Effective Hamiltonians and quantum magnetism of ultra cold atoms

Ehud Altman

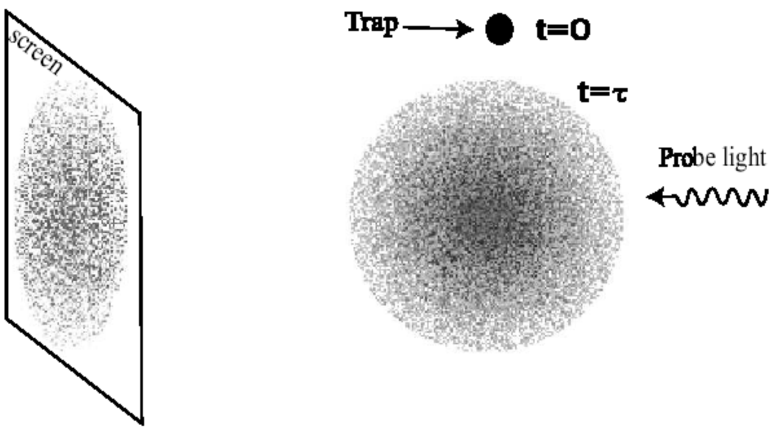
Harvard university

Collaborators:

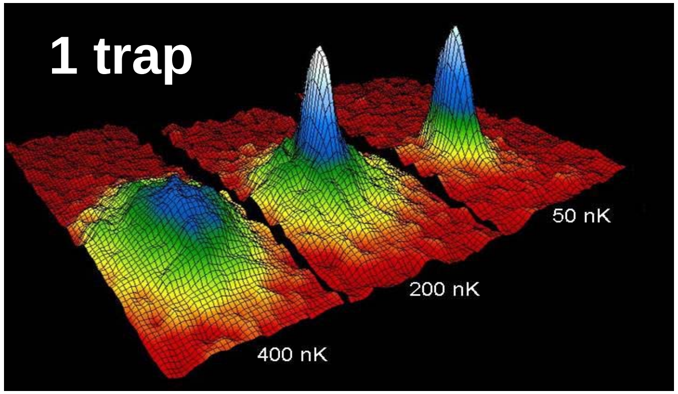
E. Demler, M. Lukin, W. Hofstetter, A. Polkovnikov, A. Auerbach

Overview

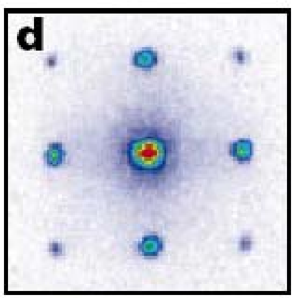
BEC and Matter Waves

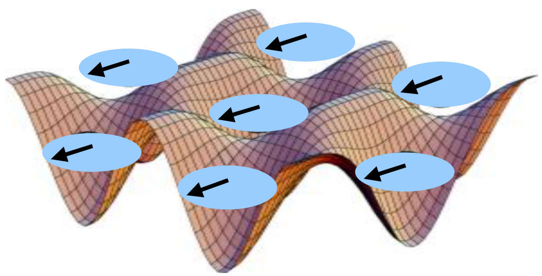


Anderson etal, Science (95)



Lattice





Greiner etal., Nature (02)

Macroscopic occupation of a single particle wave function.

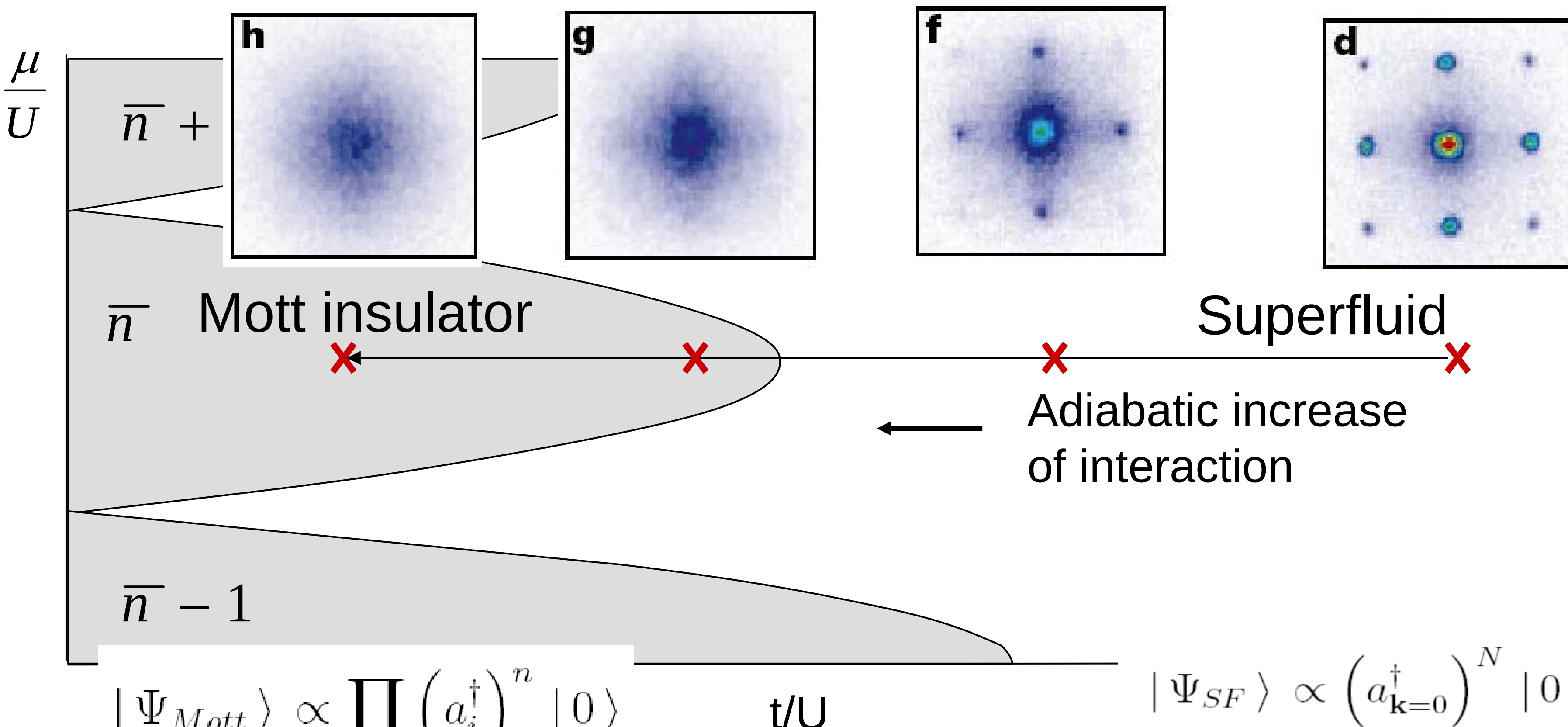
$$\psi(\mathbf{k})^N$$

Cloud density ⟷ momentum distribution in trap

From matter waves to particles

Greiner etal., Nature 415 (02) Proposal: *D. Jaksch, et al. PRL (98)*

$$H = \frac{U}{2}\sum_i (n_i - \bar{n})^2 - t\sum_{\langle ij \rangle}(a_i^\dagger a_j + \mathrm{H.c.}) - \mu \sum_i (n_i - \bar{n})$$



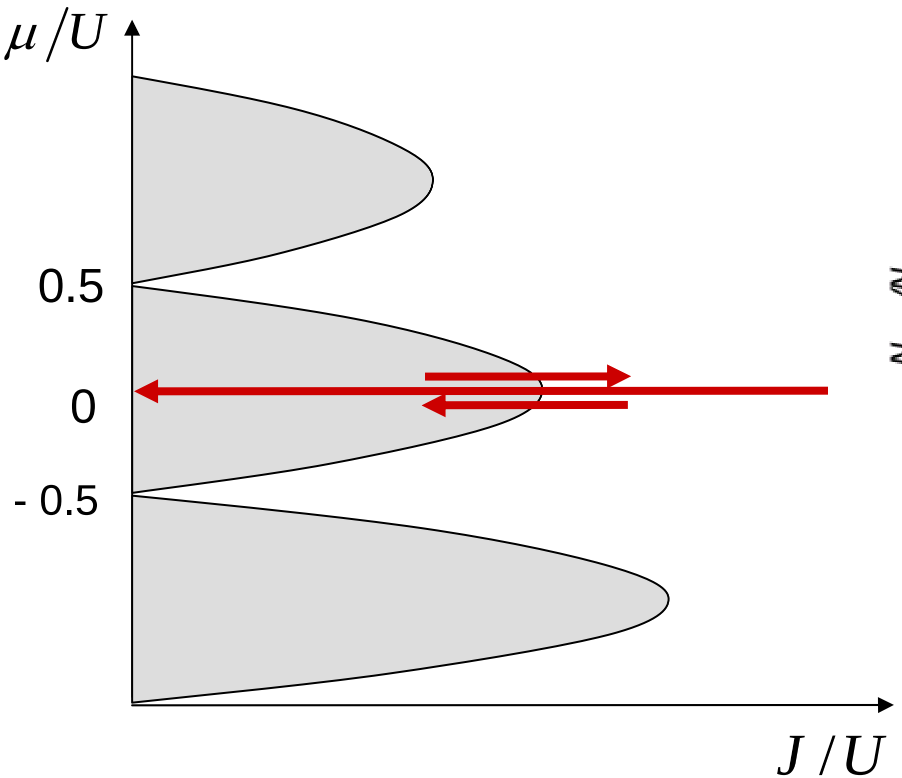
$$|\Psi_{Mott}\rangle \propto \prod_i \left(a_i^\dagger\right)^n |0\rangle$$

$$|\Psi_{SF}\rangle \propto \left(a_{\mathbf{k}=0}^\dagger\right)^N |0\rangle$$

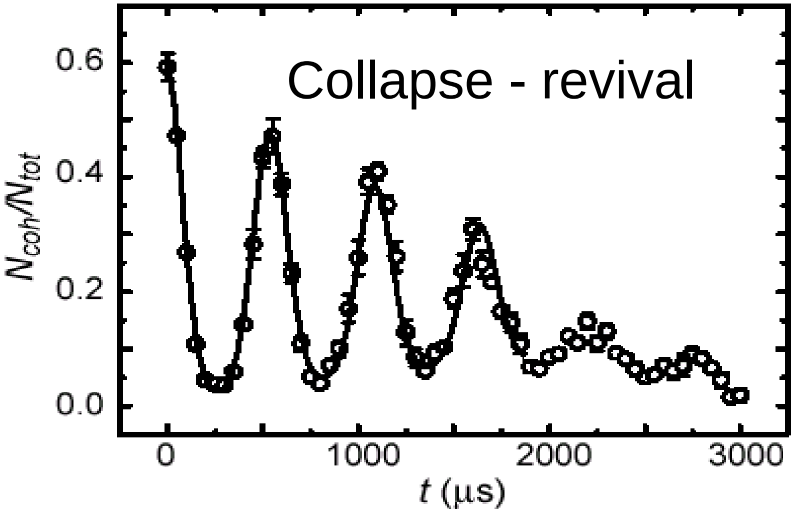
No description in terms of a single particle wavefunction

Nevertheless these systems exhibit very long coherence times

Playground for non equilibrium quantum dynamics.

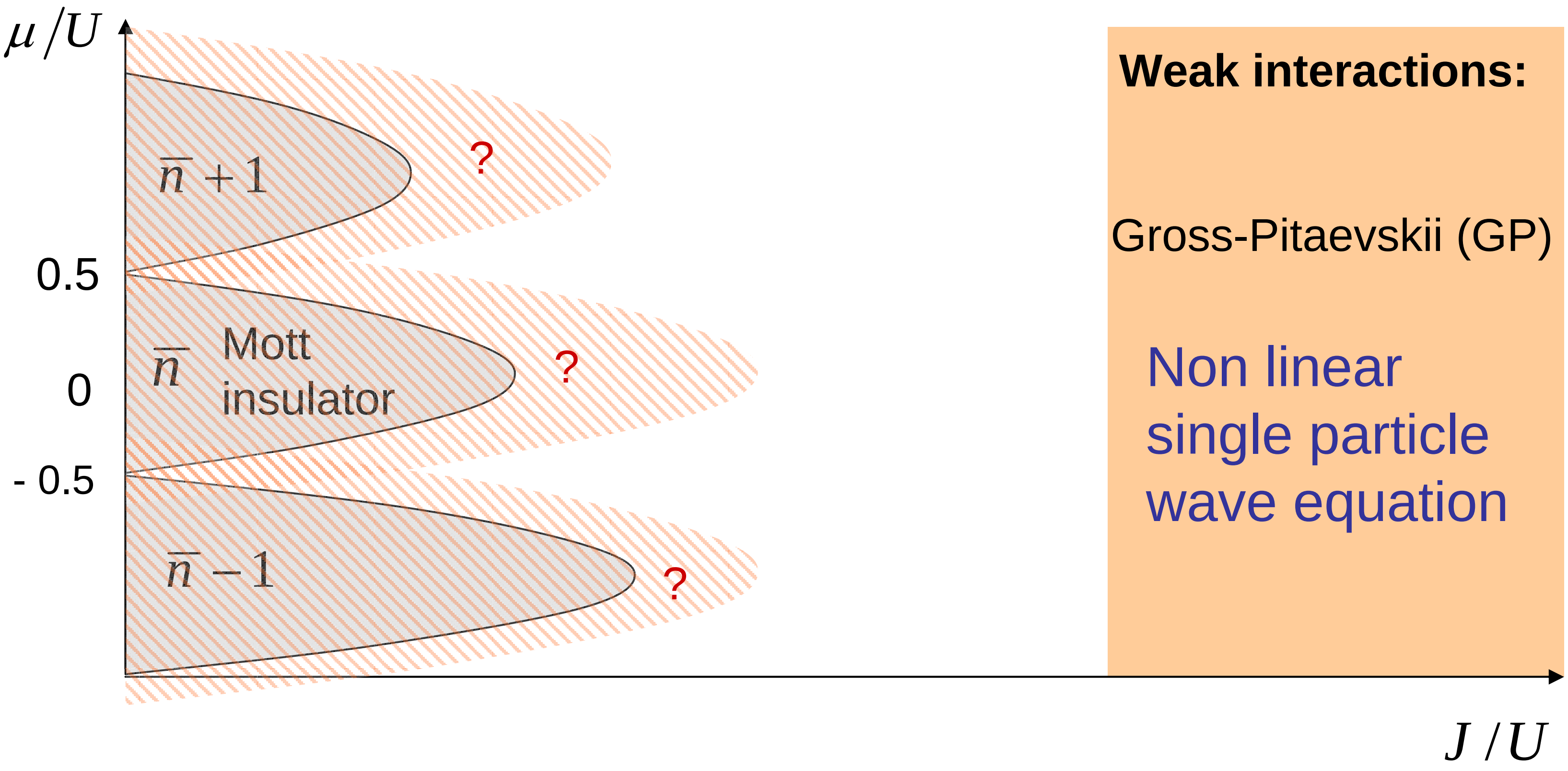


Greiner et. Al., Nature 419 (2002)



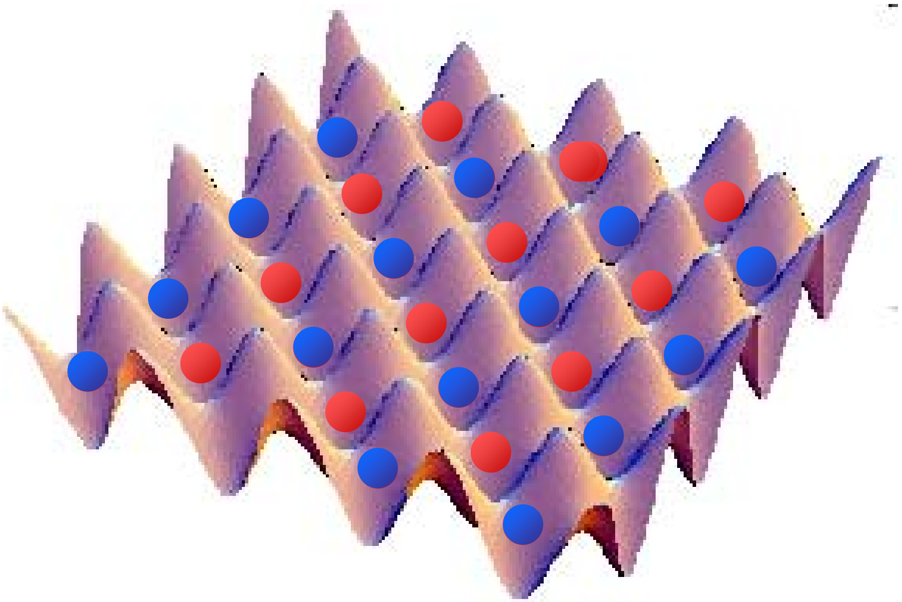
Time evolution of a Mott state when the lattice is rapidly changed to the supefluid regime?

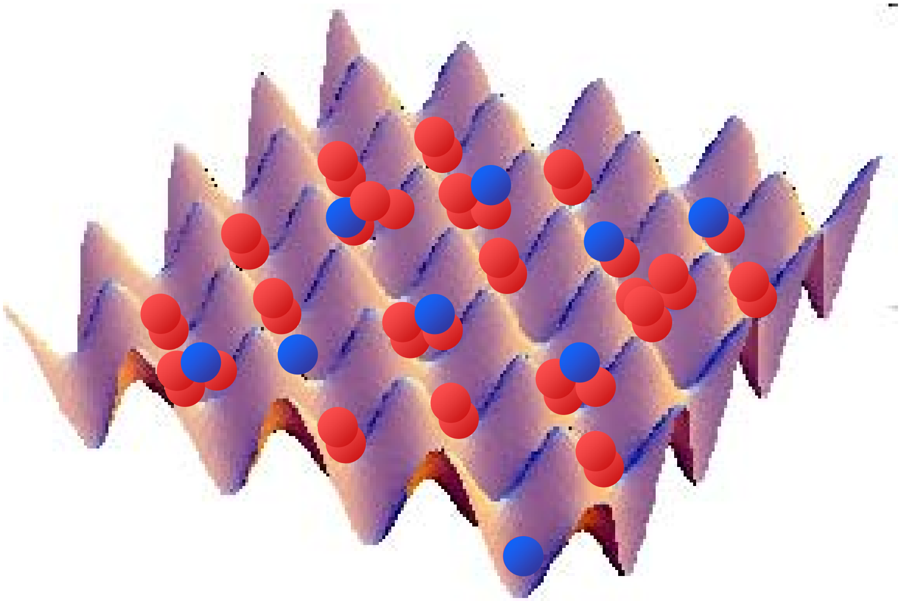
Dynamics


μ / U
$\bar{n}+1$
?
0.5
$\bar{n}$
Mott
insulator
?
0
- 0.5
$\bar{n}-1$
?
J / U
Weak interactions:
Gross-Pitaevskii (GP)
Non linear
single particle
wave equation

We understand the plain vanilla Mott transition.

