Hibert space fragmentation (HSF) and constrained dynamics

Frank Pollmann, TUM

Overview:
1 Recab: Eigenstate Thermalization Hypothesis
2 What is HSF?
3 Examples: Dipole conserving models
Pair flip models

Reading:

Review by Moudgalya et al.
https://arxiv.org/abs/2109.00548

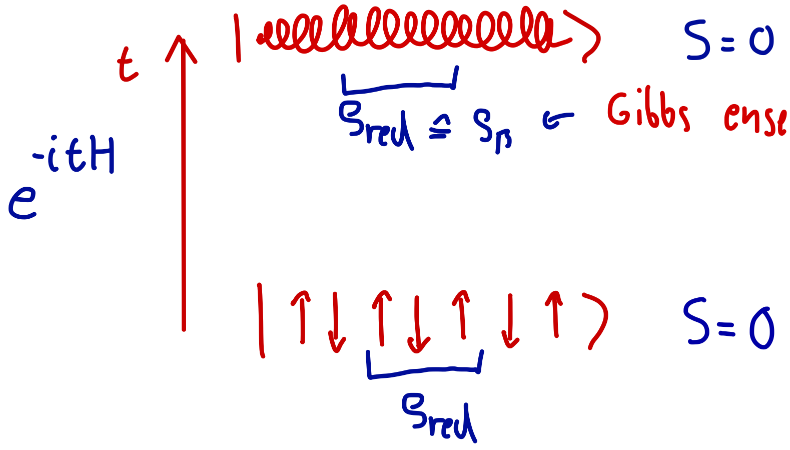
Sala et al.
https://arxiv.org/abs/1904.04266

Khemani et al.
https://arxiv.org/abs/1910.01137

1 Recab : Eigenstate Thermalization Hypothesis

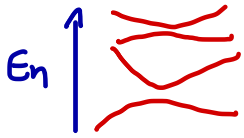
Characterizing the dynamics of quantum many-body systems:

i) Eigenstate thermalization hypothesis (ETH)



$\rho_{red} \triangleq \rho_\beta$ ← Gibbs ensemble, ETH $\langle O\rangle = f(O_{\alpha\alpha})$

ii) Level spacing statistics



$$s_n = \frac{E_{n+1} - E_n}{\bar{E}}$$

sorted energy levels

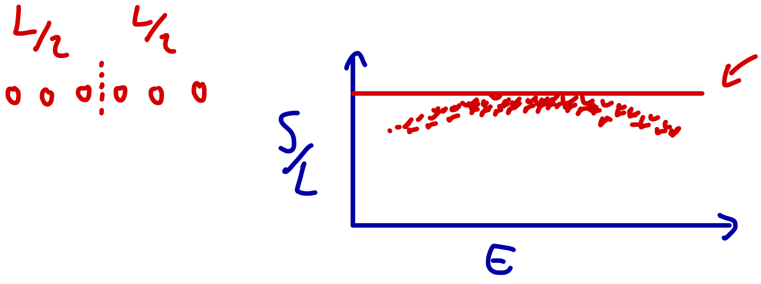
mean energy spacing

⤳ Ergodic : s_n follows Wigner Dyson dist.

⤳ non-Ergodic : s_n follows Poisson dist.

iii) Entanglement of eigenstates (at finite energy density)

Ergodic systems show volume law scaling!



Page: $S = \frac{L}{2}\log 2 - \frac{1}{2}$

For $S=\frac{1}{2}$.

Sub-volume law entanglement is a sign of (weak) ergodicity breaking.

Distinguish different types of dynamics in many-body systems:

	Strong/Weak ETH	Level statistics	Entanglement
Ergodic	✓ | ✓	WD	Volume
Integrable	× | ×	P	Volume/sub.V.
MBL	× | ×	P	Area
Quantum scars	× | ✓	WD	Volume/sub.V.
Weak HSF	× | ✓	WD	Volume / sub. V.
Strong HSF	× | ×	P	Volume / sub. V.

I What is HSF?

