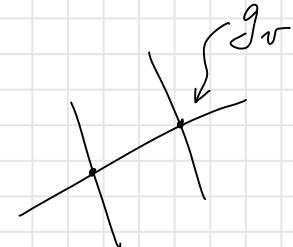


The model

prefers no charge

$$H = - \sum_v \hat{A}_v - \sum_p \hat{B}_p$$

$$\hat{A}_v |\psi\rangle = \frac{1}{|G|} \sum_g g_v |\psi\rangle$$



$$\hat{B}_p \left| \begin{array}{c} a_{e_4} \\ a_{e_1} \\ a_{e_2} \\ a_{e_3} \end{array} \right\rangle = S_{a_{e_4} a_{e_3} a_{e_2} a_{e_1}} \left| \begin{array}{c} a_{e_4} \\ a_{e_1} \\ a_{e_2} \\ a_{e_3} \end{array} \right\rangle$$

prefers no gauge flux

S -function:

$$S_g = \frac{1}{|G|} \sum_{p \in \text{irreps}} \text{tr}_p(g)$$

identity element

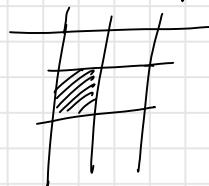
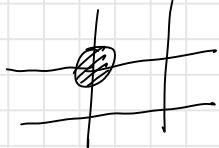
$$= \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

- excitations are described by violations of local constraints A_v, B_p .

Construct Excited States

How does excitations look like?

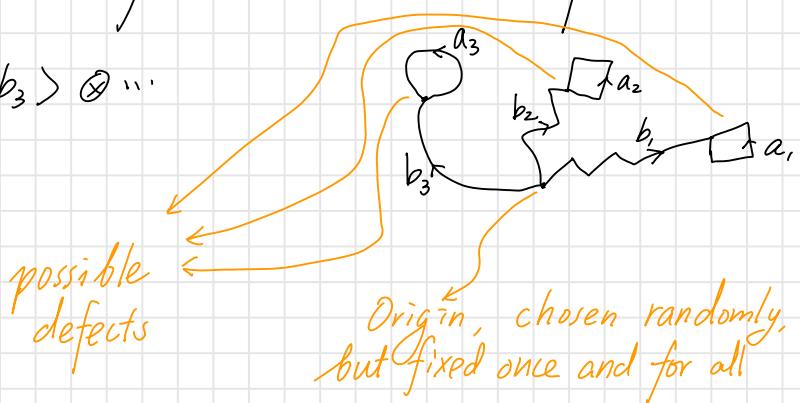
- Think of violations of A_v, B_p as defects or "punctures"



Excitations with violations of A_v, B_p \Rightarrow Ground states on the graph with defects at v, p

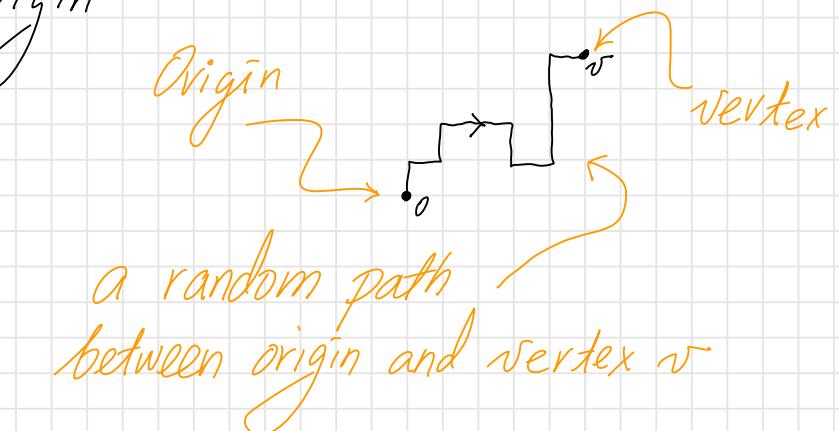
Ground States with defects in terms of basis

$$|a_1, b_1\rangle \otimes |a_2, b_2\rangle \otimes |a_3, b_3\rangle \otimes \dots$$



Ground States on a graph with defects :

- Step 1 :
- Choose one vertex as origin
 - For any vertex v other than the origin O , choose a path ($O \rightarrow v$) connecting it to the origin



Step 2 :