What about chocolate vanilla?





Time of flight image is not sensitive to many body correlations. How to detect them ?

Outline

- **Superfluid-Mott insulator transition – plain vanilla:**
  mean field theory, collective modes, dynamics

- **Chocolate vanilla Mott insulator – “spin ½” bosons**
  spin ordered phases
  How the ice-cream melts

- **Detecting correlations via noise in time of flight image**
  What does time of flight imaging measure?
  How boring is the vanilla Mott state?
  Detecting spin correlations.
  Pairing correlations.

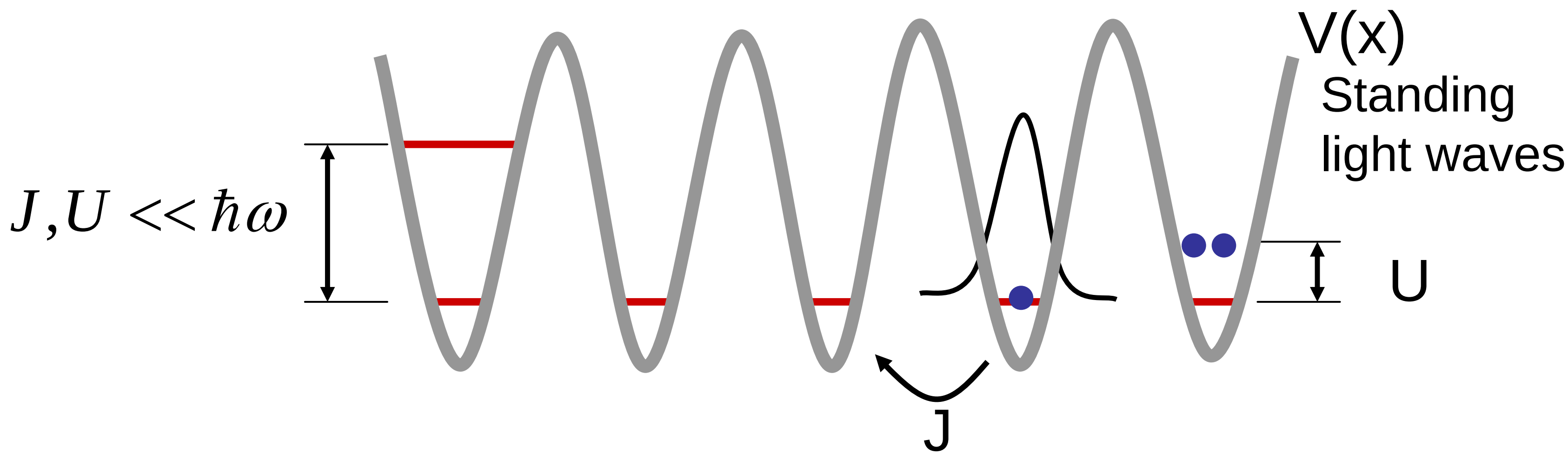
I

Mott insulator – Superfluid transition
Vanila flavor

Atoms in an optical lattice and the BHM

D. Jaksch, *et al.* PRL (98)

$$H = \int d^3x \psi^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) \right) \psi(\mathbf{x}) + \frac{2\pi a_s \hbar^2}{m} \int d^3x \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x})\psi(\mathbf{x})\psi(\mathbf{x})$$



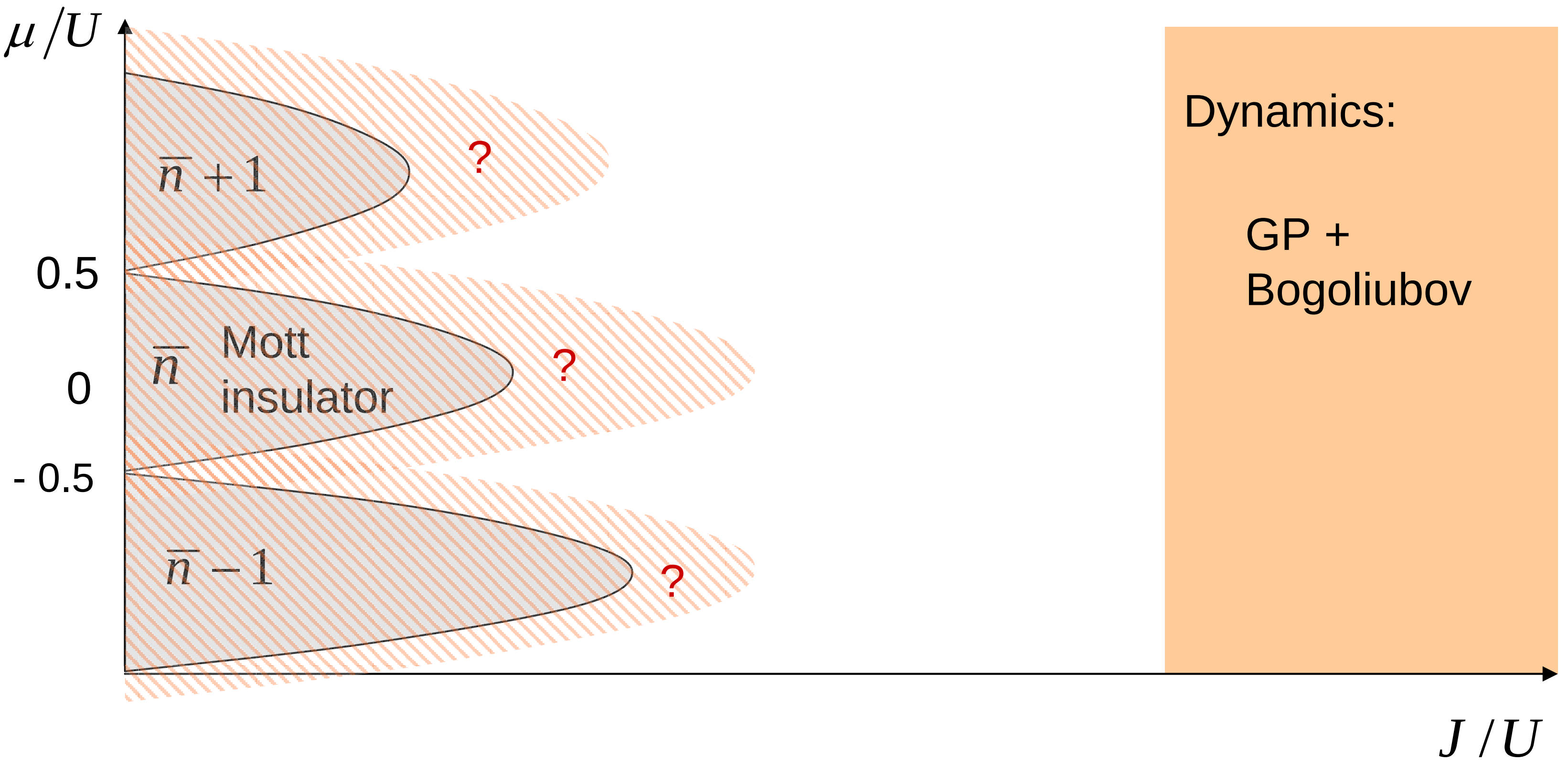
Projection to lowest Bloch band $\Longrightarrow$ **Bose Hubbard Model:**

$$H = \frac{U}{2}\sum_i (n_i - \bar{n})^2 - J \sum_{\langle ij \rangle} (a_i^\dagger a_j + \mathrm{H.c.}) - \mu \sum_i (n_i - \bar{n})$$

Enhanced interaction effects !

Phase diagram

Mean field theory: Fisher *et. al.* PRB (1989)



How to calculate excitations and dynamics in the vicinity of the insulating phase?

Breakdown of the Gross-Pitaevskii description

Variational state: $$|\,\Psi\,\rangle = \prod_i e^{\psi_i a_i^\dagger}\,|\,0\,\rangle$$

Amounts to replacing the operator a_i by the c-number ψ_i :

$$\langle H\rangle = -J\sum_{\langle ij\rangle}(\psi_i^\star\psi_j + \text{c.c.}) + \frac{U}{2}\sum_i |\psi_i|^4$$

Uniform solution: $$\psi_i = \sqrt{\bar{n}}$$

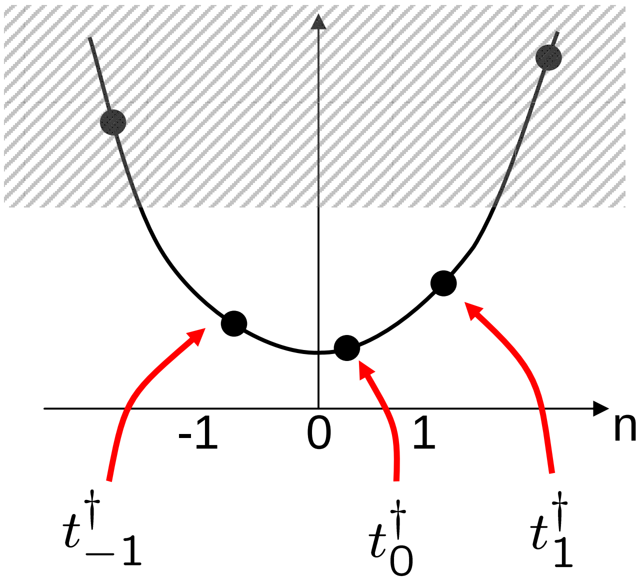
Classical equation of motion: $$i\frac{d\psi_j}{dt} = -J(\psi_{j+1} + \psi_{j-1}) + U|\psi_j|^2\psi_j$$

Cannot describe Mott transition:

1. Particle number fluctuation $\langle \delta n_i^2\rangle \sim \bar{n}$
2. Nothing special at integer filling
3. Quadratic fluctuations (Bogoliubov theory) don’t help.

Variational mean field theory (I)

Factorizable state: $|\Omega\rangle = \prod_i C_i(n)|n\rangle$ Rokhsar & Kotliar PRB (91)



Near the transition $\langle \delta n^2 \rangle << 1$

Keep 3 states

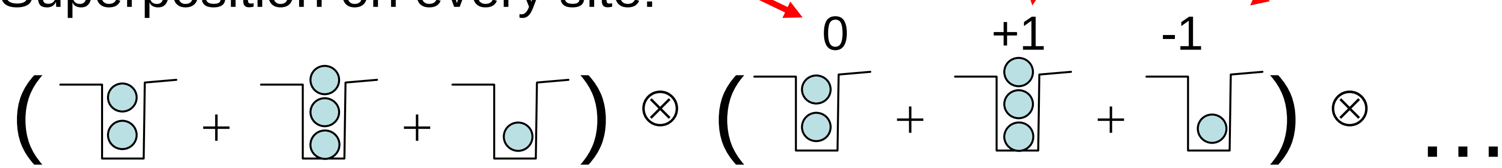
Constraint: $\sum_m t^\dagger_{mi} t_{mi} = 1$

$$|\,\Omega\,\rangle = \prod_i \left[\cos\frac{\theta_i}{2} t^\dagger_{0i} + e^{i\eta_i}\sin\frac{\theta_i}{2}\left(e^{i\varphi_i}\cos\frac{\chi_i}{2} t^\dagger_{1i} + e^{-i\varphi_i}\sin\frac{\chi_i}{2} t^\dagger_{-1i}\right)\right]|\,vac\,\rangle$$

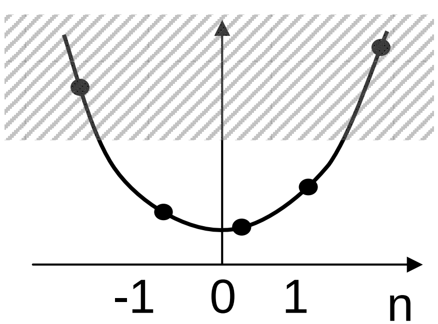
Variational mean field theory (II)

$$|\,\Omega\,\rangle = \prod_i \left[\cos\frac{\theta_i}{2} t^\dagger_{0i} + e^{i\eta_i}\sin\frac{\theta_i}{2}\left(e^{i\varphi_i}\cos\frac{\chi_i}{2} t^\dagger_{1i} + e^{-i\varphi_i}\sin\frac{\chi_i}{2} t^\dagger_{-1i}\right)\right]|\,vac\,\rangle$$

Superposition on every site:



Easy to calculate expectation values:



$$\langle n - \bar{n}\rangle = \sin^2(\theta/2)\cos\chi$$

$$\langle (n - \bar{n})^2\rangle = \sin^2(\theta/2)$$

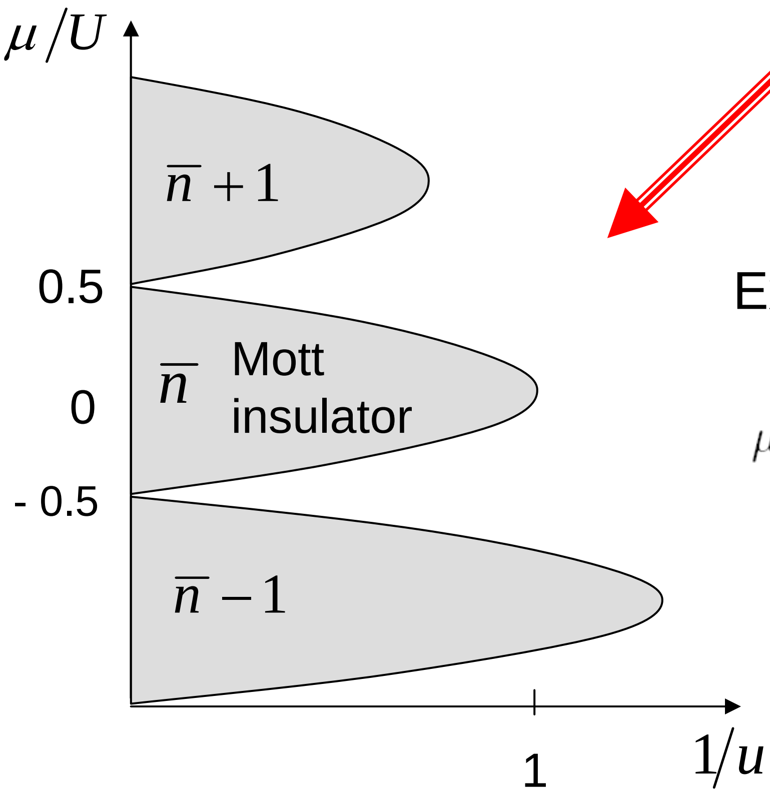
Order parameter (commensurate filling): $\langle a_i \rangle \approx e^{i\varphi}\frac{\sqrt{\bar{n}}}{2}\sin\theta$

Variational mean field theory (III)

Minimize variational energy:

$$\langle \Omega | H | \Omega \rangle$$

$$\frac{\mathcal{H}}{N} = \left(\frac{U}{2} + \mu\cos\chi\right)\sin^2\left(\frac{\theta}{2}\right) - \frac{Jz\bar{n}}{4}\sin^2\theta\left(1 + \frac{\sin^2(\chi/2)}{\bar{n}} + \sqrt{1+\bar{n}^{-1}}\sin\chi\cos 2\eta(\right)$$



Expression for the Mott lobes:

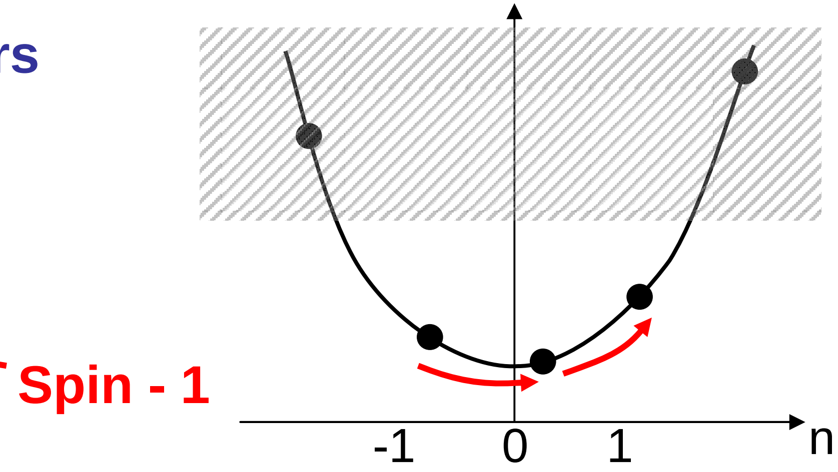
$$\mu_c/U = -\frac{1}{8\bar{n}u} \pm \frac{1}{2}\sqrt{1 - \frac{1}{u}\left(1+\frac{1}{2\bar{n}}\right) + (4\bar{n}u)^{-2}}$$

$$u = U/(4J\bar{n}z)$$

Is the dimensionless interaction

Fluctuations (I)

Rewrite H in terms of the new operators



$$a^\dagger = \sqrt{\bar{n}+1}\; t_1^\dagger t_0 + \sqrt{\bar{n}}\; t_0^\dagger t_{-1}$$
$$\approx \sqrt{\bar{n}}(t_1^\dagger t_0 + t_0^\dagger t_{-1}) = \sqrt{\bar{n}/2} S^+$$

$$n - \bar{n} = t_1^\dagger t_1 - t_{-1}^\dagger t_{-1} = S^z$$

$$(n-\bar{n})^2 = t_1^\dagger t_1 + t_{-1}^\dagger t_{-1} = (S^z)^2$$

$$H = -J\bar{n}\sum_{\langle ij\rangle}(t_{1i}^\dagger t_{0i} t_{0j}^\dagger t_{1j} + t_{1i}^\dagger t_{-1j}^\dagger t_{0i} t_{0j} + \ldots) + \frac{U}{2}\sum_i (t_{1i}^\dagger t_{1i} + t_{-1i}^\dagger t_{-1i}) - \mu \sum_i (t_{1i}^\dagger t_{1i} + t_{-1i}^\dagger t_{-1i})$$

Can also be written as the pseudo spin – 1 model (n>>1):

