Quantum Physics in One Dimension
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DISORDERED SYSTEMS

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Abstract and Keywords

This chapter discusses the effects of disorder in fermionic systems, including Anderson localization. There are important differences for the disorder effects between the one-dimensional world, where localization occurs because electrons bump back and forth between impurities, and the higher dimensional world, where Anderson's localization is a rather subtle interference mechanism. The discussion looks at one-dimensional electrons subject to weak and dense impurities, in which the disorder can be replaced by its Gaussian limit. The application of disordered systems to quantum wires, one of the ultimate weapons to study individual one-dimensional systems, is considered.

Keywords: disordered systems, fermionic systems, fermions, Anderson localization, quantum wires, one-dimensional systems, collective pinning, individual pinning, quantum impurities

> One of the advantages of being disorderly is that one is constantly making exciting discoveries.
>
> A. A. Milne

So far we have been interested in infinite and clean systems. Many of the experimental realizations of Luttinger liquids however present additional challenges. The first complication comes from the presence of the disorder. Disorder is ever present, or can be artificially introduced in a controlled way. As we will see it has particularly powerful and interesting effects in one

dimension. Even if we have by now a good understanding of the effects of disorder in a Fermi liquid, we are far from having reached the same level of understanding in a Luttinger liquid.

Other challenging problems come from the size of the devices itself and the constraints that one is able to impose on them. Indeed, because many realizations of one-dimensional systems are made using nanotechnology, the size of the sample is usually relatively small. This is, for example, the case for quantum wires and nanotubes. In such systems it is also possible to impose special boundary conditions at a given point (boundary, single impurity, etc.). I examine these effects in the two coming chapters and discuss their consequences for some realizations of Luttinger liquids.

9.1 Effect of disorder; Anderson localization

The question of disorder in Luttinger liquids is an extremely challenging one. Indeed, even in higher dimensions the effects of disorder are mostly to be understood. For free electrons we know that disorder can give rise to very strong effects such as Anderson's localization. When interactions are included the question is largely open. Before we move on to Luttinger liquids let us examine some of the questions and techniques for disordered systems in more details.

9.1.1 Generalities on disordered systems

Quite remarkably disorder in fermionic systems exhibits marked differences with disorder in classical systems. Indeed, the very existence of a Fermi energy E_F 'reduces' the effects of disorder since the relative strength of the disorder compared to the Fermi energy D/E_F or the mean free path compared to the Fermi length $k_F l$ now becomes the relevant parameter. Of course, nature would not remain as simple as that and quantum effects lead in fact also to reinforcement of disorder effects and turn in low dimensions a free electron system into an insulator, as pointed out in Anderson (1958) and Mott and Twose (1961). We **(p.271)** have now gained a very good understanding of the properties of such disordered free electron systems. To tackle them an arsenal of methods ranging from diagrams (Lee and Ramakhrishnan, 1985) to scaling theory (Abrahams *et al.*, 1979), replicas (Wegner, 1979; Efetov *et al.*, 1980), supersymmetry (Efetov, 1983) have been developed. Because of quantum effects, disorder, if strong enough, can lead to a complete localization of the electrons. The wave function decays exponentially with a characteristic length known as localization length ξ_{loc}.

The system is thus an insulator with a resistance increasing exponentially with the size of the system as $\rho \sim e^{L/\xi}$. In three dimensions, the disorder needs to be larger than a certain threshold to cause the localization. Weak disorder leaves the system metallic with a finite conductivity. Disorder can thus induce a metal insulator transition. In two dimensions, the scaling theory predicts that a free electron system is always localized. However, the localization length can be very large, of the form (9.1)

$$\xi loc \sim l \exp\left[\frac{\pi}{2} k_F l\right]$$

where l is the mean free path. In one dimension, the situation is much more drastic, and one finds from exact solutions (Berezinskii, 1974; Abrikosov and Rhyzkin, 1978) that the localization length is of the order of the mean free path itself. It means that after bumping a couple of times on the impurities the electrons are localized. This shows already that there are important differences for the disorder effects between the one-dimensional world, where localization occurs because electrons bump back and forth between impurities and the higher dimensional world where Anderson's localization is a rather subtle interference mechanism.