We decompose the Hilbert space $\mathcal{H}$ for a given Hamiltonian H into dynamically connected Krylov subspaces K_j:

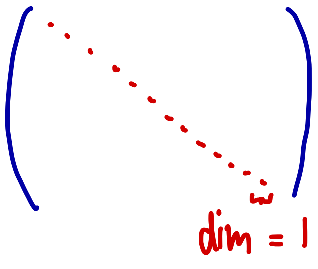
$$\mathcal{H} = \bigoplus_j K_j \quad \text{with} \quad K_j = \{|\psi_j\rangle, H|\psi_j\rangle, H^2|\psi_j\rangle, \dots\}.$$

(Note that $|\psi_j\rangle$ are chosen such that the K_j are distinct).

Some general remarks:

⤳ We find d^L sectors if $|\psi_j\rangle$ are eigenstates

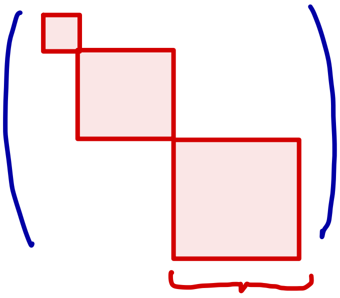
$H =$



$\#\text{sectors} = d^L$

(local Hilbert space dimension d and L sites)

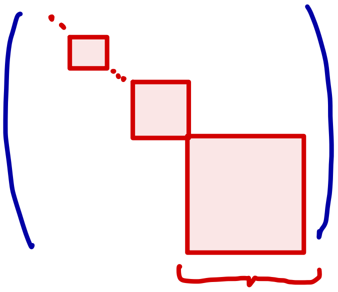
⇝ In presence of global symmetries (e.g., spin conservation), we find different sectors labeled by quantum numbers (e.g., S^2_{total} for S^2 eigenstates)

$H =$  $\# \text{sectors} = \text{poly}(L)$

$\dim \sim \text{poly}(L) \cdot d^L$

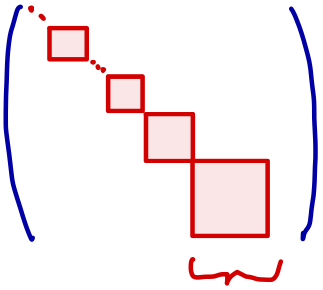
Definition of HSF:

Starting from "simple" $|\psi_i\rangle$ (e.g. product states), we find exponentially many disconnected Krylov sectors. These sectors are not labeled by any obvious local symmetry of the Hamiltonian.

"Weak HSF": $H =$  $\# \text{sectors} = \exp(L)$

$\dim \sim \text{poly}(L) \cdot d^L$

⇝ Largest sector thermalizes!

"Strong HSF": $H =$  $\# \text{sectors} = \exp(L)$

$\dim \sim \tilde{d}^L$ with $\tilde{d} < d$

⇝ None of the sectors thermalizes!

HSF arises in several models with constrained dynamics and exhibits a number of interesting physical phenomena:

⇝ Initial state dependent thermalization

⇝ Dynamical phase transitions

⇝ Protected edges modes

⇝ Subdiffusive transport

In the following, we will discuss dipole conserving and pairflip models as concrete examples that exhibit HSF.

(see slides)

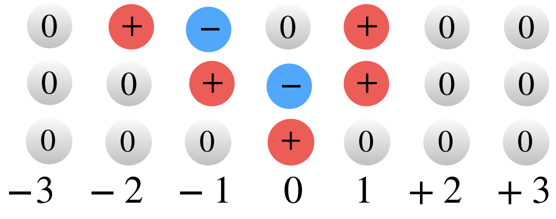
Hilbert Space Fragmentation:
Dipole conserving models

Charge and dipole conservation in spin-1 chains

Conservation of a U(1) charge Q and its associated dipole moment P in a spin-1 chain:

$$Q = \sum_n S_n^z \qquad\qquad P = \sum_n (n - n_0) S_n^z$$

- Combination of Q and P symmetry puts constrains on the mobility of excitations: **“fractons”:** $|\pm\rangle = S^{\pm}|0\rangle$ [Pretko '18]



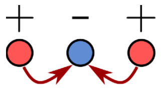
0	+	−	0	+	0	0
0	0	+	−	+	0	0
0	0	0	+	0	0	0
-3	-2	-1	0	1	$+2$	$+3$

$$Q = 1 \quad P = 0$$

- Certain random local unitary dynamics with such symmetries fail to thermalize: How robust is this phenomenon? [Pai et al '18]