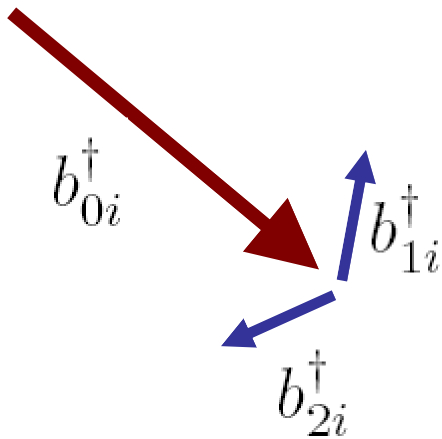
$$H_{\text{eff}} = \frac{U}{2}\sum_i (S_i^z)^2 - J\bar{n}\sum_{\langle ij\rangle}(S_i^x S_j^x + S_i^y S_j^y) - \mu \sum_i S_i^z,$$

Fluctuations (II)

Mean field state is a Fock state in terms of a new operator

$$|\,\Omega_{MF}\,\rangle = \prod_i \left[\cos(\theta/2)t^\dagger_{0i} + \frac{\sin(\theta/2)}{\sqrt{2}}\left(t^\dagger_{1i} + t^\dagger_{-1i}\right)\right]|\,vac\,\rangle \equiv \prod_i b^\dagger_{0i}\,|\,vac\,\rangle$$

Orthogonal operators create fluctuations:



$$b^\dagger_{1i} = \sin(\theta/2)t^\dagger_{0i} - \frac{\cos(\theta/2)}{\sqrt{2}}\left(t^\dagger_{1i} + t^\dagger_{-1i}\right)$$

$$b^\dagger_{2i} = \frac{1}{\sqrt{2}}\left(t^\dagger_{1i} - t^\dagger_{-1i}\right)$$

Constraint: $\sum_{\alpha=0}^{2} b^\dagger_{\alpha i} b_{\alpha i} = 1$

<u>Example:</u> Mott state

$$b^\dagger_{0i} = t^\dagger_{0i}$$

Fluctuations: $t^\dagger_{\pm 1i}$

Fluctuations (III)

Use constraint to eliminate $b_{0i}^\dagger$:

$$b_\alpha^\dagger b_0 = b_m^\dagger \sqrt{1 - b_1^\dagger b_1 - b_2^\dagger b_2} \approx b_\alpha^\dagger \left(1 - \frac{1}{2} b_1^\dagger b_1 - \frac{1}{2} b_2^\dagger b_2\right)$$

Rewrite H in terms of fluctuation operators:

$$H = \langle \Omega_{MF} \,|\, H \,|\, \Omega_{MF} \rangle + \sum_{\mathbf{k},\alpha=1}^{2} \left[f_{\mathbf{k}\alpha} b_{\mathbf{k}\alpha}^\dagger b_{\mathbf{k}\alpha} + \frac{g_{\mathbf{k}\alpha}}{2} (b_{-\mathbf{k}\alpha}^\dagger b_{\mathbf{k}\alpha}^\dagger + b_{-\mathbf{k}\alpha} b_{\mathbf{k}\alpha}) \right] + \ldots$$

Regime of validity: $\sum_{\alpha=1}^{2} \langle b_{\alpha i}^\dagger b_{\alpha i} \rangle << 1$

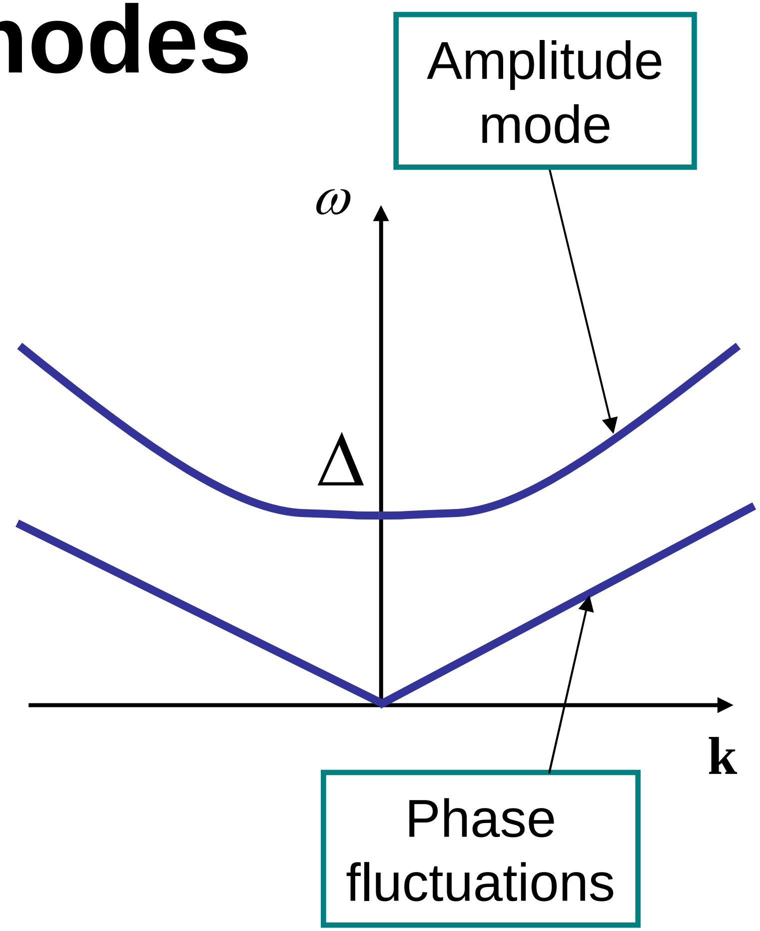
Diagonalize with Bogoliubov transformation

Collective modes

Superfluid:

$$\omega_\alpha(\mathbf{k}) = 2zJ\bar{n}\sqrt{1-u^2\gamma_\mathbf{k}} \approx \sqrt{c^2\mathbf{k}^2+\Delta^2},$$
$$\omega_\varphi(\mathbf{k}) = Jz\bar{n}(1+u)\sqrt{1-\gamma_\mathbf{k}}, \approx c|\mathbf{k}|,$$

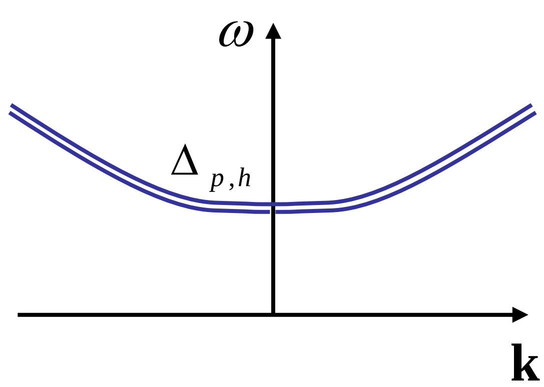
$$u \equiv U/(4J\bar{n}z) \qquad \Delta \propto \sqrt{1-u}$$



Mott:

$$\omega_p = \omega_h = \frac{U}{2}\sqrt{1-u^{-1}\gamma_\mathbf{k}}.$$

Gapped particle/hole



$$\Delta \propto \sqrt{u-1}$$

Dynamics

$$|\,\Omega\,\rangle = \prod_i \left[\cos\frac{\theta_i}{2}\,|\,0_i\,\rangle + e^{i\eta_i}\sin\frac{\theta_i}{2}\left(e^{i\varphi_i}\cos\frac{\chi_i}{2}\,|\,1_i\,\rangle + e^{-i\varphi_i}\sin\frac{\chi_i}{2}\,|\,-1_i\,\rangle\right)\right]$$

Use as (over-complete) basis to construct
a path integral for the time evolution:

$$\mathcal{U}(t) = \int \mathcal{D}\Omega \exp\left\{ i\int_0^t d\tau \left(\langle \Omega\,|\,\partial_\tau \Omega\,\rangle - \mathcal{H}[\Omega(\tau)] \right) \right\}$$

$$\langle \Omega\,|\,\partial_\tau \Omega\,\rangle = \sin^2\frac{\theta}{2}\dot{\eta} + \sin^2\frac{\theta}{2}\cos\chi\dot{\varphi} = \langle \delta n^2\rangle\dot{\eta} + \langle \delta n\rangle\dot{\varphi}$$

Canonical conjugate pairs

Close to the transition@integer filling, integrate over η to obtain:

$$S_{eff} = \frac{1}{4Jz\bar{n}^2}\int_0^t dt' \int d^d r \left\{ |\dot{\psi}|^2 - (2J\bar{n})^2 z |\nabla\psi|^2 - (2J\bar{n}z)^2(u-1)|\psi|^2 - 2(Jz)^2\bar{n}u|\psi|^4 \right\}$$

Recall effective action from Sachdev’s talk

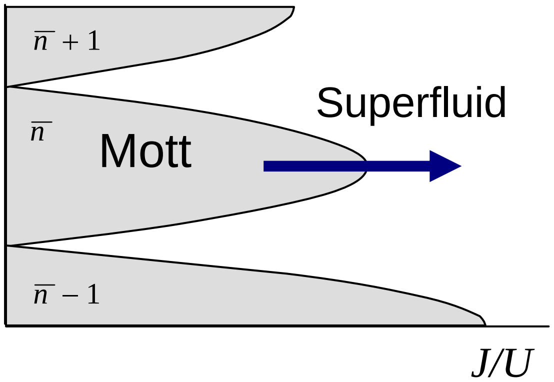
Application: sudden quench

E. Altman & A. Auerbach, PRL (2002)

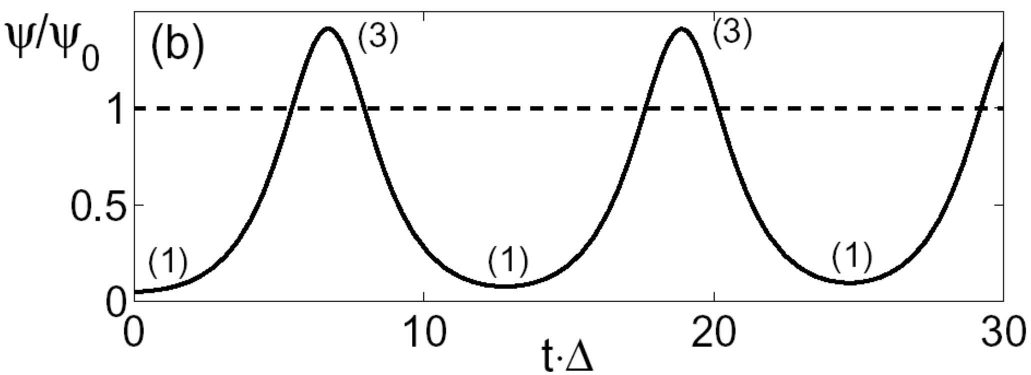
Different regime: Polkovnikov *et. al.* PRA (2002)

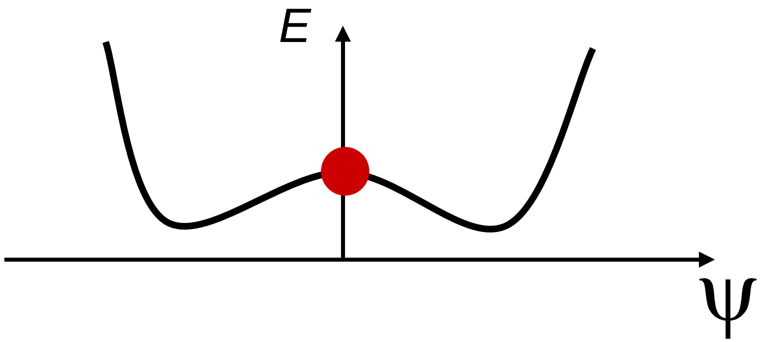
Saddle point equation of motion:

$$\ddot{\Psi} = c^2\nabla^2\Psi + \frac{1}{2}\Delta^2\Psi(1-|\Psi|^2).$$



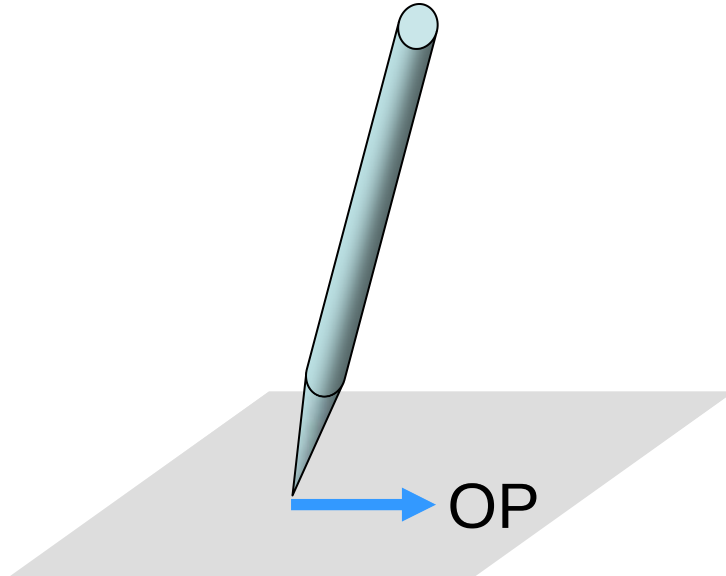
Uniform order parameter evolution:

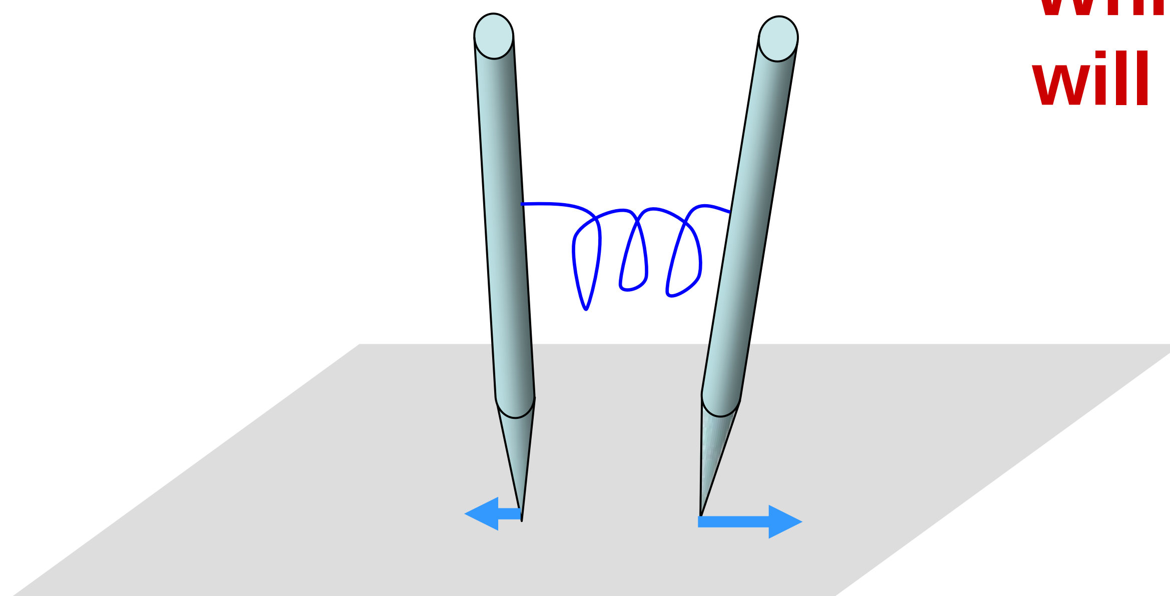




Frequency scale: $\Delta = 2\sqrt{2}Jz\bar{n}\sqrt{1-u}$

Non uniform configurations?

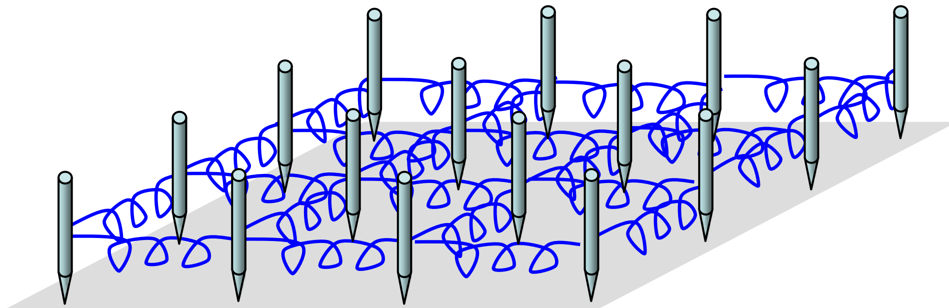




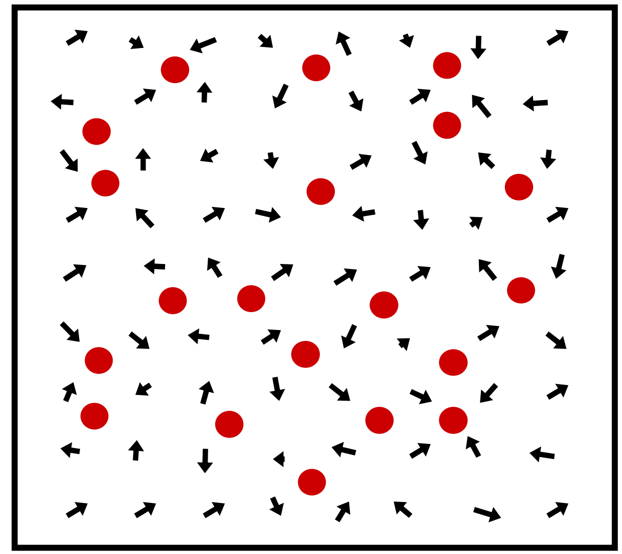
Which way will they fall?