Life becomes much less simple when interactions among fermions are taken into account. The naive approach would be to say that for a Fermi liquid interactions are not so important and the free electron picture is a good starting point. This is certainly true, at least from a practical (that is, experimental) point of view if the disorder is very weak and the localization length is large. Note that if we had considered bosons instead of fermions then interactions would have to be included from the start. I will come back to this interesting case in Section 11.1. But both from a conceptual point of view and for practical purposes, when the disorder is moderately strong this intuitive feeling that interactions should not play a major role is totally incorrect. Even if in the pure system interactions can be 'removed' from the system by resorting to Fermi Liquid theory, this is not the case when disorder is present. Because disorder renders electrons slowly diffusive rather than ballistic, they feel the interactions much more strongly, with explosive results (Finkelstein, 1984; Altshuler and Aronov, 1985; Lee and Ramakhrishnan, 1985; Abrahams et al., 2001). Effective interactions increase when looking at low-energy properties and Fermi liquid theory breaks down. In addition, when the dimension is small or the interactions strong to start with (like in systems undergoing Mott transitions) it is of course impossible to start from the non-interacting limit and one has to solve the full problem.

(p.272) In that respect one dimension is a very interesting case. The separate effects of disorder and interactions are at their strongest. The disorder gives

rise in the non-interacting system to a very strong localization. Since there is no diffusive regime the derivation used in higher dimensions for the increase of interactions cannot be applied, but one would naively expects similar effects to hold. If the electrons cannot move, then the interactions will be felt differently: either not at all if the electrons are pinned at different points in space, or extremely strongly if they are trapped at the same place. The interactions also give rise to extremely strong effects, as we saw in the rest of the book. In particular, the charge susceptibility in one dimension is divergent at $2k_F$. One can thus expect very strong response to any potential having such Fourier components, which is the case of the disorder (it is present at all wavevectors) and thus a strong reinforcement of the disorder due to the Luttinger liquid effects.

Before we move on to the study of the one-dimensional problem, let us see which tools of our theoretical toolbox one can use to tackle a disordered problem. If we put a random term in our Hamiltonian, then every observable depends on the specific realization of the random potential. What is needed is then to take an average of these observables with respect of the probability distribution of the disorder to get the average response of the system. This is of course a theoretical trick. A real experimental system has usually only one realization of the disorder, and the self-averaging comes from the fact that the system is large enough so that little pieces of the system more or less behave as independent sub-systems. Solving the problem for a given specific realization of the disorder and averaging afterwards is of course a totally impossible program. In addition of being impossible it is in general stupid, since a given realization of the disorder breaks many symmetries of the system. Since a given realization of the disorder is not invariant by translation, all correlation functions depend now on both coordinates x and x' at which they are computed and not simply on the difference $x - x'$ as in a translationally invariant system. On the other hand, the *averaged* correlation function is invariant by translation so it is much simpler. Of course, averages can be done order by order in a perturbation theory, but if one wants to go beyond perturbation one wants a method to average over disorder *from the start*. Unfortunately, many techniques that were useful for the non-interacting systems (such as supersymmetry) fail when interactions are included. I present here a quite general method that still works and is known as the replica method. We will barely use it in this book, but since it is a fairly important method I still feel it is worth having seen it at least once in one's life. The other useful method is the dynamical method (the so-called Keldysh technique) but it is of a

more complex use. For a recent review on dynamical methods see Kamenev (2002).

Let us assume we want to compute the average value of some observable *O* for a system with a random potential *V*. The average value can be written as a functional integral over the configurations of the system as **(p.273)** (9.2)

$$< O >_V = \frac{\int D\phi\, O(\phi) e^{-S_V(\phi)}}{\int D\phi\, e^{-S_V(\phi)}}$$

where $S_V(\phi)$ is the action of the system for a given realization of the random potential *V*. Of course, $\#O\#_V$ depends on *V* itself, so we have to average over all realizations of *V*. If we assume that the disorder has a probability distribution *p*(*V*) the average over disorder is (9.3)

$$\overline{< O >} = \frac{\int DV\, p(V) < O >_V}{\int DV\, p(V)}$$

In general, $S_V(\phi)$ is linear in *V*, something like (9.4)

$$S_V = S_0(\phi) + \int dx\, d\tau\, V(x) A(\phi(x,\tau))$$

Note that for a quantum problem the disorder is time independent. For the disorder one takes in general a gaussian disorder. This is very often justified by the central limit theorem.[35] For example using a distribution (9.5)

$$p(V) = e^{-\frac{1}{2D}\int dx\, V(x)^2} = e^{-\frac{1}{2D\Omega}\Sigma_q V_q^* V_q}$$

corresponds, using the techniques explained in Appendix C, to the average (9.6)