Charge and dipole conservation in spin-1 chains

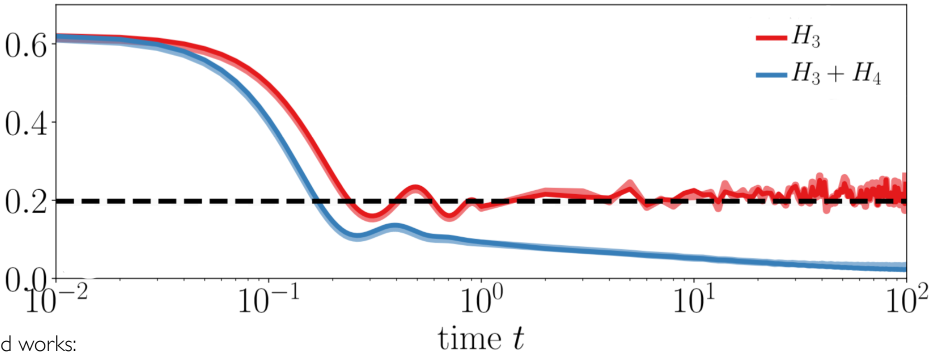
Q and P conserving spin-1 Hamiltonians



$$H_3 = -\sum_n \left[S_n^+ \left(S_{n+1}^-\right)^2 S_{n+2}^+ + \mathrm{H.c.}\right]$$

$$H_4 = -\sum_n \left[S_n^+ S_{n+1}^- S_{n+2}^- S_{n+3}^+ + \mathrm{H.c.}\right]$$

Autocorrelation function at $T = \infty: C(t) = \langle S_0^z(t) S_0^z(0) \rangle - 2/3N$

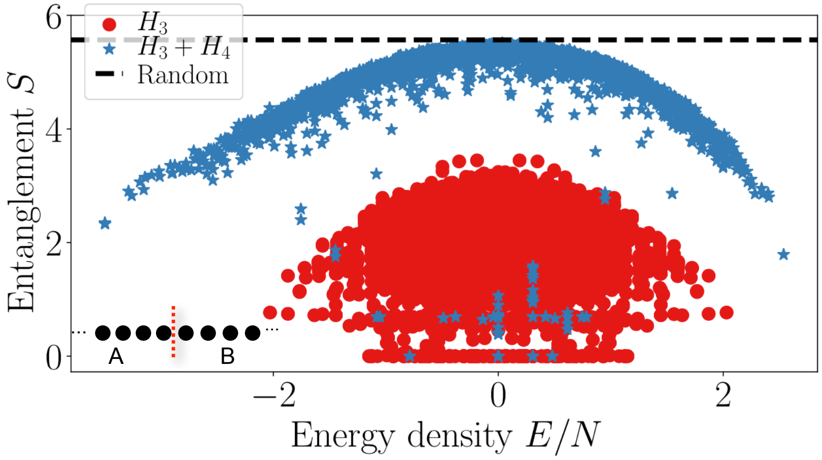


Related works:
Khemani, Hermele, and Nandkishore PRB '20
Moudgalya et al. arXiv:1910.14048

[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

Charge and dipole conservation in spin-1 chains

Half chain entanglement entropy of the eigenstates



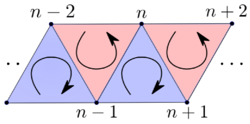
- None of the eigenstates of H_3 are thermal

[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

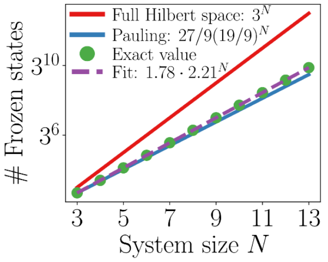
Hilbert space fragmentation

Breakdown of thermalization of H_3 due to fragmentation of the sectors defined by (Q,P)

- Exponentially many frozen states: Pauling estimate



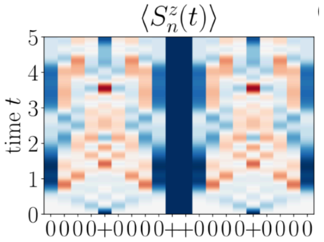
$$3^N \times (19/27)^{N-2}$$



- Sign of the right and left most charges in any region of space is conserved

+ + 0 0 + 0 0 + +

− − 0 − + 0 0 + +



[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

Hilbert space fragmentation

Largest connected sector of H_3 maps to spin-1/2 XY model

- Configurations where subsequent charges have opposite signs (arbitrary number of zeros): Spin-1/2 on bonds