Vortex trapping (I)

Initial state:

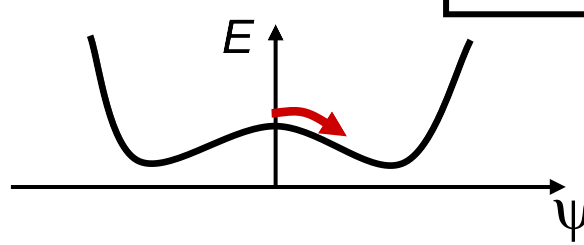


Top View with fluctuations



How many defects survive?

$$\ddot{\Psi} = c^2\nabla^2\Psi + \frac{1}{2}\Delta^2\Psi(1 - \cancel{|\Psi|^2}).$$



Growth modes:

$$\omega(k) = \sqrt{(ck)^2 - \frac{1}{2}\Delta^2}.$$

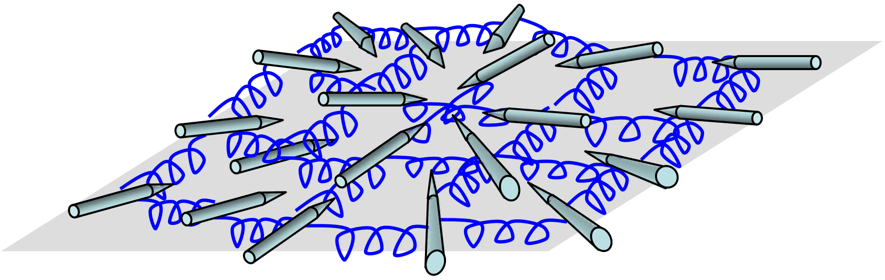
Fastest growing: k=0

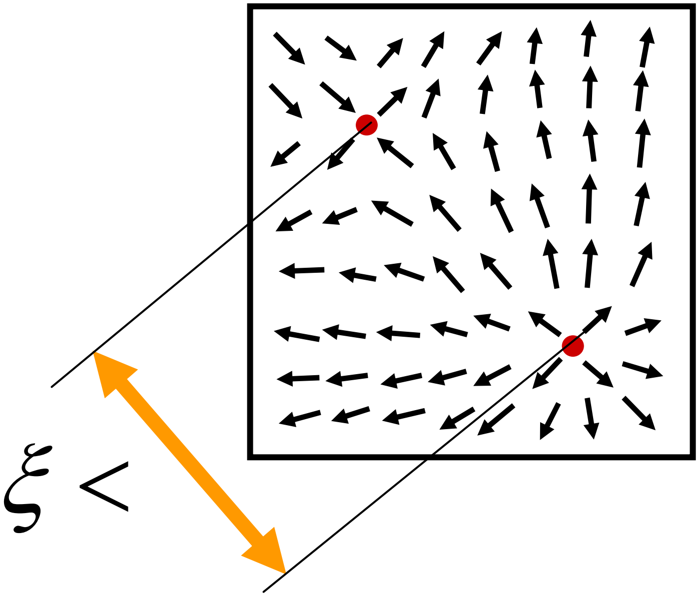
Do not grow: $k > \frac{\Delta}{c\sqrt{2}} \equiv \frac{1}{\xi\sqrt{2}}$

Vortex trapping (II)

$$k_c = \frac{\Delta}{c\sqrt{2}} = \frac{1}{\xi\sqrt{2}}$$

sets initial size of domains





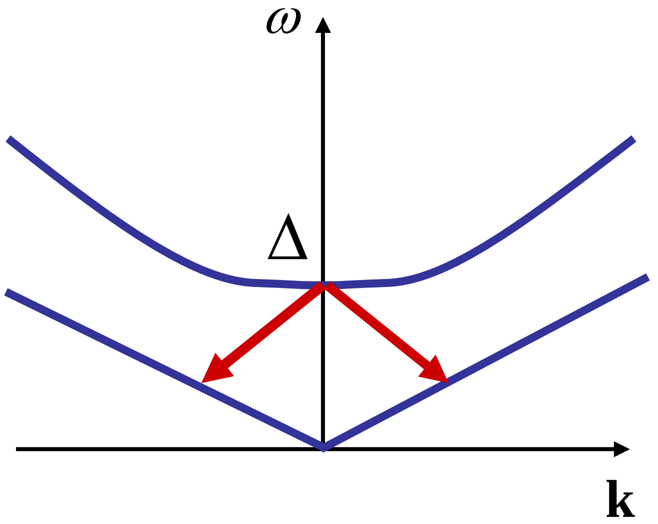
Quantum Kibble-Zurek mechanism

Quantum Corrections ⇒ Damping

Emission of phason pairs

$$H_{int} = \sum_{\mathbf{k}} V_{\mathbf{k}} (a_0 p^{\dagger}_{-\mathbf{k}} p^{\dagger}_{\mathbf{k}} + \mathrm{H.c.})$$

$$V_{\mathbf{k}} \;=\; c\mathbf{k}\sqrt{\frac{2Jz\bar{n}^2}{N\Delta|\Psi_0|^2}}$$



Damping rate (Q factor) of the oscillations:

$$\frac{\Gamma_d}{\Delta} = \frac{u}{\sqrt{2}}(1-u)^{\frac{d-3}{2}}$$

Over-damped in the critical region!

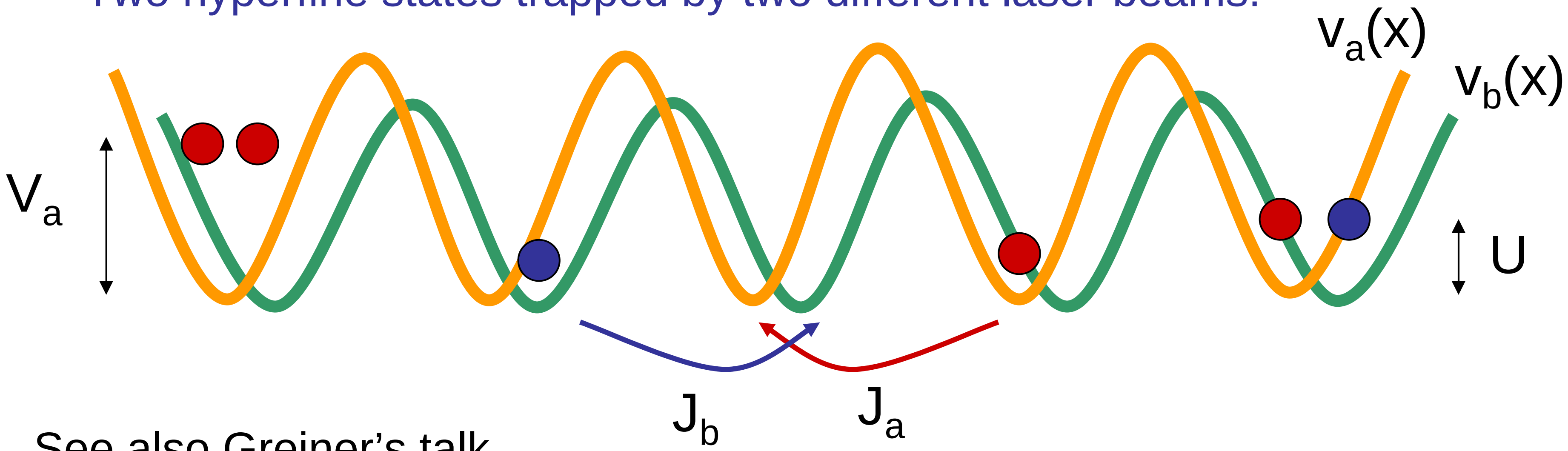
II

“Spin ½” bosons on optical lattices

Optical lattice for two species

Realized in I. Bloch's group: O. Mandel *et. al.*, PRL (2003)

Two hyperfine states trapped by two different laser beams:



See also Greiner's talk

$$H = -\sum_{\langle ij\rangle} t_a (a_i^\dagger a_j + \text{h.c.}) - t_b \sum_{\langle ij\rangle} (b_i^\dagger b_j + \text{h.c.}) + U \sum_i (n_{ai} - \tfrac{1}{2})(n_{bi} - \tfrac{1}{2})$$

$$+ \tfrac{1}{2} \sum_{i\alpha = a,b} V_\alpha n_{\alpha i}(n_{\alpha i} - 1) - \sum_{i\alpha} \mu_\alpha n_{\alpha i}.$$

Strong coupling @ integer occupation

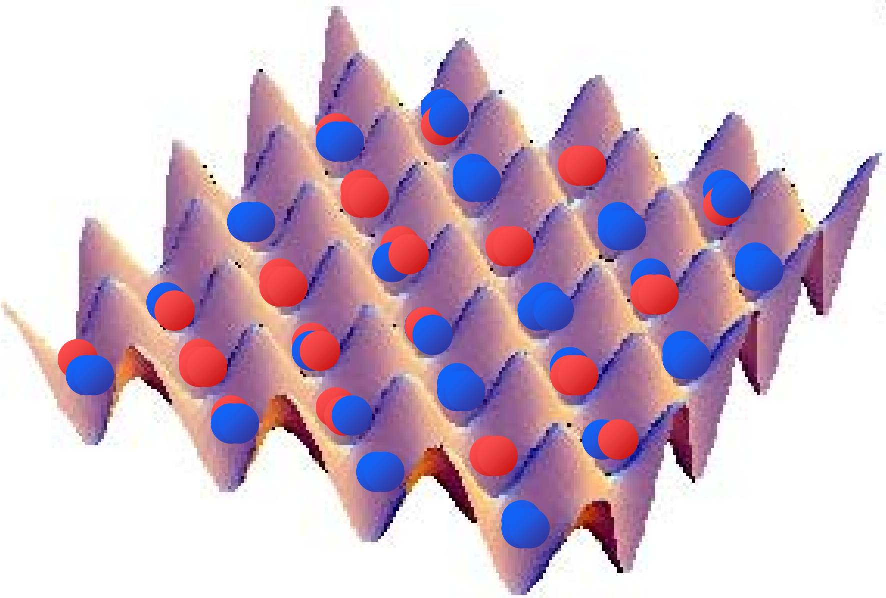
$$H = -\sum_{\langle ij\rangle} t_a (a_i^\dagger a_j + \text{h.c.}) - t_b \sum_{\langle ij\rangle} (b_i^\dagger b_j + \text{h.c.}) + U \sum_i (n_{ai} - \tfrac{1}{2})(n_{bi} - \tfrac{1}{2})$$
$$+ \tfrac{1}{2} \sum_{i\alpha=a,b} V_\alpha n_{\alpha i}(n_{\alpha i} - 1) - \sum_{i\alpha} \mu_\alpha n_{\alpha i}.$$

$t_a = t_b = 0$

$$a_1^\dagger b_2^\dagger b_3^\dagger b_4^\dagger a_5^\dagger b_6^\dagger a_7^\dagger a_8^\dagger \ldots |0\rangle$$

$$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger b_5^\dagger b_6^\dagger b_7^\dagger b_8^\dagger \ldots |0\rangle$$

$$a_1^\dagger b_2^\dagger a_3^\dagger b_4^\dagger a_5^\dagger b_6^\dagger a_7^\dagger b_8^\dagger \ldots |0\rangle$$



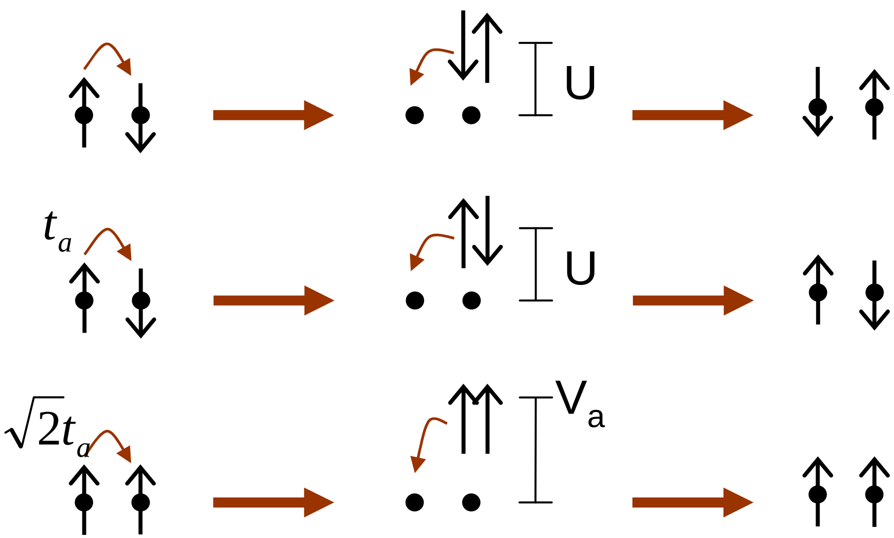
All configurations with 1 atom per site are degenerate !

Lifting the degeneracy

$t_{a,b} \ll U,\ V_{a,b}$

Hopping is frozen. Spin is the only remaining degree of freedom.

2 sites (qualitative):



$\propto\ -t_a t_b / U$

$\propto\ -t_a^2 / U$

competing

$\propto\ -2t_a^2 / V_a$

Effective spin Hamiltonian:

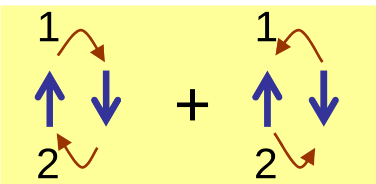
degenerate manifold

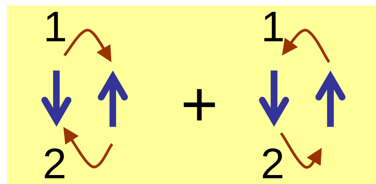
$$H_{eff} = -\sum_{\alpha\beta e} \frac{|\,\alpha\,\rangle\,\langle\,\alpha\,|\,\hat{T}\,|\,e\,\rangle\,\langle\,e\,|\,\hat{T}\,|\,\beta\,\rangle\,\langle\,\beta\,|}{E_e - E_0}$$

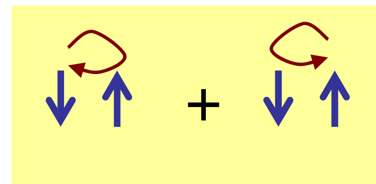
Effective spin-½ Hamiltonian

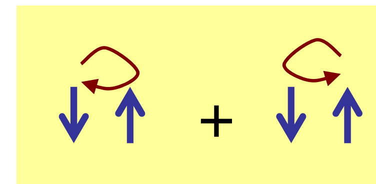
degenerate manifold

$$H_{eff} = -\sum_{\alpha\beta e} \frac{|\,\alpha\,\rangle\,\langle\,\alpha\,|\,\hat{\mathcal{T}}\,|\,e\,\rangle\,\langle\,e\,|\,\hat{\mathcal{T}}\,|\,\beta\,\rangle\,\langle\,\beta\,|}{E_e - E_0}$$

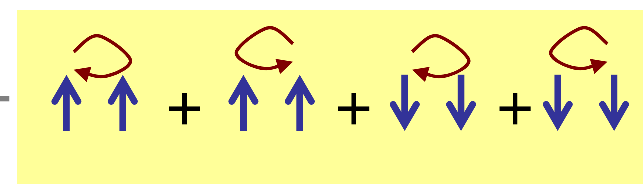








$$H_{eff} = -\sum_{\langle ij\rangle}\left(\frac{2t_at_b}{U}a_i^\dagger b_j^\dagger a_j b_i + \frac{2t_at_b}{U}b_i^\dagger a_j^\dagger b_j a_i + \frac{t_a^2+t_b^2}{U}a_i^\dagger b_j^\dagger b_j a_i + \frac{t_a^2+t_b^2}{U}b_i^\dagger \alpha_j^\dagger a_j n_i\right)$$
$$-\sum_{\langle ij\rangle}\left(\frac{2t_a^2}{V_a}a_i^\dagger a_j^\dagger a_j a_i + \frac{2t_b^2}{V_b}b_i^\dagger b_j^\dagger b_j b_i\right)$$



Recall: $a^\dagger\,|\,0\,\rangle = |\uparrow\rangle \quad b^\dagger\,|\,0\,\rangle = |\downarrow\rangle$

$$S_i^+ = a_i^\dagger b_i$$

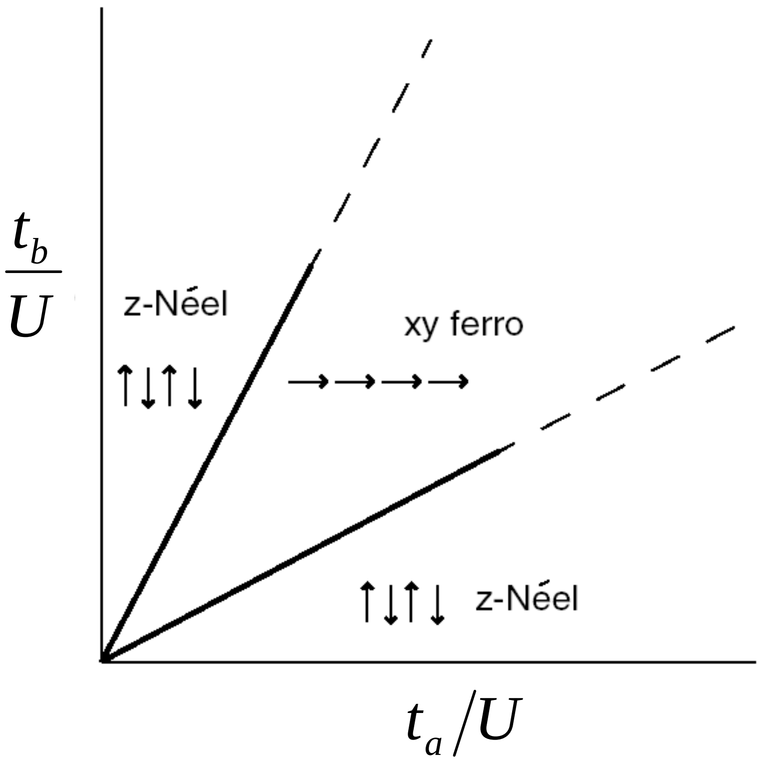
$$a_i^\dagger a_i = \frac{1}{2} + S^z$$

$$b_i^\dagger b_i = \frac{1}{2} - S^z$$

Effective spin-½ Hamiltonian

$$H_{\rm eff} = J_z \sum_{\langle ij \rangle} S_i^z S_j^z - J_\perp \sum_{\langle ij \rangle} (S_i^x S_j^x + S_i^y S_j^y) - h \sum_i S_i^z$$

Phase diagram $t_{a,b} \ll U \ll V_{a,b}$



$$J_z = 2\frac{t_b^2 + t_a^2}{U} - \frac{4t_a^2}{V_a} - \frac{4t_b^2}{V_b}$$

$$J_\perp = \frac{4t_a t_b}{U}$$

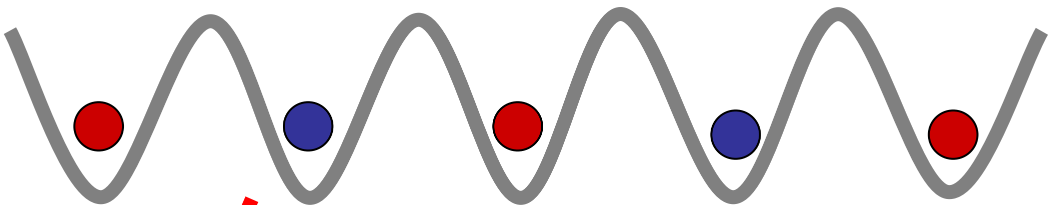
Spin operators:

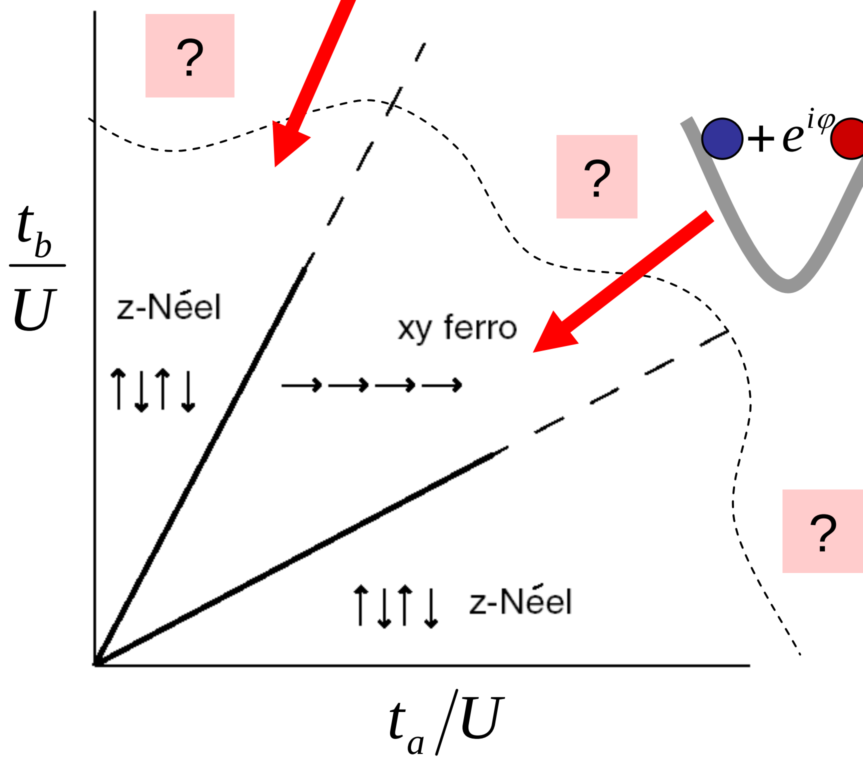
$$S_i^+ = a_i^\dagger b_i$$

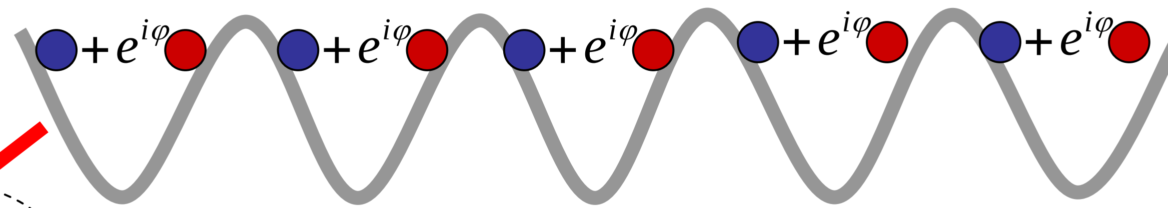
$$a_i^\dagger a_i = \frac{1}{2} + S^z$$

$$b_i^\dagger b_i = \frac{1}{2} - S^z$$

Svistunov and Kuklov (PRL 03),
Duan, Demler and Lukin (PRL 03)







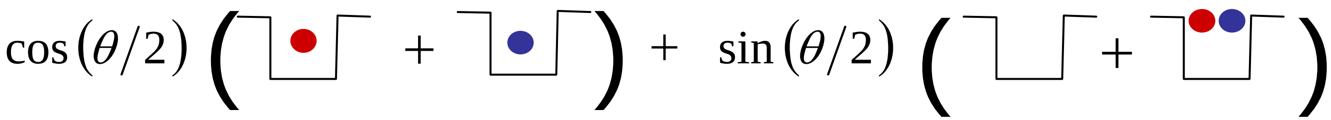
Counter-flow superfluid

Treatment valid only deep in the Mott insulating phase

Transition to superfluid?

Variational mean field theory (I)

$$|\Phi\rangle = \prod_i \left[\cos\frac{\theta_i}{2}\left(e^{i\varphi_i/2}\cos\frac{\chi_i}{2}a_i^\dagger + e^{-i\varphi_i/2}\sin\frac{\chi_i}{2}b_i^\dagger\right) + \sin\frac{\theta_i}{2}\left(e^{i\psi_i}\cos\frac{\eta_i}{2} + e^{-i\psi_i}\sin\frac{\eta_i}{2}a_i^\dagger b_i^\dagger\right)\right]|0\rangle$$



$\Rightarrow$ Mott: $\theta_i = 0$

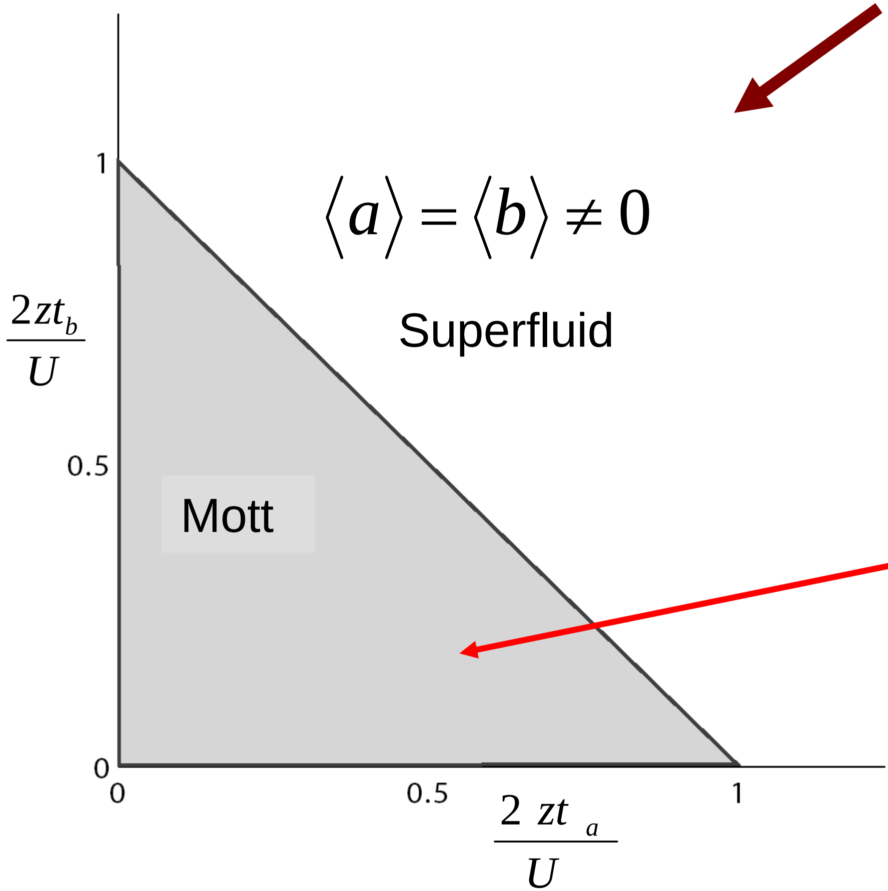
Order parameters:

Superflid of a:	$\langle a\rangle \propto \sin\theta e^{-i(\psi-\varphi/2)}$
Superflid of b:	$\langle b\rangle \propto \sin\theta e^{-i(\psi+\varphi/2)}$
Paired superfluid	$\langle a^\dagger b^\dagger\rangle = \propto \sin^2(\theta/2)e^{i2\psi}$
Counterflow superfluid / x-y ferromagnet:	$\langle a^\dagger b\rangle = \langle S^+\rangle \propto \cos^2(\theta/2)e^{i\varphi}$
Relative density wave/ z antiferromgnet	$\langle \tilde{S}^z_\pi\rangle = \frac{1}{N}\sum_{\mathbf{r}}\langle S^z_{\mathbf{r}}\rangle e^{i\mathbf{r}\cdot\pi} \propto \cos^2(\theta/2)(\cos\chi_A - \cos\chi_B)$

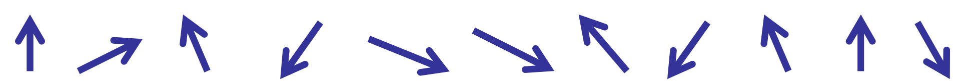
Note: superfluid of both a and b is necessarily also a paired and a counterflow SF because both total and relative phases are fixed.

Variational mean field theory (II)

Minimize variational energy: $\langle \Phi | H | \Phi \rangle$



$$|\Phi_{MI}\rangle = \prod_i \left(e^{i\varphi/2} \sin\frac{\chi_i}{2} a_i^\dagger + e^{-i\varphi/2} \cos\frac{\chi_i}{2} b_i^\dagger \right) |0\rangle$$



$E_{MI} = 0$ independent of local spin orientation.

→ Massive degeneracy

Where are the spin phases?

Dilema

Effective Hamiltonian in low energy subspace captures spin ordering but cannot access the superfluid phase

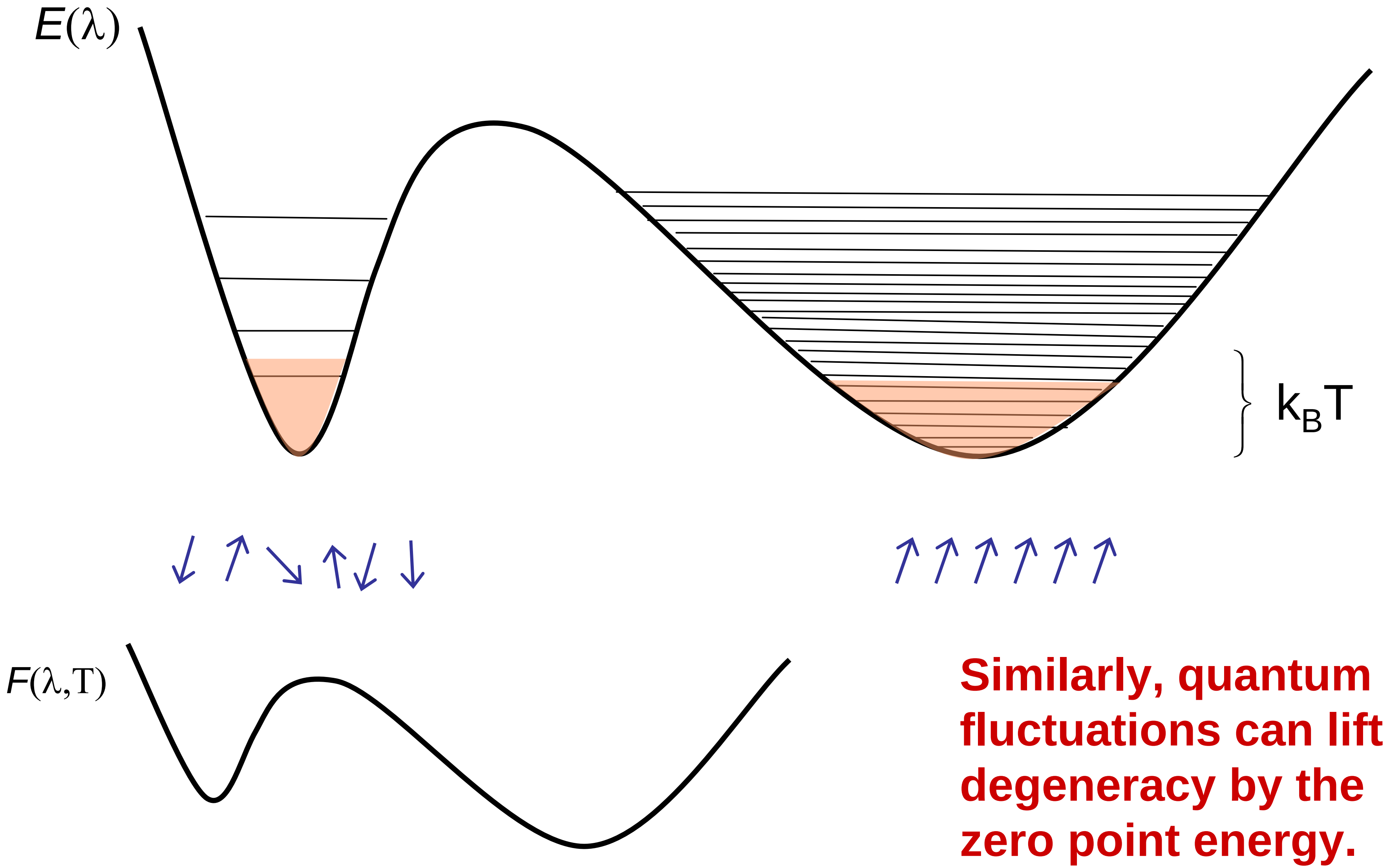
Variational mean field theory captures transition to superfluid but spin orders degenerate in the Mott phase.

Solution

Quantum fluctuations about the variational states lift the degeneracy selecting specific ordered states

$\Rightarrow$ **"Order from disorder" mechanism**

Digression: "classical order from disorder"

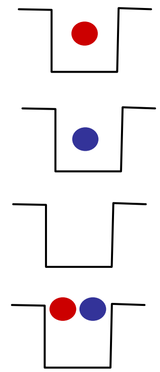


Similarly, quantum fluctuations can lift degeneracy by the zero point energy.

Fluctuations (I)

As before, define second quantized operators that create the local Hilbert space:

$$
\begin{aligned}
a_i^\dagger\,|\,0\,\rangle &\rightarrow \alpha_{1i}^\dagger\,|\,vac\,\rangle \\
b_i^\dagger\,|\,0\,\rangle &\rightarrow \alpha_{2i}^\dagger\,|\,vac\,\rangle \\
|\,0\,\rangle &\rightarrow h_i^\dagger\,|\,vac\,\rangle \\
a_i^\dagger b_i^\dagger\,|\,0\,\rangle &\rightarrow p_i^\dagger\,|\,vac\,\rangle
\end{aligned}
$$



Constraint:

$$\alpha_{1i}^\dagger\alpha_{1i} + \alpha_{2i}^\dagger\alpha_{2i} + p_i^\dagger p_i + h_i^\dagger h_i = 1$$

Rotate the basis:

$$
\begin{pmatrix} \psi_{0i} \\ \psi_{1i} \\ \psi_{2i} \\ \psi_{3i} \end{pmatrix} =
\begin{pmatrix}
\cos\frac{\theta_i}{2}\cos\frac{\chi_i}{2} & \cos\frac{\theta_i}{2}\sin\frac{\chi_i}{2} & \sin\frac{\theta_i}{2}\cos\frac{\eta_i}{2} & \sin\frac{\theta_i}{2}\sin\frac{\eta_i}{2} \\
-\sin\frac{\theta_i}{2}\cos\frac{\chi_i}{2} & -\sin\frac{\theta_i}{2}\sin\frac{\chi_i}{2} & \cos\frac{\theta_i}{2}\cos\frac{\eta_i}{2} & \cos\frac{\theta_i}{2}\sin\frac{\eta_i}{2} \\
-\sin\frac{\chi_i}{2} & \cos\frac{\chi_i}{2} & 0 & 0 \\
0 & 0 & -\sin\frac{\eta_i}{2} & \cos\frac{\eta_i}{2}
\end{pmatrix}
\begin{pmatrix} \alpha_{1i} \\ \alpha_{2i} \\ h_i \\ p_i \end{pmatrix}
$$

Fluctuation operators

$$|\Phi\rangle = \prod_i \psi_{0i}^\dagger|\Omega\rangle$$

Fluctuations (II)

Eliminate ψ_{0i} using the constraint:

$$\psi_{0i} \rightarrow \sqrt{1 - \sum_{\alpha=1}^{3} \psi_{\alpha i}^{\dagger} \psi_{\alpha i}}$$

$$H = \langle \Phi \,|\, H \,|\, \Phi \rangle + \frac{1}{2} \sum_{\mathbf{k}} \left\{ \mathbf{\Psi}_k^{\dagger} \begin{pmatrix} \mathcal{F}_k & \mathcal{G}_k \\ \mathcal{G}_k^{\star} & \mathcal{F}_k^{\star} \end{pmatrix} \mathbf{\Psi}_k - \mathrm{tr}\, \mathcal{F}_k \right\} + \ldots$$

$$\mathbf{\Psi}_{\mathbf{k}}^{\dagger} \equiv (\psi_{1,\mathbf{k}}^{\dagger} \quad \psi_{2,\mathbf{k}}^{\dagger} \quad \psi_{3,\mathbf{k}}^{\dagger} \quad \psi_{1,-\mathbf{k}} \quad \psi_{2,-\mathbf{k}} \quad \psi_{3,-\mathbf{k}})$$

(Precise form of the Hamiltonian depends on the variational state)

Diagonalize with Bogoliubov transformation

Zero point energy: $\Delta E = \frac{1}{2} \sum_{k} \left\{ -\mathrm{tr}\, \mathcal{F}_k + \sum_{\alpha} \omega_{\alpha k} \right\}$

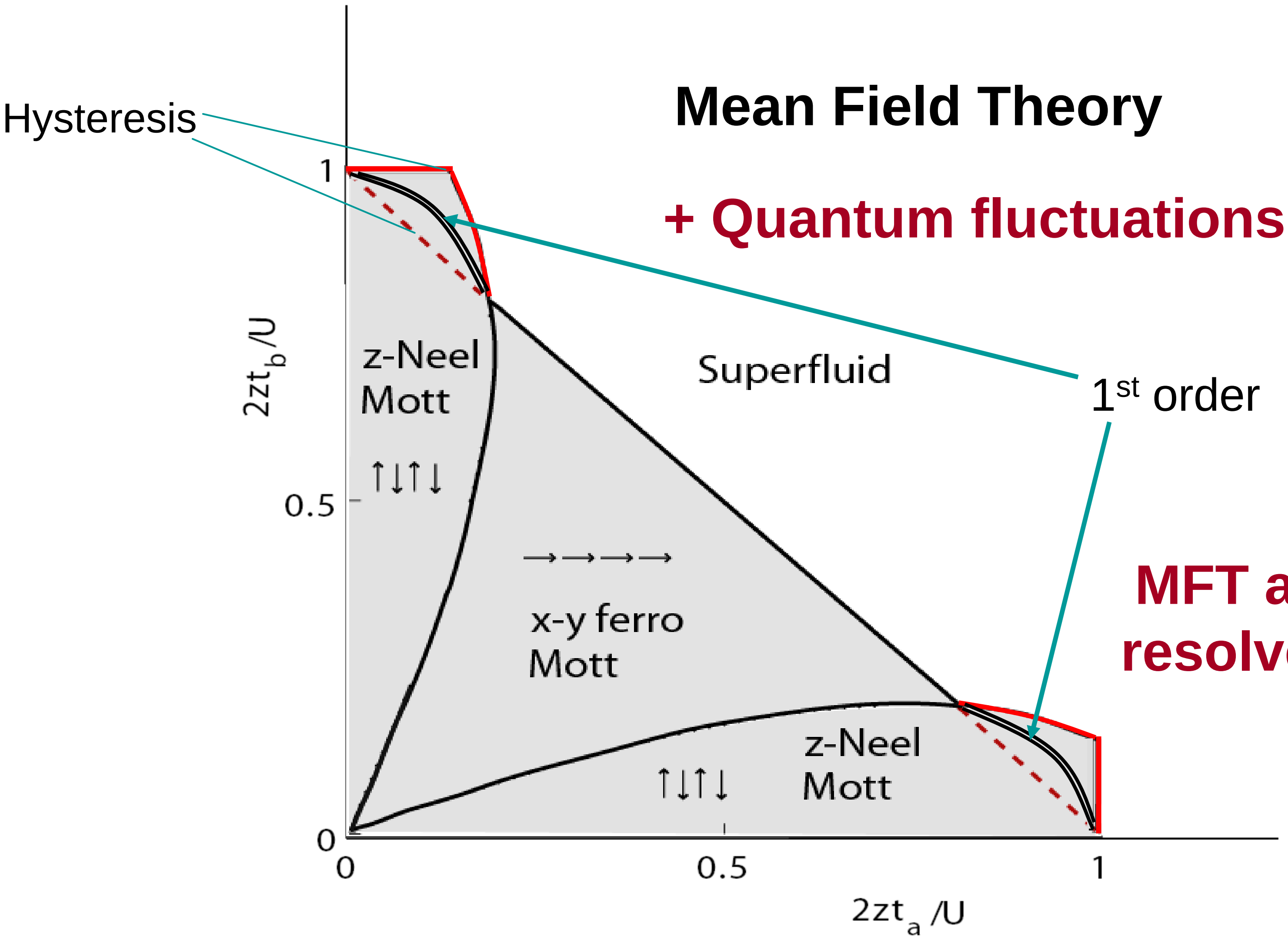
Depends on the assumed spin ordering

Mott-SF transition of 2 comp. bosons

More detail: E. Altman, W. Hofstetter, E. Demler and M. Lukin, NJP 5 (03)

