Equation for the PDF of the statistics

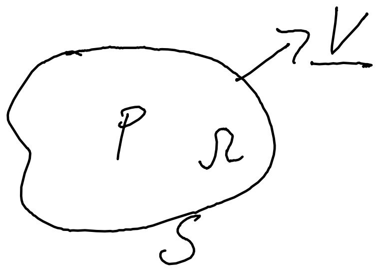
Consider Ordinary 1st order Differential Equation set

$$\frac{dx_i}{dt} = \underbrace{V_i(x_1, \dots, x_N)}_{\text{deterministic}} + \underbrace{\eta_i(t)}_{\text{noise}}$$

$$\frac{d\underline{x}}{dt} = \underline{V} + \underline{\eta}$$

"Handwaving" argument for derivation of eqn for probability distribution

[see slide picture]



Flux of P across surface is sum of deterministic and stochastic flux

$$\int_S -\underline{V}P + \Gamma\nabla P \; dS$$

$$= \int_V -\underline{\nabla}\cdot\left(\underline{V}P - \Gamma\nabla P\right) d\Omega$$

$$\Rightarrow \frac{d}{dt}\int_{\Omega} P(\underline{x},t)\, d\Omega = \int_{\Omega} -\underline{\nabla}\cdot\left(\underline{V}P - \Gamma\nabla P\right) d\Omega$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial t}(\underline{x},t) = -\underline{\nabla}\cdot\left(\underline{V}P\right) + \Gamma\nabla^2 P}$$

- Note this argument is easily formalised
- It is not-so-easily extended to PDEs and infinite dimensions.

Hopf Functional Technique for ODEs

(eg Mat Marston 2005)

- Consider set of 1st order differential eqns

$$\boxed{\frac{dx_i}{dt} = F_i(x_1, \ldots, x_N) \qquad i = 1, N}$$

$$\left(\frac{d\underline{x}}{dt} = \underline{F}\right) \qquad \text{(finite dimensional)}$$

- let u_i ($\underline{u}$) be variables conjugate to $\underline{x}$

and write

$$\psi[\underline{u}, \underline{x}(t)] = e^{i\underline{u}\cdot\underline{x}(t)}$$

Define

$$\hat{H} = -\sum_{j=1}^{N} u_j F_j\left(\frac{-i\partial}{\partial u_1}, \frac{-i\partial}{\partial u_2}, \ldots, \frac{-i\partial}{\partial u_N}\right)$$

is the Hopf 'Hamiltonian' operator

then

$$\boxed{i\frac{d\psi}{dt} = \hat{H}\psi}$$

is the Hopf Eqn.

Exercise: Check the Hopf Eqn $\Rightarrow$ EOM.

NOTE

i) $\hat{H}$ generally non-Hermitian

ii) We have turned a <u>nonlinear</u> system for $\underline{x}$ into a much higher dimensional <u>linear</u> system for the "wavefunction" ψ.

iii) Equation can be averaged over initial conditions to get statistics ...

Ensemble average

$$\overline{\psi}(\underline{u},t) = \overline{e^{i\underline{u}\cdot\underline{x}(t)}}$$
$$= \int_{-\infty}^{\infty} e^{i\underline{u}\cdot\underline{x}} \mathbb{P}(\underline{x}(0))\, d^N \underline{x}(0)$$

· average over all initial conditions.

<u>Now</u>
$$-i\frac{\partial\overline{\psi}}{\partial u_j}\bigg|_{\underline{u}=0} = \int_{-\infty}^{\infty} x_j\, \mathbb{P}(\underline{x}(0))\, d^N\underline{x}(0)$$
$$= \overline{x_j(t)}$$

$$-\frac{\partial^2\overline{\psi}}{\partial u_j \partial u_k}\bigg|_{\underline{u}=0} = \int_{-\infty}^{\infty} x_j(t)\, x_k(t)\, \mathbb{P}(\underline{x}(0))\, d^N\underline{x}(0)$$

$$= \overline{x_j(t)\, x_k(t)}$$

- Note: The characteristic functional is the Fourier transform of the PDF.
- n-point correlation functions are more easily extracted from the characteristic functional than the PDF.
- Becomes particularly clear for PDEs

- Hopf + Fokker-Planck approaches are dual to one another.

$$x_j \longrightarrow -i\frac{\partial}{\partial u_j}$$

$$\frac{\partial}{\partial x_j} \longrightarrow -iu_j$$

- BOTH APPROACHES TRADE LINEARITY FOR AN INCREASE IN DIMENSIONALITY.

Thinking about writing down equations for the statistics; an example from dynamo theory.

Remember induction eqn

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{u} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

Write $\underline{u} = \overline{\underline{u}} + \underline{u}'$; $\underline{B} = \overline{\underline{B}} + \underline{b}'$

traditional to set $\overline{\underline{u}} = 0$ (for simplicity)

$$\Rightarrow \frac{\partial \overline{\underline{B}}}{\partial t} = \underline{\nabla} \times \overline{(\underline{u}' \times \underline{b}')} + \eta \nabla^2 \overline{\underline{B}}$$

$$\underline{\mathcal{E}} = \overline{\underline{u}' \times \underline{b}'}$$ (mean EMF) cf Reynolds stress

$$\frac{\partial \underline{b}'}{\partial t} = \underline{\nabla} \times (\underline{u}' \times \overline{\underline{B}}) + \underbrace{\underline{\nabla} \times \underline{G}}_{\text{pain term}} + \eta \nabla^2 \underline{b}'$$

$$\underline{G} = \underline{u}' \times \underline{b}' - \overline{\underline{u}' \times \underline{b}'}$$

We really want to know the EMF $\underline{\mathcal{E}}$
(the fluc/fluc → mean interaction).

What could we do?
We could say it is going to be some function of the mean field with sensible properties $\underline{\mathcal{E}} = f(\overline{\underline{B}})$
(cf Baylor's talk
– remember Baylor had different functionalities depending on his system + perhaps scale)

Actually as the induction eqn is formally linear one can write a linear Taylor Expansion for $\underline{\mathcal{E}}$

$$\mathcal{E}_i = \underbrace{\alpha_{ij} \overline{B}_j}_{\text{generational term.}} + \underbrace{\beta_{ijk} \frac{\partial \overline{B}_j}{\partial x_k}}_{\text{diffusive term}} + \dots$$

(Steenbeck et al 1966)

Symmetry considerations ⇒ $\alpha_{ij} = 0$ if the system is reflectionally symmetric. $\alpha = 0$ @ equator

But we have at this point no theory for how α_{ij}, β_{ijk} behave as functions of other variables

- Can choose them plausibly

Now the key thing is the EMF $\underline{\mathcal{E}}$

we'd like an equation for it.

Of course $\dfrac{\partial \underline{\mathcal{E}}}{\partial t} = \dfrac{\partial}{\partial t} \overline{\underline{u}' \times \underline{b}'}$

$$= \overline{\left(\frac{\partial \underline{u}'}{\partial t} \times \underline{b}' \right)} + \overline{\left(\underline{u}' \times \frac{\partial \underline{b}'}{\partial t} \right)}$$

subs in from NS (or take as given)

have eqn.

Get evolution equations that involve linear, quadratic + higher order terms.

If we generalised we could write

$$\underline{\mathcal{E}} = C_{\underline{ub}} \Big|_{\underline{r}_1 = \underline{r}_2} = \overline{\underline{u}'(\underline{r}_1) \times \underline{b}'(\underline{r}_2)} \Big|_{\underline{r}_1 = \underline{r}_2}$$