− 0 0 0 + 0 0 0 − + 0 0 0 0

↓ ↓ ↓ ↓ ↑ ↑ ↑ ↑ ↓ ↑ ↑ ↑ ↑

- N-1 free spins → 2^{N-1} states (x4)
- Hamiltonian $S^+_{n-1}\left(S^-_n\right)^2 S^+_{n+1} + \text{h.c.} \rightarrow \sigma^+_{n-1,n}\sigma^-_{n,n+1} + \text{h. c.}$

Largest connected sector has dimension $D = \binom{N-1}{(N-1)/2}$

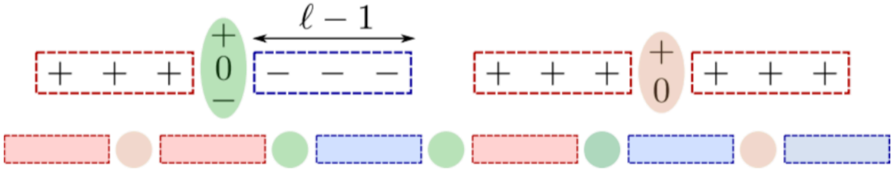
Vanishingly small fraction of states: "Strong Fragmentation"

[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

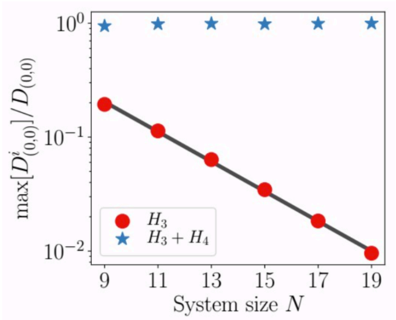
Hilbert space fragmentation

How is the Hamiltonian $H_3 + H_4$ different?

- Still exponentially many frozen states



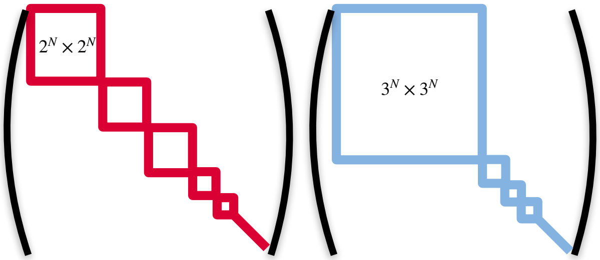
- Largest connected sector has almost all states in it: **"Weak Fragmentation"**



[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

Hilbert space fragmentation

	H_3	H_3+H_4
Fragmentation	Strong	Weak
# of sectors	~exp[N]	~exp[N]
Size of largest	~ $2^N \times 2^N$	~ $3^N \times 3^N$

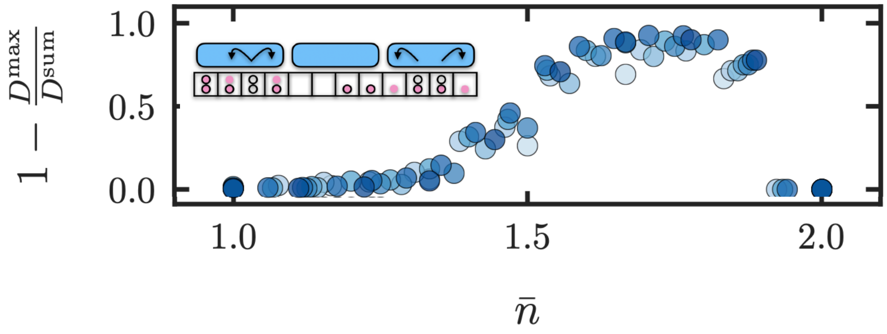


[**Sala**, Rakovszky, Verresen, Knap, FP, PRX **10**, 011047 (2020)]

Hilbert space fragmentation

Fragmentation transition as function of the density

[Morningstar, Khemani, Huse PRB '20]



- As the charge density is tuned away from half filling there is a phase transition to a frozen phase