If we know a_e on each edge

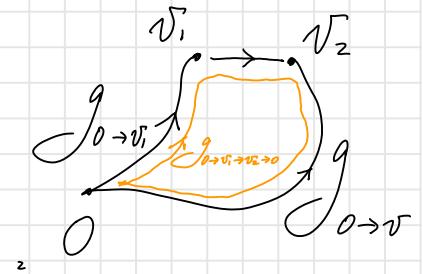
- \Rightarrow
- can know $g_{O \rightarrow v}$ as the product of all group elements along the path $O \rightarrow v$
 - know holonomy along arbitrary loops

but from r.h.s.,

if we start with $\mathcal{G}_{0 \rightarrow v_i}$
 $\mathcal{G}_{0 \rightarrow v_2}$

and the holonomy \mathcal{G}

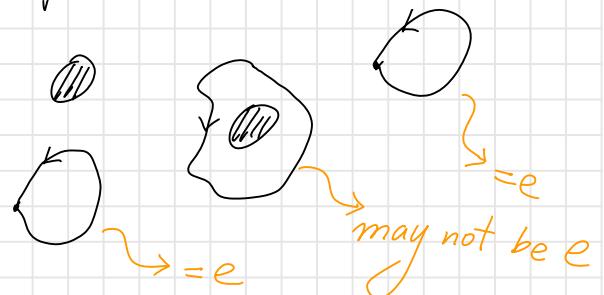
$\mathcal{G}_{0 \rightarrow v_i \rightarrow v_2 \rightarrow 0}$



then we can find out the group element on the edge $v_i \rightarrow v_2 =$

$$a_{e(v_i \rightarrow v_2)} = \mathcal{G}_{0 \rightarrow v_2} \mathcal{G}_{0 \rightarrow v_i \rightarrow v_2 \rightarrow 0} (\mathcal{G}_{0 \rightarrow v_i})^{-1}$$

Step 3. For ground state, holonomy along any loop without defects included is the identity element e



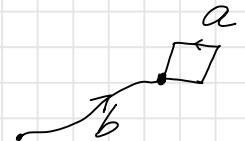
only holonomy along a defect is important.

Step 4: Ground States are in basis of

$(|a_1, b_1\rangle \otimes |a_2, b_2\rangle \otimes \dots)$ tensored by $\frac{1}{|G|} \sum_g |g\rangle$
on all trivial paths.

* Local actions on excitations

$$\begin{cases} \hat{P}_h |a, b\rangle = S_{a,h} |a, b\rangle & \text{flux operator} \\ \hat{g} |a, b\rangle = |g a g^{-1}, b\rangle & \text{charge operator} \end{cases}$$



These actions form an algebra generated by $\{P_h, \mathcal{G}\}$ obeying the relations

$$\left\{ \begin{array}{l} \hat{P}_{h_1} \hat{P}_{h_2} = S_{h_1 h_2} \hat{P}_{h_1}, \quad , \quad \hat{g}_1 \cdot \hat{g}_2 = \underbrace{\hat{g}_1 g_2}_{\text{group multiplication}} \\ \hat{g} \cdot \hat{P}_h = \hat{P}_{ghg^{-1}} \hat{g} \end{array} \right.$$

called Quantum Double of G , denote $D(G)$

For Abelian group G , $gP_h = P_hg \Rightarrow D(G) \cong G \times G$

Example = \mathbb{Z}_2 case: anyons are $\{1, e, \epsilon, e\epsilon\}$

Excitations are "anyons"

- "particle-like":

 - in Quantum mechanics, particles are labelled by irreducible representations of a Group (Symmetry)
 - excitations here are labelled by irrep's of Quantum Double of G denote $D(G)$

Flux/Charge eigenvectors described by:

- Unitary Irreducible representations
of Quantum Double of S_3

Flux (conjugacy class)	Centralizer	Charge	Basis vectors
$C_1 = \{e\}$	$Z(e) = S_3$	[+]	$ +\rangle$
		[-]	$ -\rangle$
		[2]	$ 2_+\rangle, 2_-\rangle$
$C_2 = \{(12), (23), (31)\}$	$Z((12)) = \{e, (12)\}$	+	$ (12)\rangle, (23)\rangle, (31)\rangle$
		-	$ (12), -\rangle, (23), -\rangle, (31), -\rangle$
$C_3 = \{(123), (132)\}$	$Z((123)) = \{e, (123), (132)\}$	1	$ (123)\rangle, (132)\rangle$
		ω	$ (123), \omega\rangle, (132), \omega\rangle$
		$\bar{\omega}$	$ (123), \bar{\omega}\rangle, (132), \bar{\omega}\rangle$