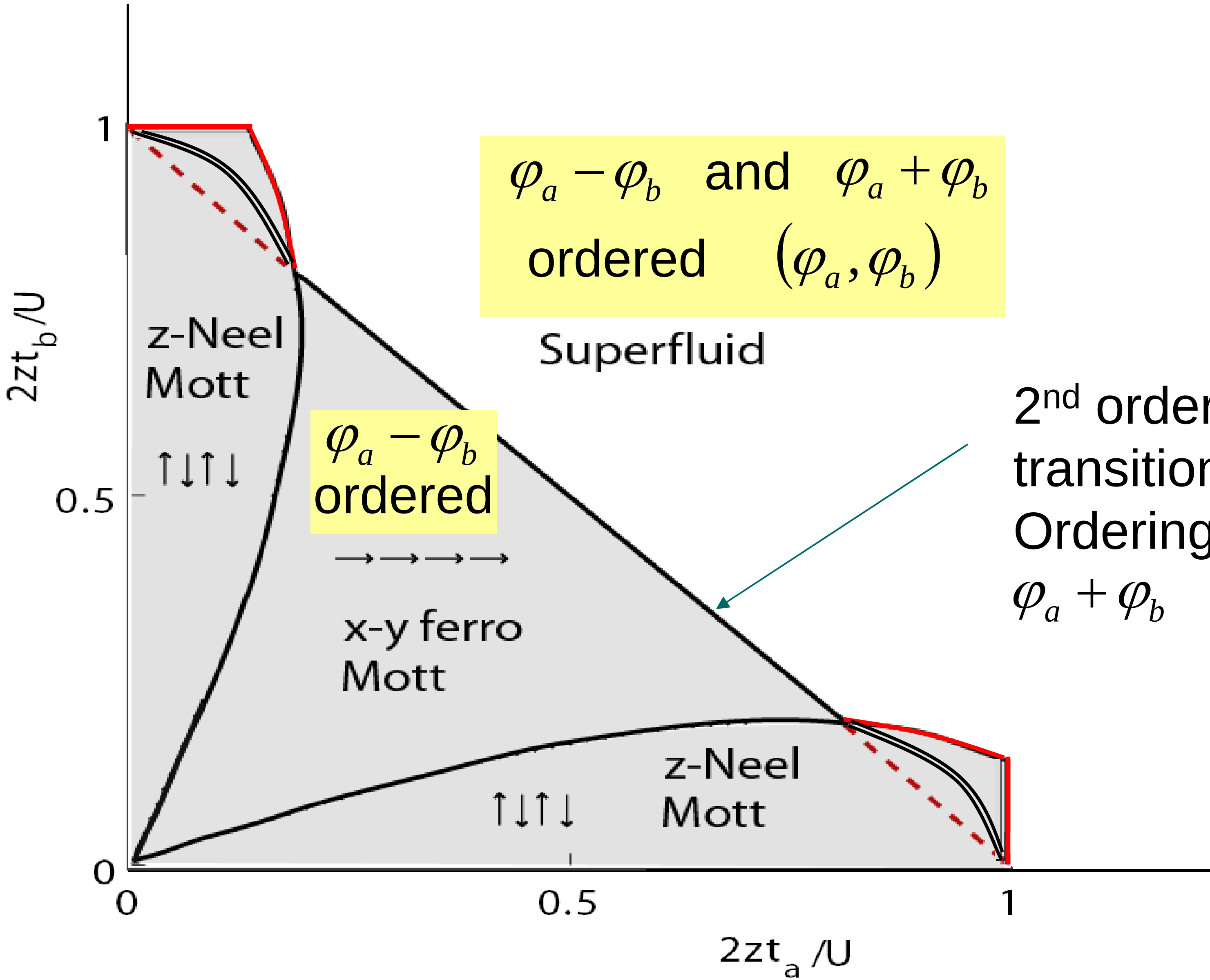
Mean Field Theory

+ Quantum fluctuations



MFT alone cannot resolve spin order!

Mott-SF transition of 2 comp. bosons



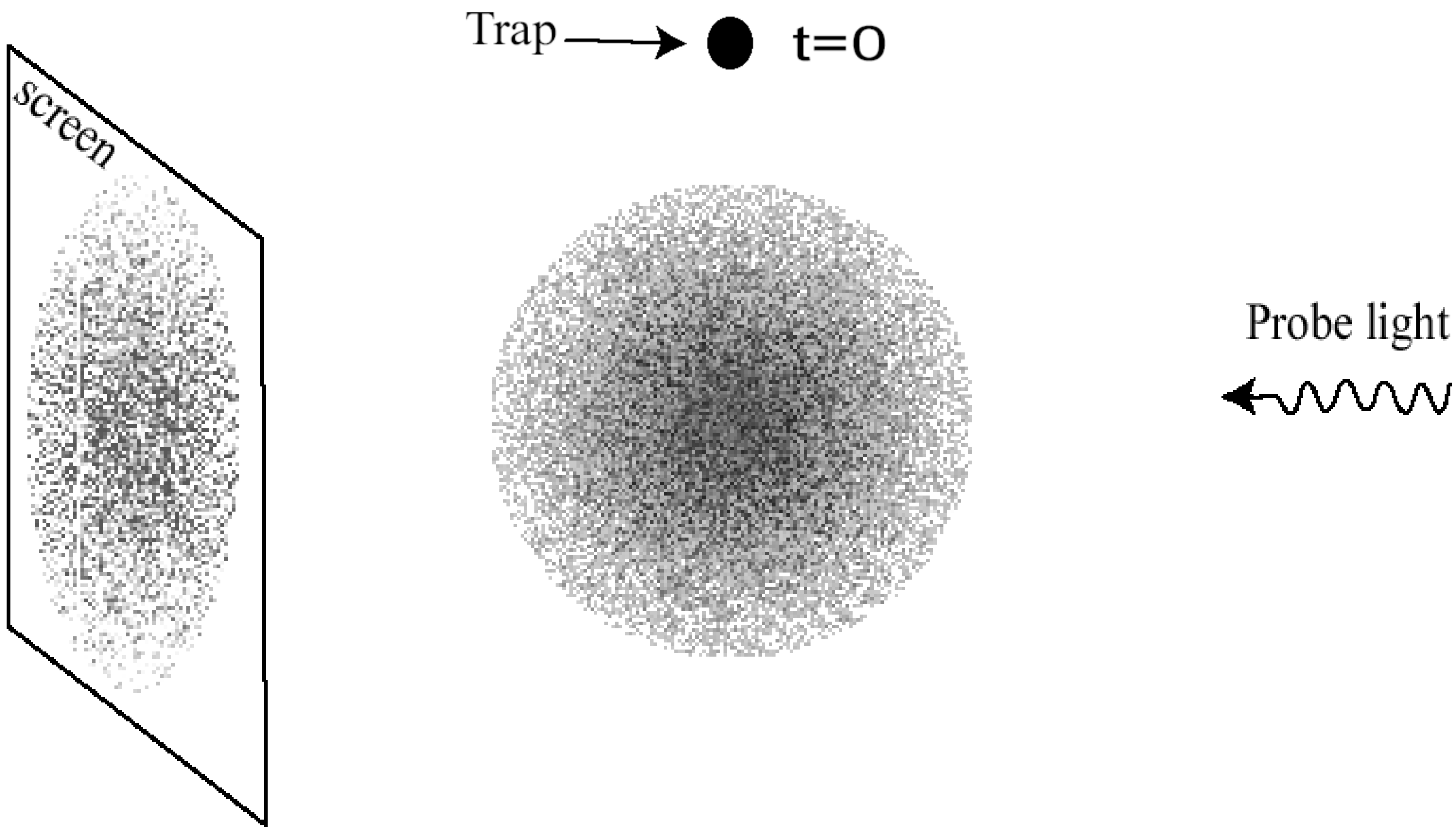
2nd order *x-y* transition. Ordering of $\varphi_a + \varphi_b$

How to detect spin order experimentally?

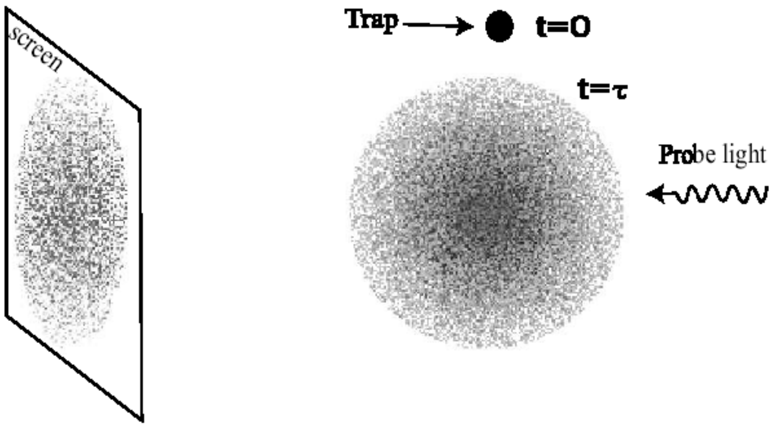
III

Probing many body states of ultra-cold atoms via noise correlations

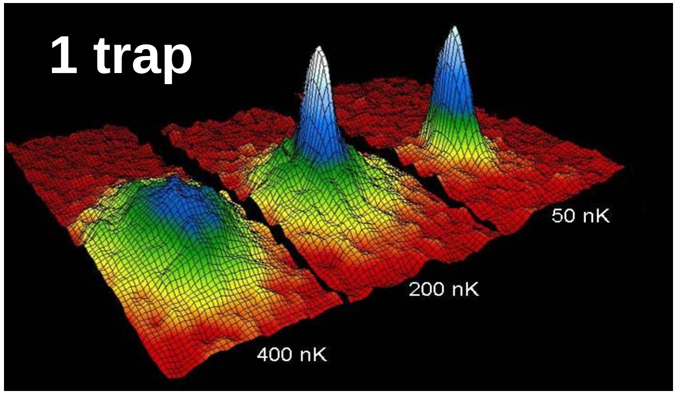
Time of flight experiment



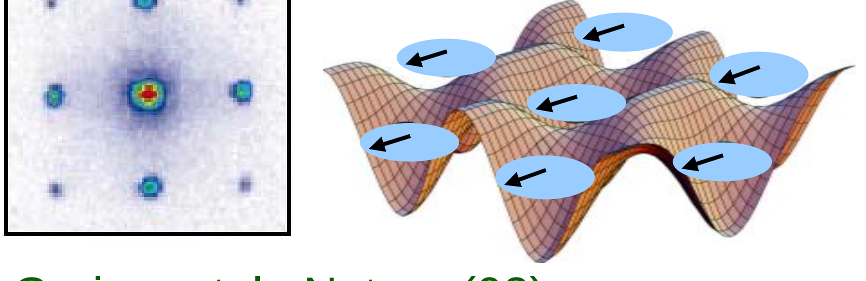
BEC and Matter Waves



Anderson etal, Science (95)



Lattice



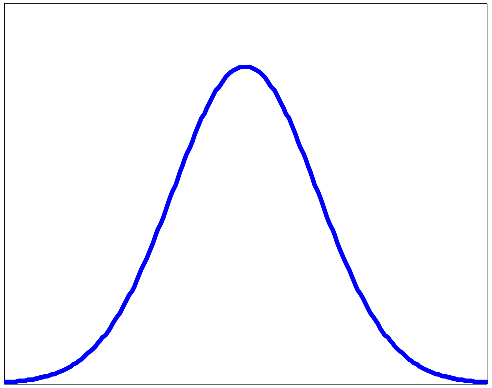
Greiner etal., Nature (02)

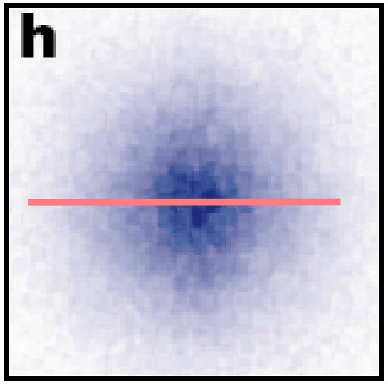
Macroscopic occupation of a single particle wave function.

$$\psi(\mathbf{k})^N$$

What if the state is not a product of single particle wave functions?

Time of flight image





Average cloud density after long expansion:

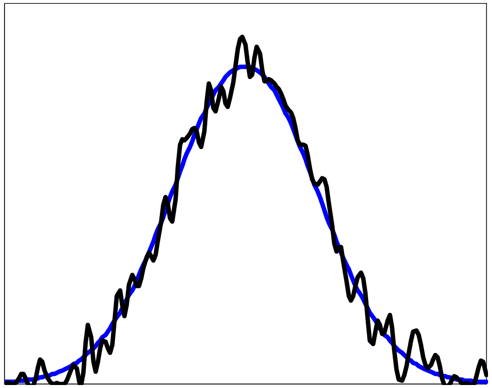
$$\langle \hat{n}_\alpha(\mathbf{r}) \rangle_t = \langle \Phi \,|\, U_0^\dagger(t) \psi_\alpha^\dagger(\mathbf{r}) \psi_\alpha(\mathbf{r}) U_0(t) \,|\, \Phi \rangle \sim \langle n_\mathbf{k} \rangle_{t=0}$$

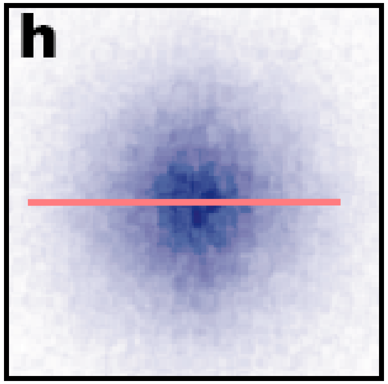
After expansion

In trap

But a single shot does not measure an expectation value !

$\Rightarrow$ There are fluctuations (shot noise).





<u>Proposal:</u> extract information from the noise

E. Altman, E. Demler and M. Lukin, PRA (04)

Correlations in the noise:

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \langle n(\mathbf{r})n(\mathbf{r}')\rangle_t - \langle n(\mathbf{r})\rangle_t \langle n(\mathbf{r}')\rangle_t \sim \langle n_{\mathbf{k}} n_{\mathbf{k}'}\rangle_0 - \langle n_{\mathbf{k}}\rangle_0 \langle n_{\mathbf{k}'}\rangle_0$$

After expansion

In trap

Time of flight image from an optical lattice (I)

Density expectation value after free expansion for time t :

$$\mathcal{I}(\mathbf{r},t) = \langle\, \Phi_0 \,|\, U_t^\dagger \psi^\dagger(\mathbf{r})\psi(\mathbf{r}) U_t \,|\, \Phi_0 \,\rangle \;=\; \langle\, \Phi_0 \,|\, \psi^\dagger(\mathbf{r},-t)\psi(\mathbf{r},-t) \,|\, \Phi_0 \,\rangle$$

Lives in lowest Bloch band

Projection on lowest Bloch band:

$$\psi(\mathbf{r},-t)\,|\,\Phi_0\,\rangle \;=\; \sum_i w(\mathbf{r}-\mathbf{R}_i,t)a_i\,|\,\Phi_0\,\rangle \;\equiv\; A(\mathbf{r},t)\,|\,\Phi_0\,\rangle$$

Time evolved Wannier function

$$\mathcal{I}(\mathbf{r},t) = \langle A^\dagger(\mathbf{r},t)A(\mathbf{r},t)\rangle_0$$

$$\begin{aligned}\psi(\mathrm{r},-t)\,|\,\Phi_0\,\rangle &= \int d\mathrm{r}' G(\mathrm{r}-\mathrm{r}',t)\psi(\mathrm{r}')\,|\,\Phi_0\,\rangle = \int d\mathrm{r}' G(\mathrm{r}-\mathrm{r}',t)\sum_i w(\mathrm{r}-\mathrm{R}_i)a_i\,|\,\Phi_0\,\rangle \\ &= \sum_i w(\mathrm{r}-\mathrm{R}_i,t)a_i\,|\,\Phi_0\,\rangle = A(\mathrm{r},t)\,|\,\Phi_0\,\rangle\end{aligned}$$

Time of flight image from an optical lattice (II)

Assuming Gaussian Wannier function and long time of flight ($r \gg R_i$):

$$w_i(\mathbf{r},t) \approx \left(\frac{2\pi\hbar^2 t^2}{a^2 m^2}\right)^{-\frac{1}{4}} e^{-\left(\frac{amr}{2\hbar t}\right)^2} e^{i\frac{mr^2}{2\hbar t} - i\frac{m\mathbf{r}\cdot\mathbf{R}_i}{\hbar t}}$$

$$\mathcal{I}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi W(t)^2}} e^{-\frac{r^2}{2W^2}} \sum_{i,j} e^{i(\mathbf{R}_i - \mathbf{R}_j)\cdot\mathbf{Q}(\mathbf{r})} \langle\, \Phi_0 \,|\, a_i^\dagger a_j \,|\, \Phi_0 \,\rangle$$