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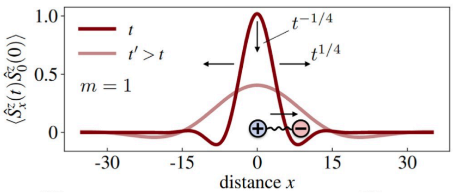
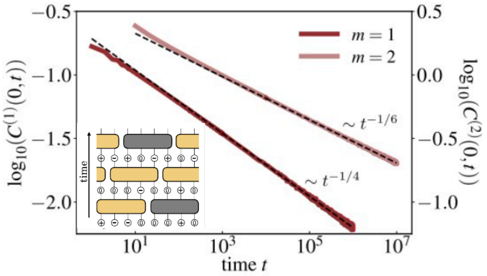
Hydrodynamics with higher-moment conservation

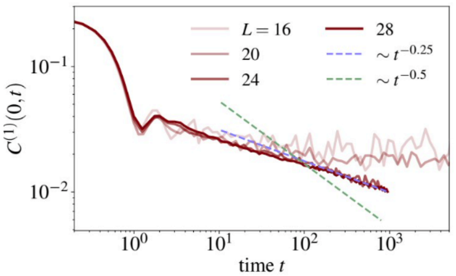
Universal late-time features at infinite temperature: Conservation of m'th moment $\mathcal{Q}^{(m)} = \sum_x x^m S^z_x$

$$\partial_t \langle \hat{S}^z_x \rangle = -D(-1)^{m+1} \partial_x^{2(m+1)} \langle \hat{S}^z_x \rangle$$

$$C_0(t) \sim t^{-1/2(m+1)}$$



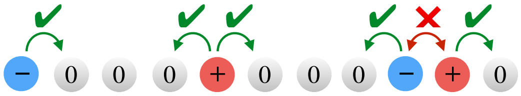




cf. [Gromov et al. 2020], [Zhang 2020] [**Feldmeier**, Sala, De Tomasi, FP, Knap, PRL **125**, 245303, 2020]

Statistically localized integrals of motions (SLIOMs)

Illustrative example of SLIOMS (tJ_z-model)



Spin of the k-th particle from the left $\hat{q}_k \equiv \sum_{i=1}^{L} \mathcal{O}_i^k = \sum_{i=1}^{L} \hat{\mathcal{P}}_i^k S_i^z$

For i>k and averaged over the Haar ensemble:

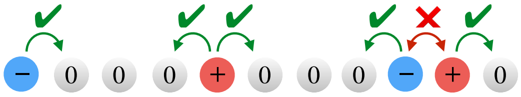
$$p_{\text{Haar}}(i;k) \equiv \mathbb{E}_{\text{Haar}}[\langle\psi|\mathcal{O}_i^{k\dagger}\mathcal{O}_i^k|\psi\rangle] = \nu^k(1-\nu)^{i-k}\binom{i-1}{k-1}$$

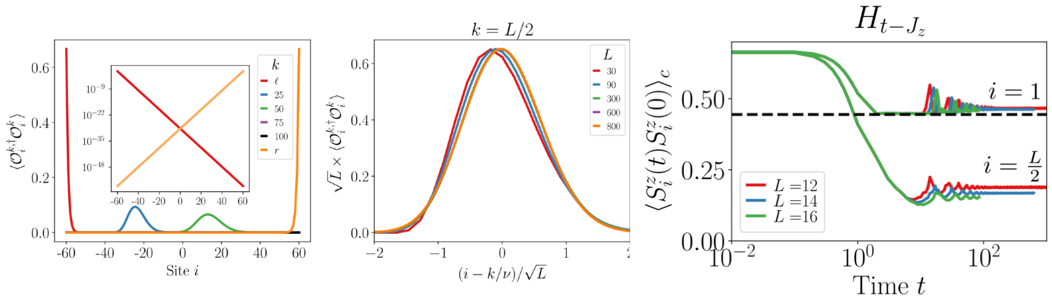
- Peaked around the position $k\nu$
- Left most charge (k=1) decays exponentially
- Standard deviation in the bulk is $\sim \sqrt{L}$

[**Rakovszky, Sala**, Verresen, Knap, FP, PRB **101**, 125126 (2020)]

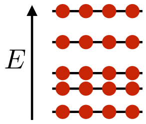
Statistically localized integrals of motions (SLIOMs)