How to read the Table:

1. Find the Conjugacy classes ${}^A C = \{{}^A h_1, {}^A h_2, \dots\}$ of S_3 :

$$C_1 = \{e\}, C_2 = \{(12), (23), (31)\}, C_3 = \{(123), (132)\}$$

with a representative ${}^A h_i$ in each C_A

$${}^1 h_1 = e, {}^2 h_1 = (12), {}^3 h_1 = (123)$$

2. Find out Centralizer of each Representative ${}^A h_i$,

$${}^A N = \{g \mid g {}^A h_i = {}^A h_i, g, g \in S_3\}$$

3. Find out all irreducible representations of ${}^A N$ labelled as ${}^A \rho_\alpha$

Anyons : $\begin{cases} [+]: \text{trivial representation} \\ [-] \end{cases}$

	ρ_1	ρ_2	ρ_3
C_1	$ e,+\rangle$	$ e,-\rangle$	$ e,2+\rangle$ $ e,2-\rangle$
C_2	$ (12),+\rangle$ $ (23),+\rangle$ $ (31),+\rangle$	$ (12),-\rangle$ $ (23),-\rangle$ $ (31),-\rangle$	
C_3	$ (123),+\rangle$ $ 132,+\rangle$	$ (123),\omega\rangle$ $ 132,\omega\rangle$	$ (123),\omega^*\rangle$ $ 132,\omega^*\rangle$

charges:
(on sites)

no violation
of flux operator
 $P_e |e, \alpha_V\rangle = |e, \alpha_V\rangle$

α_V is a vector
in Representation space
 ρ_α in class C_i

pure vortices: (on plaquetts)

no violation of
charge operator

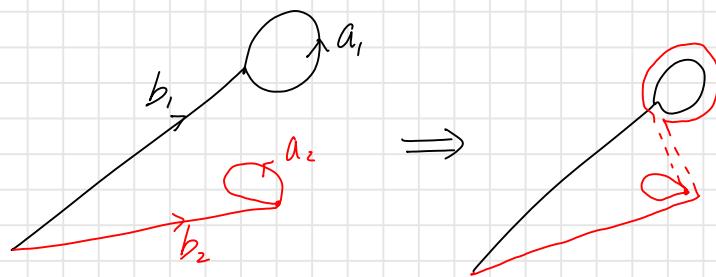
$$\hat{g} |{}^A h_i, +\rangle = |{}^A h_i, +\rangle$$

dyons:

occupies a site
and a plaquette

Quantum Statistics:

* Braiding:



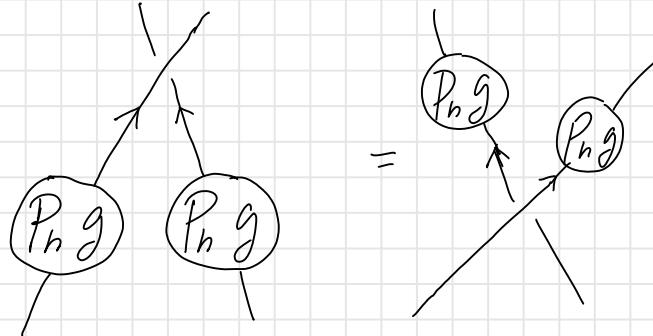
R-matrix:

$$R_{12} = |a_1, b_1\rangle \otimes |a_2, b_2\rangle$$

$$\mapsto \sum_{g,h} (P_h g |a_2, b_2\rangle) \otimes (P_g |a_1, b_1\rangle)$$

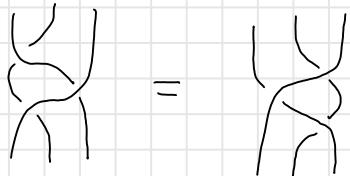
Properties:

(1)



Braiding commutes with global symmetry transformation g and conserve the flux measured by P_h

(2) Yang-Baxter equation



$$\text{twist (Spin value)}: e^{i\theta} = \frac{1}{\dim(\mathcal{A}_\alpha^F)} \operatorname{tr}_\alpha({}^A h_1)$$

Only dyons have non-trivial statistics:

	ρ_1	ρ_2	ρ_3
C_1	1	1	1
C_2	1	-1	N/A
C_3	1	$e^{\frac{2i\pi}{3}}$	$e^{-\frac{2i\pi}{3}}$

$$e^{i\theta} :$$