Gaussian envelope

Interesting part:

Width of the gaussian envelope: $W(t) = \hbar t/(am)$

$$\mathbf{Q}(\mathbf{r}) \equiv \frac{m\mathbf{r}}{\hbar t}$$

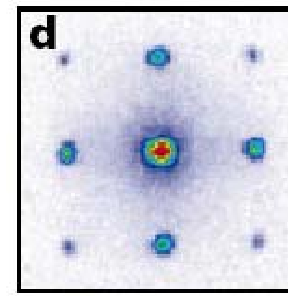
Defines correspondance between position in the cloud and lattice momentum in the trap

What to expect from the expectation value

$$F(\mathbf{r}) = \sum_{i,j} e^{i(\mathbf{R}_i - \mathbf{R}_j)\cdot\mathbf{Q}(\mathbf{r})} \langle \Phi_0 | a_i^\dagger a_j | \Phi_0 \rangle$$

<u>Superfluid:</u>

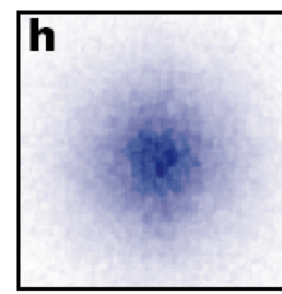
$$\langle a_i^\dagger a_j \rangle = |\Psi|^2$$



$$F(\mathbf{r}) = N(\bar{n} - |\psi|^2) + N\psi|^2 \left(\frac{2\pi a}{l}\right)^2 \sum_{\mathbf{G}} \delta\left(\mathbf{r} - \frac{\hbar t}{m}\mathbf{G}\right)$$

<u>Mott:</u>

$$\langle a_i^\dagger a_j \rangle \approx \delta_{ij} \qquad F(\mathbf{r}) = N\bar{n}$$

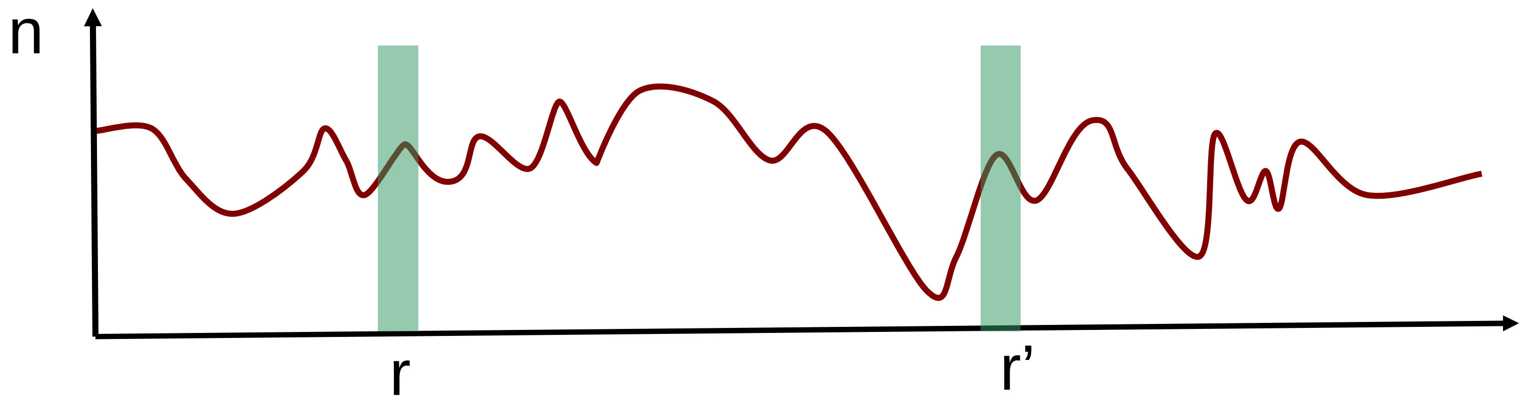


$$\langle a_i^\dagger a_j \rangle = e^{-|R_i - R_j|/\xi}$$

$$F(r) = \left(\frac{2\pi a}{l}\right) N\bar{n} \left(\frac{2\pi a}{l}\right)^2 \sum_{\mathbf{G}} \frac{\xi/\pi}{(r - \frac{\hbar t}{m}\mathbf{G})^2 + \xi^2}$$

Second order correlations (Noise)

After time of flight t: $\mathcal{G}(\mathbf{r},\mathbf{r}') = \langle n(\mathbf{r})n(\mathbf{r}')\rangle_t - \langle n(\mathbf{r})\rangle_t \langle n(\mathbf{r}')\rangle_t$



After normal ordering we can replace again: $\psi(\mathbf{r},-t) \longrightarrow A(\mathbf{r},t)$

And after long time of flight as before :

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = g(\mathbf{r})g(\mathbf{r}')\left[\sum_{ii'jj'} e^{i(\mathbf{R}_j-\mathbf{R}_{j'})\mathbf{Q}(\mathbf{r})+i(\mathbf{R}_i-\mathbf{R}_{i'})\mathbf{Q}(\mathbf{r}')}\langle a_i^\dagger a_j^\dagger a_{j'} a_{i'}\rangle - F(\mathbf{r})F(\mathbf{r}')\right]$$

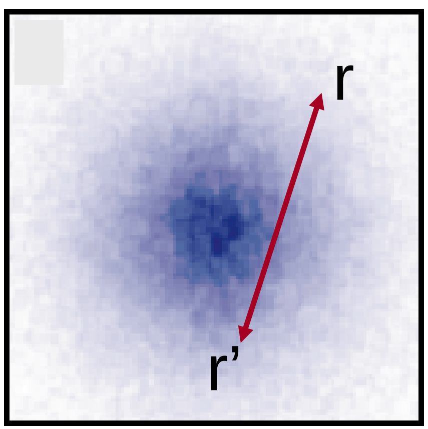
Bose →
Fermi →

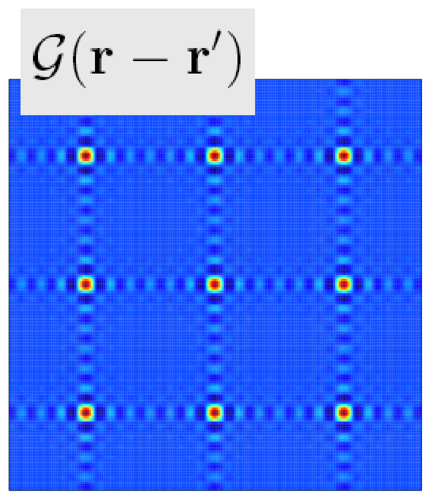
$$\pm\delta(\mathbf{r}-\mathbf{r}')\mathcal{I}(\mathbf{r})$$

Plain vanilla Mott state

$$\langle a_i^\dagger a_j^\dagger a_{j'} a_{i'} \rangle = \bar{n}^2 (\delta_{ii'}\delta_{jj'} \pm \delta_{ij'}\delta_{jix'})$$

$$\mathcal{G}_{Mott}(\mathbf{r},\mathbf{r}') \approx \frac{N}{W^d}\left(\frac{2\pi a_0}{l}\right)^d \sum_{\mathbf{G}} \tilde{\delta}^d \left(\mathbf{r}-\mathbf{r}'+\frac{\hbar t}{m}\mathbf{G}\right)$$



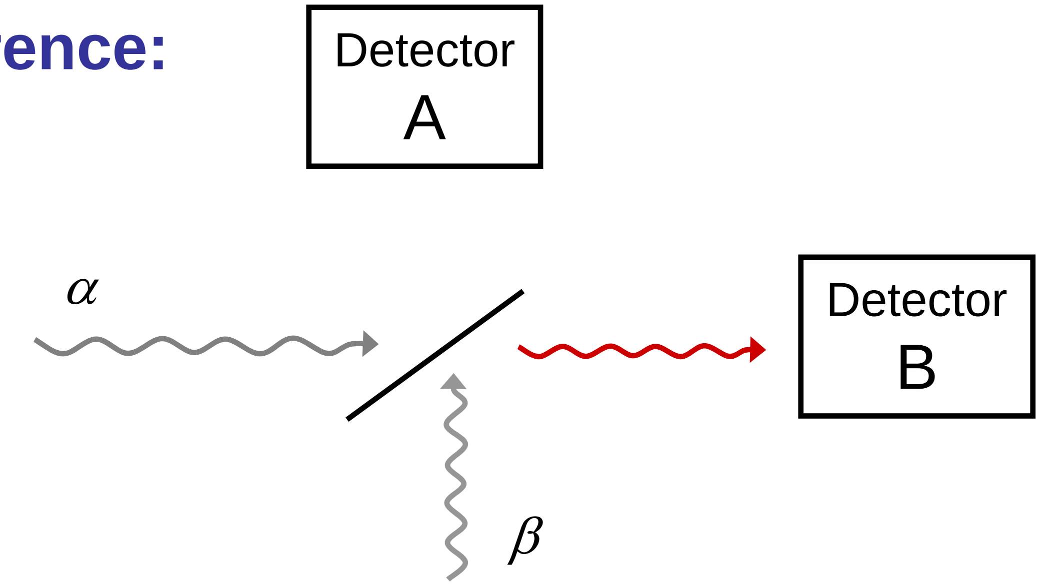


Sharp Bragg peaks in 2nd order coherence!

This is simply bunching/antibunching.

Relation to Hanbury-Brown-Twiss Effect

Standard interference:



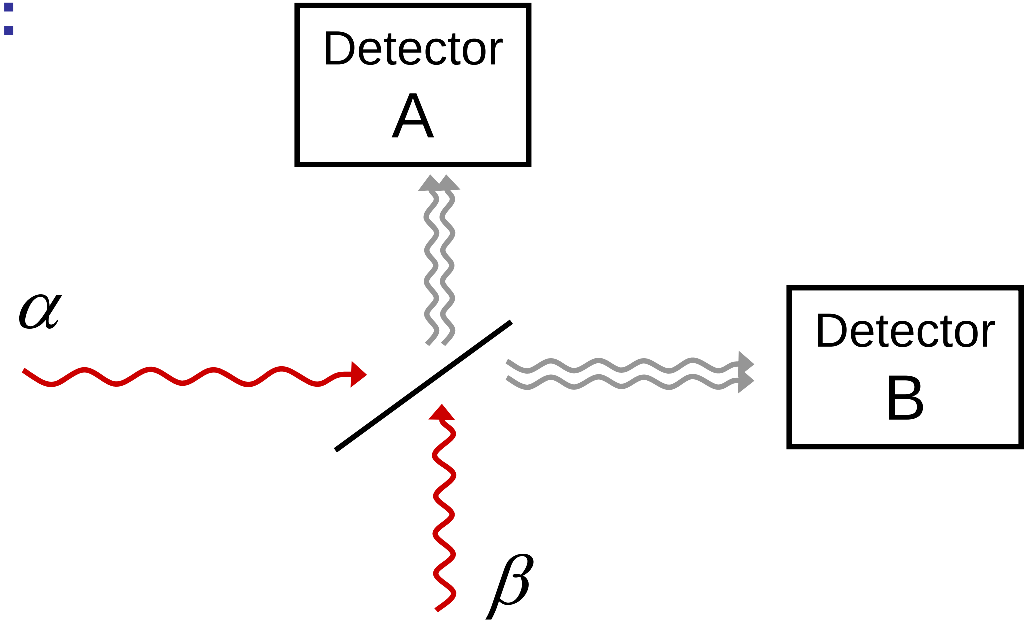
$$(\alpha^\dagger + i\beta^\dagger)\,|\,0\,\rangle \rightarrow (\cancel{A^\dagger} + iB^\dagger + iB^\dagger + \cancel{i^2 A})\,|\,0\,\rangle^\dagger \propto B^\dagger\,|\,0\,\rangle$$

$$(\alpha^\dagger + i\beta^\dagger)^N\,|\,0\,\rangle \rightarrow (B^\dagger)^N\,|\,0\,\rangle$$

Single particle coherence!

Relation to Hanbury-Brown-Twiss Effect

HBT interference:

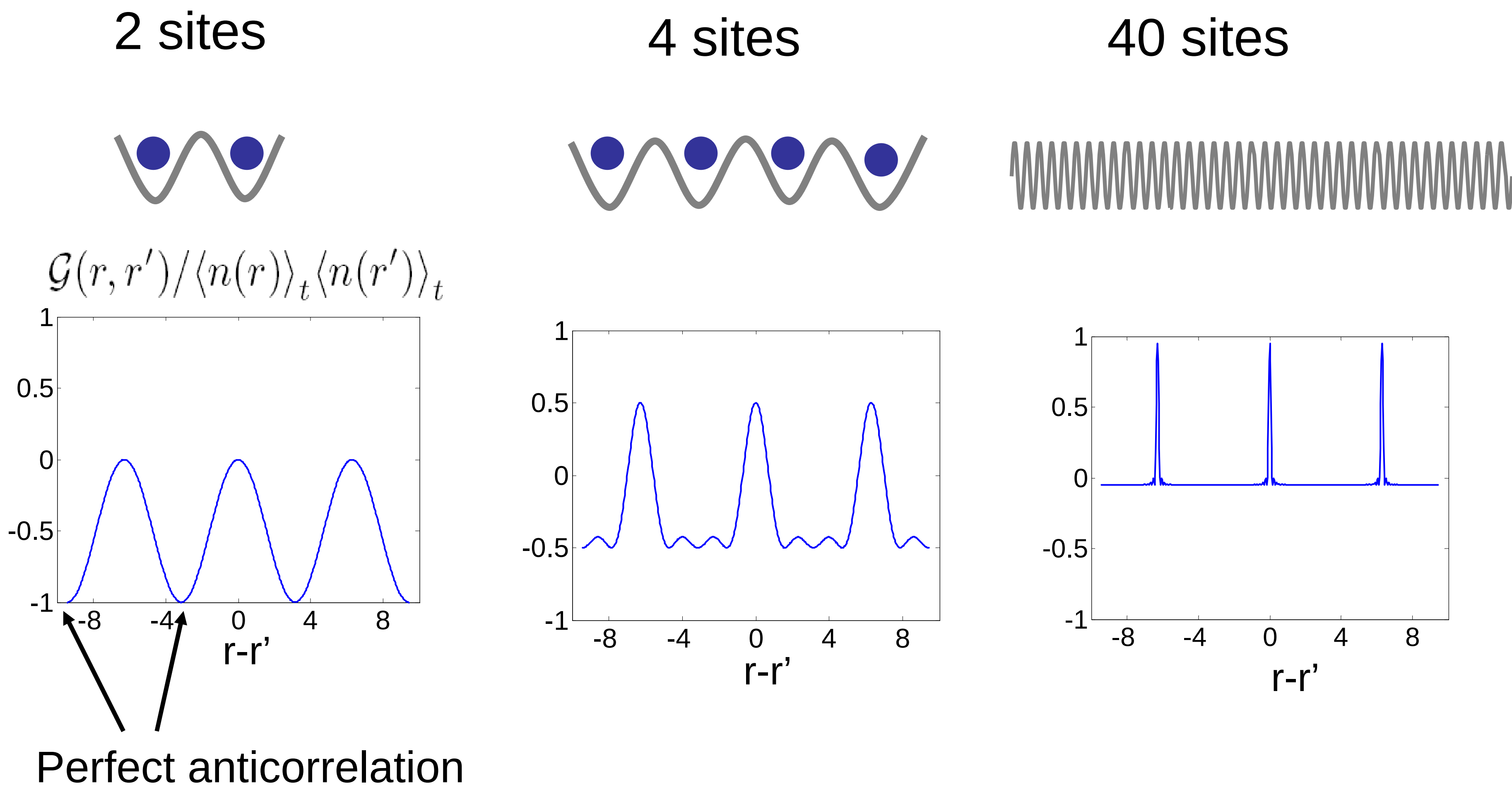


Number state as in the Mott insulator.

$$\alpha^\dagger \beta^\dagger \,|\,0\,\rangle \rightarrow (A^\dagger + iB^\dagger)(B^\dagger + iA^\dagger)\,|\,0\,\rangle \;=\; \left(\cancel{A^\dagger B^\dagger} + \cancel{i^2 B^\dagger A^\dagger} + (A^\dagger)^2 + (B^\dagger)^2\right)\;|\,0\,\rangle$$

Two particle interference ! $\Rightarrow$ Bunching

Apparent only in the correlation between detectors.
Not in the average count.



Notes: 1. For fermions the result should be inverted

2. Identical result for a high temperature gas confined to the lowest Bloch band

Can we learn more about the state other than quantum statistics?

Need to relax indistinguishability ! → **Spin**

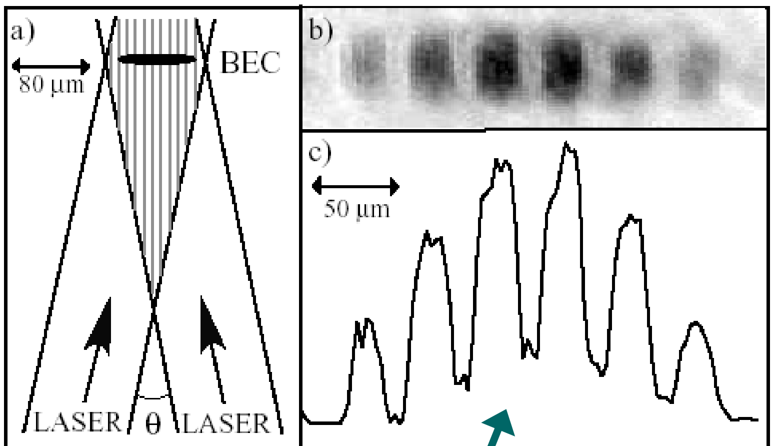
Experiment

Cond-mat/0405113

Interference of an array of independent Bose-Einstein condensates

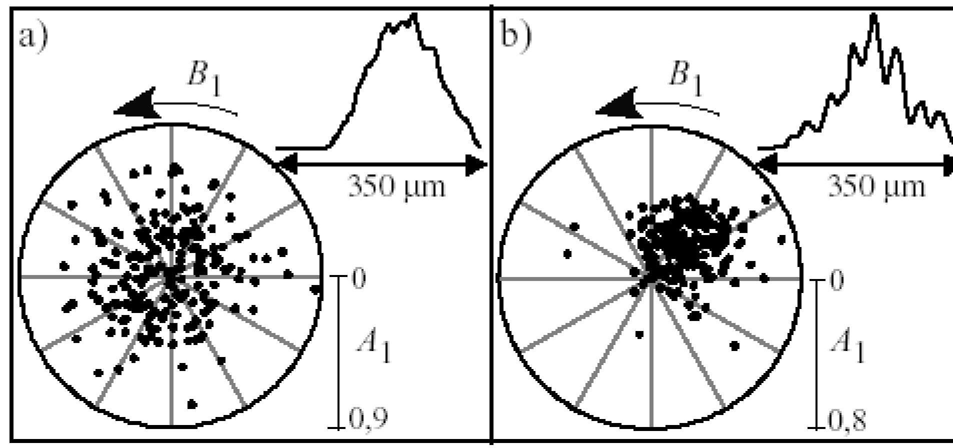
Zoran Hadzibabic, Sabine Stock, Baptiste Battelier, Vincent Bretin, and Jean Dalibard
Laboratoire Kastler Brossel, 24 rue Lhomond, 75005 Paris, France*
(Dated: May 19, 2004)

30 independent wells:



Smooth structure is a result of low experimental resolution (filtering)

Fringe phase and amplitude:



Coupled wells decoupled wells

Classical limit of our prediction !