Illustrative example of SLIOMS





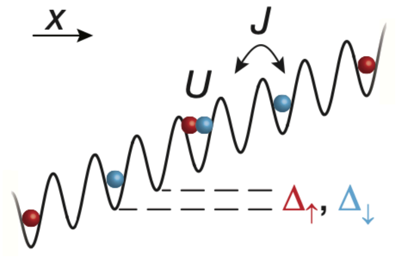
Exponentially localized edge spins yield **strong zero modes** in the presence of additional symmetries (e.g., $R_x = \prod_j e^{i\pi S_j^x}$) [Fendley '16]



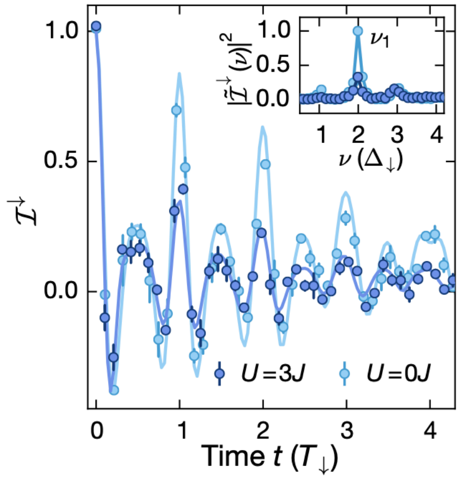
[**Rakovszky, Sala**, Verresen, Knap, FP, PRB **101**, 125126 (2020)]

Kinetic constraints in tilted Fermi-Hubbard chains

Experimental setup and short time dynamics



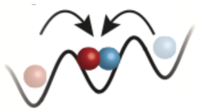
$$\hat{H} = \sum_{i,\sigma=\uparrow,\downarrow} \left(-J\hat{c}^{\dagger}_{i,\sigma}\hat{c}_{i+1,\sigma} + \text{h.c.} + \Delta_\sigma i\hat{n}_{i,\sigma}\right) + U\sum_{i}\hat{n}_{i,\uparrow}\hat{n}_{i,\downarrow}\,,$$



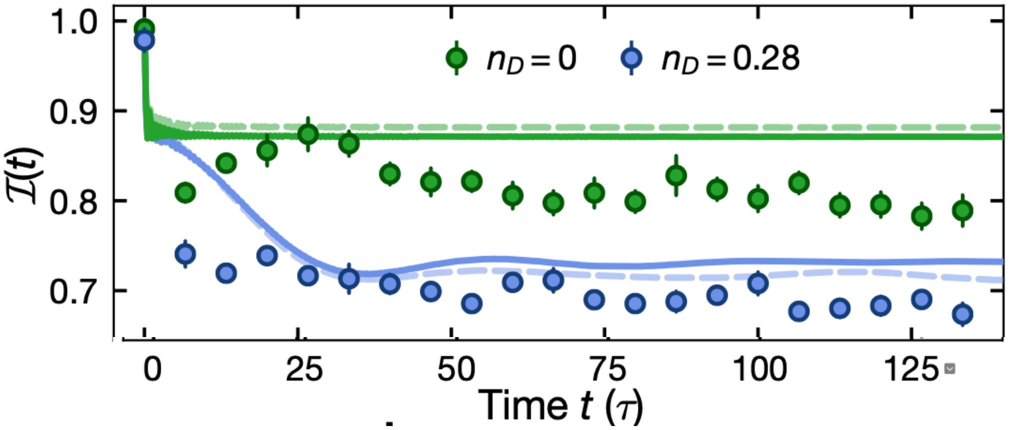
[Scherg et al., Nat. comm. **12**, 4490 (2021)]

