Boulder notes by Victor V. Albert.

I. EUN-AH KIM

We focus on experimentally realizing topological superconductors (TSCs), with Hamiltonian

$$H = T + H_{int}\,.$$

Two strategies are manipulating T to produce exotic non-interacting systems (e.g., arXiv:1606.00857) or H_{int} for interactions. TSCs are similar to TIs in their exotic properties, but have also particle-hole symmetry and (as a result) Majorana zero modes.

If we study the Hubbard model using mean-field and picking the superconducting channel ($c^\dagger c^\dagger$) as opposed to the magnetic/charge channel ($c^\dagger c$), we get a superconductor, but it is s-wave and not p-wave. So BdG is insufficient for guiding materialization.

Consider a fermionic Cooper pair wavefunction

$$\Psi_{CP}\left(r_1, r_2, \sigma_1, \sigma_2\right) = g\left(r_1 - r_2\right)\chi\left(\sigma_1, \sigma_2\right) \qquad \Psi_{CP}\left(r_2, r_1, \sigma_2, \sigma_1\right) = -\Psi_{CP}\left(r_1, r_2, \sigma_1, \sigma_2\right)\,.$$

1. In singlet SCs (s, d-wave), g is symmetric and χ is antisymmetric.
2. In triplet SCs (p-wave), g is antisymmetric: $g_k = -g_{-k}$. Now our wavefunction is entangled:

$$\Psi_{CP}\left(r_1, r_2, \sigma_1, \sigma_2\right) = g_{\uparrow\uparrow}\left(r_1 - r_2\right)|\uparrow\uparrow\rangle + g_{\downarrow\downarrow}\left(r_1 - r_2\right)|\downarrow\downarrow\rangle + g_{\uparrow\downarrow}\left(r_1 - r_2\right)\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

   and the order parameter is matrix-valued:

$$\Delta = \begin{pmatrix} \Delta_{\uparrow\uparrow} & \Delta_{\uparrow\downarrow} \\ \Delta_{\downarrow\uparrow} & \Delta_{\downarrow\downarrow} \end{pmatrix}\,.$$

   Realizations of this are discussed in ^{3}He (either time-reversal invariant or not) and StRu.

   (a) In the ^{3}He ABM phase (TRS T broken),

$$\Delta_{ABM} = \Delta_0\left(k_x + ik_y\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

   where Δ_0 is an odd function of k. We get a chiral state:

$$T\Delta_{\uparrow\uparrow}(k) = \Delta^\star_{\downarrow\downarrow}(-k) = -\Delta_0\left(k_x - ik_y\right) \neq \Delta_{\uparrow\uparrow}(k) = \Delta_0\left(k_x + ik_y\right)\,.$$

   (b) In the BW phase (TRS upheld),

$$\Delta_{BW} = \Delta_0\begin{pmatrix} -\left(k_x - ik_y\right) & 0 \\ 0 & k_x + ik_y \end{pmatrix}\,.$$

   There is "spin-orbit locking".

$$T\Delta_{\uparrow\uparrow}(k) = \Delta^\star_{\downarrow\downarrow}(-k) = -\Delta_0\left(k_x - ik_y\right) = \Delta_{\uparrow\uparrow}(k)\,.$$

3. We consider a BdG with equal spin pairing (as opposed to $c_{k,\uparrow}c_{-k,\downarrow}$ as in Tinkham),

$$H = \sum_k\left[\xi_k c_k^\dagger c_k + \frac{1}{2}\left(\Delta^\star c_{-k}c_k + H.c.\right)\right] \qquad \xi_k = \frac{|k|^2}{2m} - \mu \qquad \Delta_k = \Delta_0\left(k_x + ik_y\right)\,.$$

   This was diagonalized in Nick Read's talk:

$$H = \sum_k E_k\gamma_k^\dagger\gamma_k \qquad E_k = \pm\sqrt{\xi_k^2 + |\Delta_k|^2}\,,$$

and we get the weak and strong pairing phases. The wavefunction is

$$|\Psi_{BCS}\rangle = \prod_k u_k \left(1 + g_k c_k^\dagger c_{-k}^\dagger\right)|0\rangle$$

with $g_k = v_k/u_k$. In position space, we obtain the Pfaffian[1]

$$\langle r_1 ... r_N|\Psi_{BCS}\rangle = a\left\{g\left(r_i - r_j\right)\right\} = \mathrm{Pf}\left\{g\left(r_i - r_j\right)\right\} \qquad\qquad g\left(r\right) = \int dk g\left(k\right) .$$

We now calculate the topological index (defined for $\mu \neq 0$, where map is continuous)

$$M\left(\hat{n}\right) = \int_{S^2} d^2 k \epsilon_{ij} \hat{n}_k \times \left(\partial_i \hat{n}_k \times \partial_j \hat{n}_k\right) ,$$

which is the Pontryagin index/Chern number. The vector $\hat{n}_k = \frac{1}{E_k}\langle \Re\Delta_k, -\Im\Delta_k, \xi_k\rangle \in S^2$ is a map from *non-translationally invariant* k-space (the 2D base space S^2, since we've identified $k \to \infty$ with the North Pole[2]) to the sphere (target space S^2). This is then just an element of the second homotopy group, i.e., space of equivalence classes of maps from S^2 to S^2:

$$\pi_2\left(S^2\right) = \mathbb{Z}\,.$$

Calculating this invariant, which basically tells you how much you've wrapped around the sphere.

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[1] Antisymmetrized products of $g\left(r_i - r_j\right)$ do not correspond to a Slater determinant since that is of the form $f_i\left(r_j\right)$.

[2] We can only do this if the limit as $\langle k_x, k_y\rangle \to \infty$ is defined, i.e., if all limits to ∞ yield the same value of $\hat{n}$. If this was not the case and we instead had, e.g., $E \to$ constant as $k_y \to \infty$, then large k_y's would be participating in the behavior at long timescales (since they are constant, they are not high frequency and would not be integrated out at large times).