Detection of spin order

Two species: $\alpha = \uparrow, \downarrow$

$$\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}') = \langle n_\alpha(\mathbf{r}) n_\beta(\mathbf{r}')\rangle_t - \langle n_\alpha(\mathbf{r})\rangle_t \langle n_\beta(\mathbf{r}')\rangle_t$$

$$\begin{aligned}\mathcal{G}_{\alpha\beta}(\mathbf{r},\mathbf{r}') &= g(\mathbf{r})g(\mathbf{r}')\left[\sum_{ii'jj'} e^{i(\mathbf{R}_j-\mathbf{R}_{j'})\mathbf{Q}(\mathbf{r})+i(\mathbf{R}_i-\mathbf{R}_{i'})\mathbf{Q}(\mathbf{r}')}\langle a^\dagger_{\alpha i} a^\dagger_{\beta j} a_{\beta j'} a_{\alpha i'}\rangle - F_\alpha(\mathbf{r})F_\beta(\mathbf{r}')\right] \\ &\pm \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\mathcal{I}(\mathbf{r})\end{aligned}$$

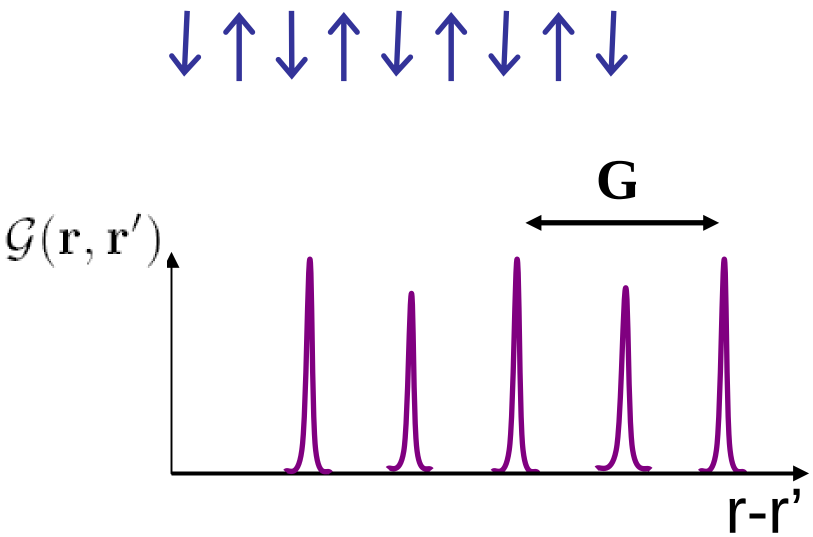
Mott insulator 1 particle per site: (Spin insensitive detection)

$$\begin{aligned}\mathcal{G}(\mathbf{r},\mathbf{r}') &= \sum_{\alpha\beta}\mathcal{G}_{\alpha\beta} = \frac{\eta}{2}N\left(\frac{2\pi a_0}{l}\right)^d \sum_{\mathbf{G}}\tilde{\delta}^d\left(\mathbf{r}-\mathbf{r}'+\frac{\hbar t}{m}\mathbf{G}\right) \\ &+2\eta\sum_{ij} e^{i(\mathbf{Q}(\mathbf{r})-\mathbf{Q}(\mathbf{r}'))\cdot\mathbf{R}_{ij}}\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle\end{aligned}$$

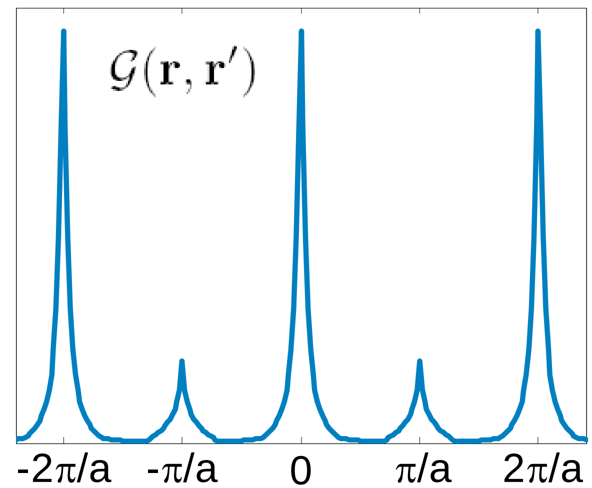
$$\mathbf{S}_i = \tfrac{1}{2} a^\dagger_{\alpha i}\vec{\sigma}_{\alpha\beta} a_{\beta i}$$

Direct measurement of the spin structure factor !

Example: antiferromagnet



One dimension (quasi LRO)



How can we detect spin order without spin sensitivity?

Condensed matter analog: unpolarized neutron scattering.

Another way to understand:

Two interpenetrating Mott states of indistinguishable particles each has a doubled unit cell.

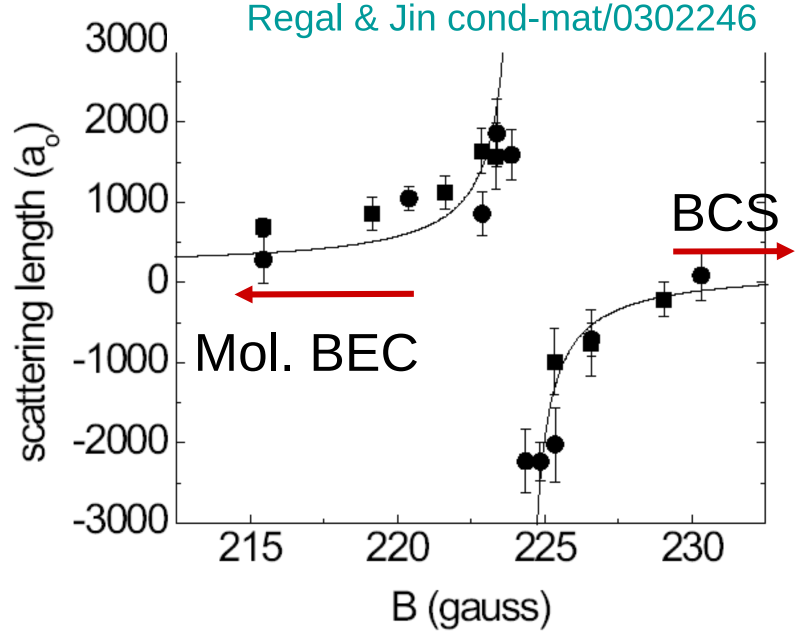
$\Longrightarrow$ twice the number of brag peaks

Fermion superfluidity

Use Feshbach resonance to induce superfluidity.

JILA (Jin)
MIT (Ketterle)
Innsbruck (Grim)
Rice (Hulet)
ENS (Solomon)

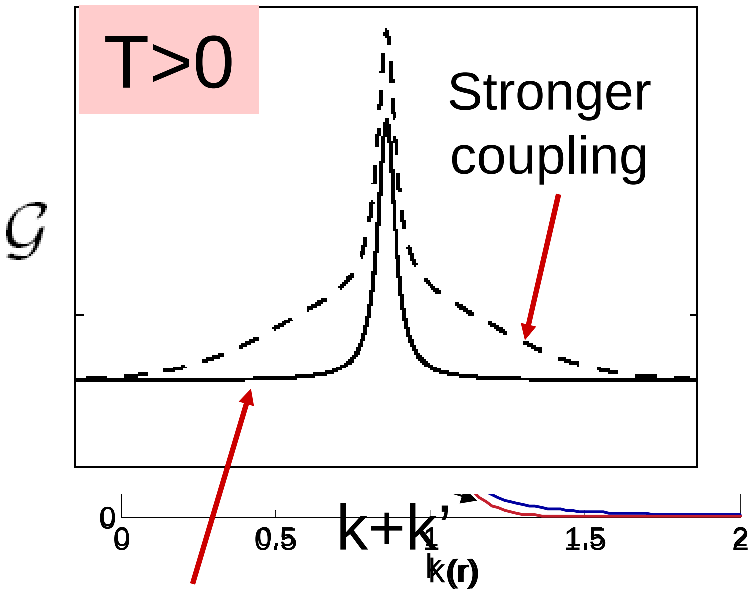
Regal & Jin cond-mat/0302246



How to directly detect pair condensation in the attractive regime ?

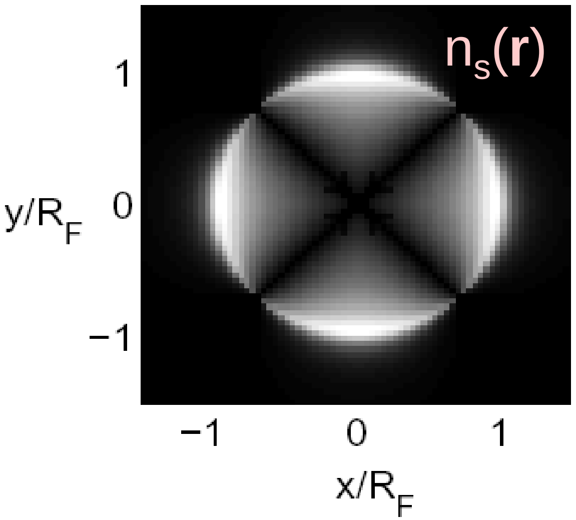
(see Jin talk for indirect method)

Why is superfluidity hard to detect?



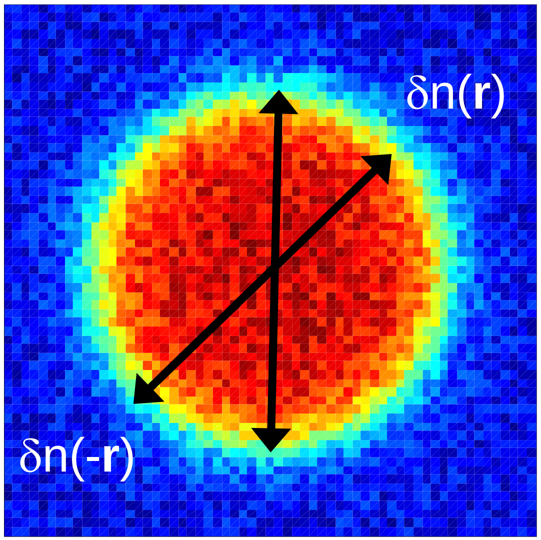
Weak coupling BCS

e.g. d-wave



But the fluctuations are distinct:

(k,-k) pairs



$$\mathcal{G}(\mathbf{r},\mathbf{r}') = 2|u_{\mathbf{Q}(\mathbf{r})}|^2|v_{\mathbf{Q}(\mathbf{r}')}|^2\tilde{\delta}(\mathbf{r}+\mathbf{r}')$$

$n_s(\mathbf{r})$

Analogue of BEC peak

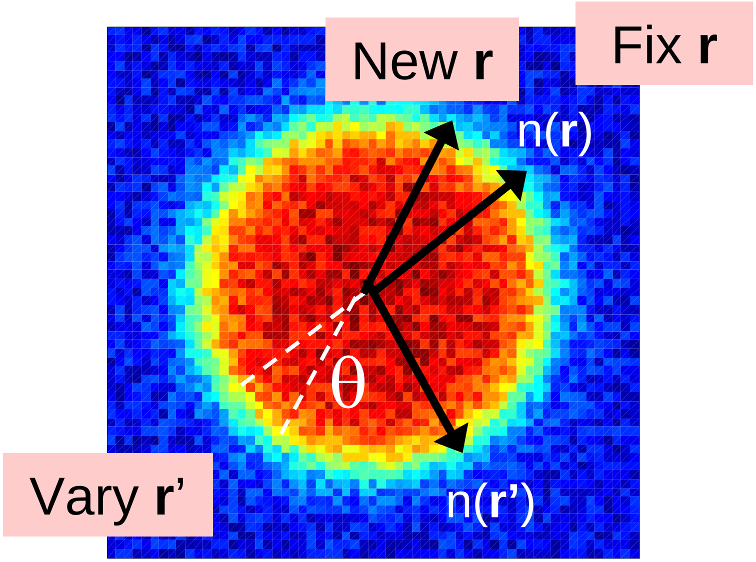
Width: $\hbar t/mL$

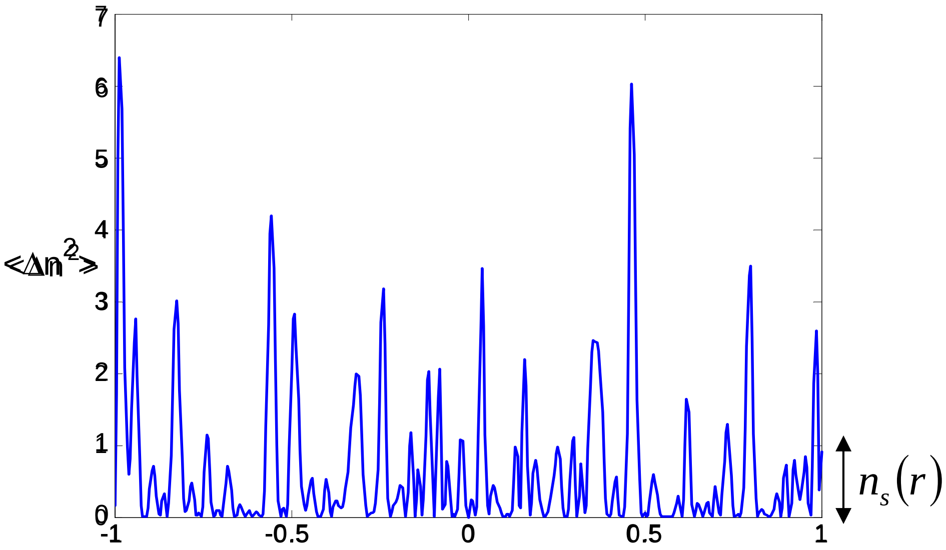
Sensitive to pairing symmetry

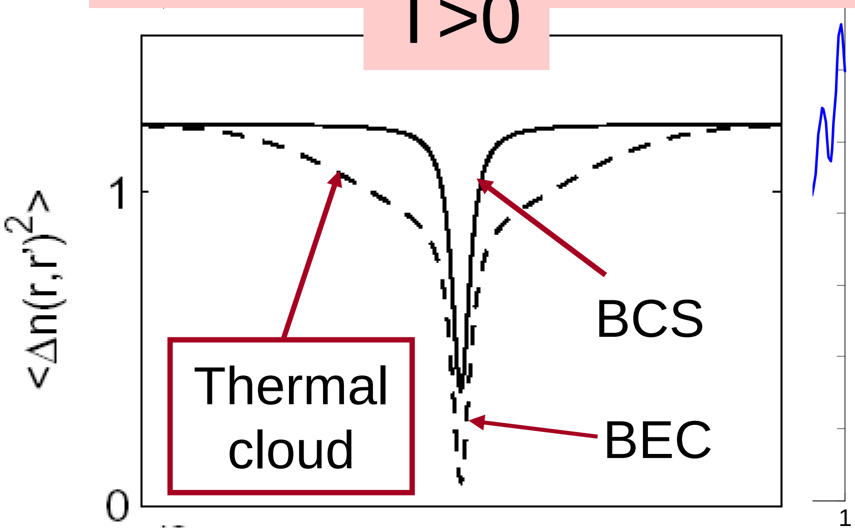
Alternative measurement

$$\Delta n(\mathbf{r},\mathbf{r}') \equiv n(\mathbf{r}) - n(\mathbf{r}')$$

$$\Delta n(\mathbf{r},-\mathbf{r})\left|\Psi_{BCS}\right\rangle = 0$$



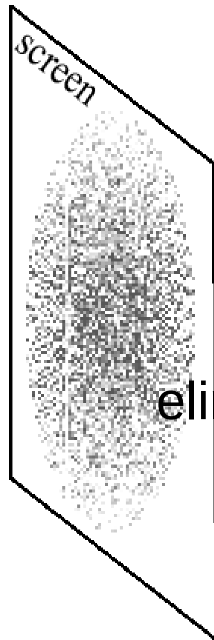




$$N_{average} = Lk_F \sim N^{1/3}$$

independent Positions on a perimeter

Atom shot noise versus other noise sources



Number of detected photons:

$$\langle p_{\rm col}\rangle = \eta p_{in} e^{-\kappa}$$

$$\kappa = \kappa_0 N_A$$

$$\langle p_0\rangle = \eta p_{in}$$

Empty image

$$\langle p\rangle = \langle p_{\rm col}\rangle - \langle p_0\rangle$$

eliminates noise originating from
from the optical aparatus:

Shot Noise:

Demand

$$\langle \delta p^2\rangle_{atom} = \eta^2 p_{in}^2 e^{-2\kappa}\kappa^2/\langle N_A\rangle \quad > \quad \langle \delta p^2\rangle_{photon} \approx \langle \delta p_0^2\rangle = \eta p_{in}$$

$$\boxed{p_{in} > \frac{e^{2\kappa}}{\eta\kappa^2}\langle N_A\rangle} \quad \Rightarrow \quad \text{Optimal} \quad \kappa \approx 1$$

Typically: $p_{in}/N_A \sim 10\text{-}100$ ✓

Conclusions

- **Quantum dynamics beyond Gross-Pitaevskii.**
  Rapid quench from localized Mott to superfluid.
  How does the order parameter develop?

- **Two component bosons on optical lattice.**
  Spin ordered phases
  SF-Insulator transition qualitatively different from spinless

- **Detection of many-body correlations via noise.**
  Plain vanilla Mott: **peaks due to** bunching/antibunching
  With spin: detect static spin structure factor
  Fermions: detect pairing correlations