Kinetic constraints in tilted Fermi-Hubbard chains

Effective Hamiltonian dynamics for $|U| \ll \Delta, J$

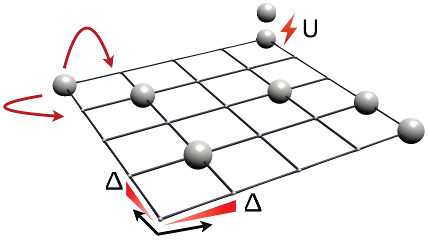


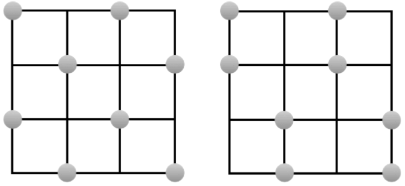
$\Delta/J = 8.0(2)$, $U/J = 2.7(2)$

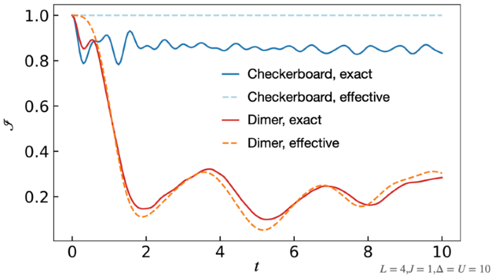


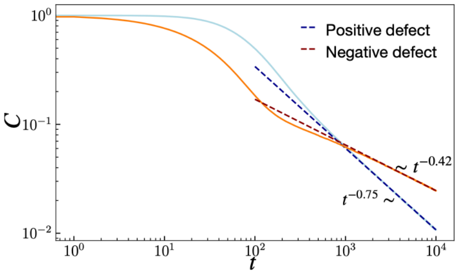
[Kohlert et al., PRL 130, 010201 (2023)]

HSF Fragmentation in a 2D Bose Hubbard model









[👉 Poster by Melissa Will]

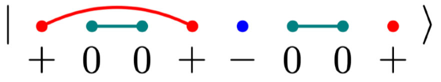
Hilbert Space Fragmentation: Pair-flip models

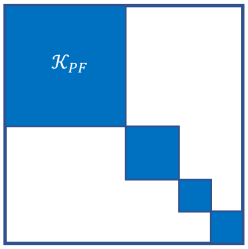
From Classical to Quantum Fragmentation

Fragmentation: Spin-1 Pair-flip model [Moudgalya & Motrunich '22]

$$H_{\mathrm{PF}} = \sum_{j=1}^{L-1} \sum_{\alpha,\beta=1}^{3} [g_{j,j+1}^{\alpha\beta} (|\alpha\alpha\rangle\langle\beta\beta|)_{j,j+1} + \mathrm{h.c.}]$$

Fragmentation in product state basis: "Classical Fragmentation"

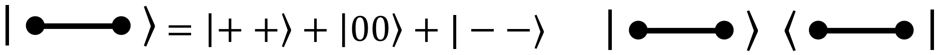




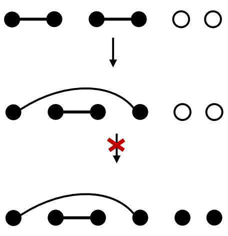
From Classical to Quantum Fragmentation

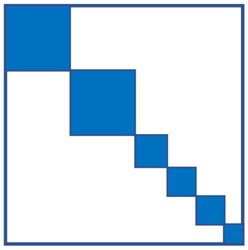
Fragmentation: Spin-1 Temperly-Lieb model [Moudgalya & Motrunich '22]

$$H_{\mathrm{TL}} = \sum_{j} \sum_{\alpha,\beta=1}^{3} J_j (|\alpha\alpha\rangle\langle\beta\beta|)_{j,j+1} .$$



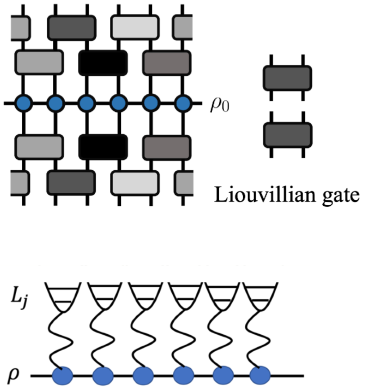
Fragmentation in entangled state basis: "Quantum Fragmentation"

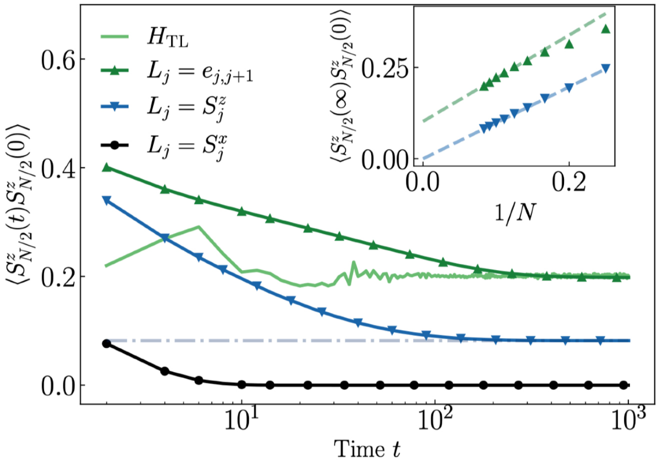




Open System: Temperly-Lieb model

Dephasing noise (Z_j) versus structure preserving noise (e_j)

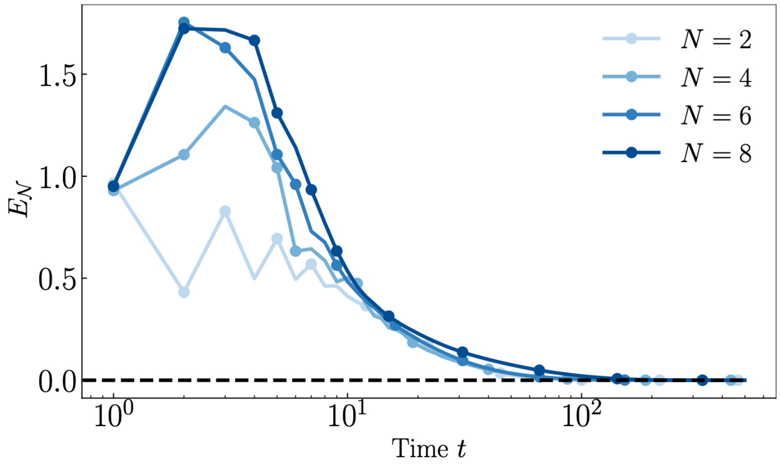




[Li, Sala, FP, arxiv:2305.06918]

Open System: Temperly-Lieb model

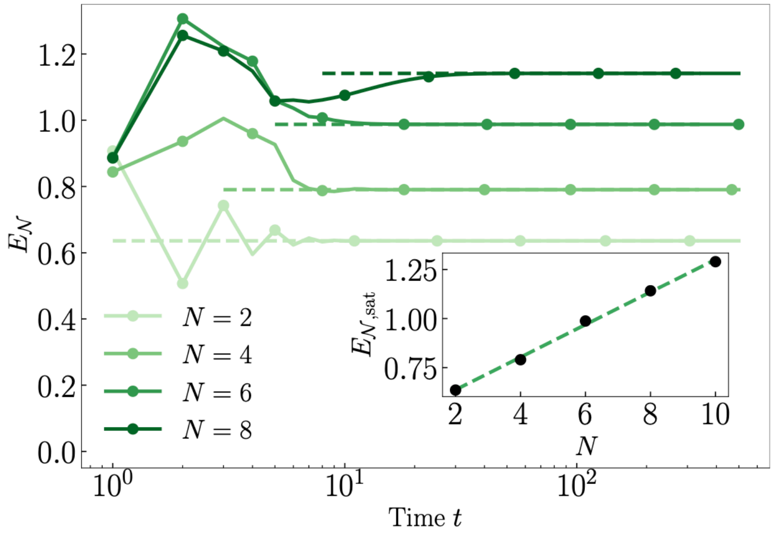
Logarithmic negativities of with dephasing noise
→ Separable ($T \to \infty$) steady state!



[Li, Sala, FP, arxiv:2305.06918]

Open System: Temperly-Lieb model

Logarithmic negativities of with structure preserving noise
→ Entangled steady state! Volume law?

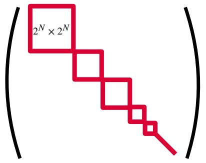


[Li, Sala, FP, arxiv:2305.06918]

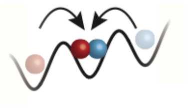
Summary

Constrained Dynamics and Hilbert space fragmentation

- Fragmentation in dipole conserving models
  [Sala, Rakovszky, Verresen, Knap, FP, Phys. Rev. X **10**, 011047 (2020)]
  [Rakovszky, Sala, Verresen, Knap, FP, PRB **101**, 125126 (2020)]



- Experimental realization: Fermi-Hubbard chains
  [Scherg et al., Nat Commun **12**, 4490 (2021)]
  [Kohlert et al., PRL **130**, 010201 (2023)]



- Quantum HSF in open systems
  [Sala, Li, FP (in progress)]


T. Rakovszky

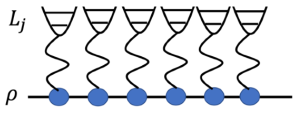

P. Sala


R. Verresen


M.Knap


Y. Li


P. Sala



Thank You!