$$\overline{V(x)V(x')} = D\delta(x - x')$$

I will come back to the physical significance of this type of disorder in the next section. With distribution (9.5) it would be very easy to perform the average (9.3) if it were not for the denominator in (9.2). Indeed, in the absence of such denominator one has (9.7)

$$\frac{1}{\int DV\, e^{-\frac{1}{2D}\int dx V(x)^2}} \int DV\, e^{-\frac{1}{2D}\int dx V(x)^2} e^{-\int dx \int d\tau\, V(x) A(\phi(x,\tau))}$$

$$= e^{\frac{D}{2}\int dx \int d\tau \int d\tau' A(\phi(x,\tau)) A(\phi(x,\tau'))}$$

One would end up with an effective action where the disorder has been eliminated and has given after average an interaction term so the action would be (9.8)

$$S_{\text{eff}} = S_0(\phi) - \frac{D}{2}\int dx \int d\tau \int d\tau' A(\phi(x,\tau)) A(\phi(x,\tau'))$$

we could then treat this problem using our favorite method since it would not be more complicated than the type of problems that we already encountered

in this book. Compare in particular with the integration over phonons for the spin-Peierls transition in Section 6.2.

(p.274) Unfortunately, the presence of the denominator in (9.2) prevents us from doing the average as in (9.7). The idea of the replica method (Edwards and Anderson, 1975) is thus to get rid of the denominator and to transform it into a numerator. As with any really great method the idea is very simple. One can rewrite (9.9)

$$\frac{1}{\int \mathrm{D}\phi\, e^{-S_V(\phi)}} = \left[\int \mathrm{D}\phi\, e^{-S_V(\phi)}\right]^{n-1}$$

with $n = 0$. If we forget $n = 0$ for a moment and consider n as a positive integer $n = 2, 3, 4, \ldots$, then (9.10)

$$\left[\int \mathrm{D}\phi\, e^{-S_V(\phi)}\right]^{n-1} = \left[\int \mathrm{D}\phi_2\, e^{-S_V(\phi_2)}\right]\cdots\left[\int \mathrm{D}\phi_n\, e^{-S_V(\phi_n)}\right]$$

where we have introduced the fields ϕ_2, ϕ_3 etc. The denominator can thus be rewritten as the product of $n - 1$ copies. The average (9.2) can thus be rewritten (9.11)

$$<O>_V = \left[\int \mathrm{D}\phi_1\, O(\phi_1)\, e^{-S_V(\phi_1)}\right]\left[\int \mathrm{D}\phi_2\, e^{-S_V(\phi_2)}\right]$$

$$\left[\int \mathrm{D}\phi_3\, e^{-S_V(\phi_3)}\right]\cdots\left[\int \mathrm{D}\phi_n\, e^{-S_V(\phi_n)}\right]$$

$$= \int \mathrm{D}\phi_1 \mathrm{D}\phi_2 \ldots \mathrm{D}\phi_n\, O(\phi_1)\, e^{-\Sigma_{a=1}^{n} S_V(\phi_a)}$$

There is no denominator anymore. The price to pay is the introduction of n copies of the system. Of course, one would only recover (9.2) if one can take the limit $n \to 0$ at the end. Before averaging over disorder in (9.11) all replicas (copies) are independent. Since there is no denominator in (9.11) one can do the average over disorder in the manner described above. One thus finds (9.12)

$$<\overline{O}> = \int \mathrm{D}\phi_1 \mathrm{D}\phi_2 \ldots \mathrm{D}\phi_n\, O(\phi_1)\, e^{-S_{\mathrm{eff}}}$$

where the effective action is now (9.13)

$$S_{\mathrm{eff}} = \sum_{a=1}^{n} S_0(\phi_a) - \frac{D}{2}\sum_{a=1,b=1}^{n} \int dx \int d\tau \int d\tau'\, A(\phi_a(x,\tau)) A(\phi_b(x,\tau'))$$

This is nearly the same form except that now one has n fields and the interaction couples all fields together. We have thus traded a disordered system with only one field for a clean interacting problem of n coupled fields. Of course, the second one is more complicated due to the presence of the n fields, but as explained before we should be more equipped to tackle it. One important difficulty is of course that we should obtain a good enough solution for any n to be able to make the analytic continuation to $n \to 0$ at the end, since it is only in this limit that one **(p.275)** recovers the disordered solution.

Taking this limit is far from being obvious and contains hidden difficulties that I will briefly allude to later in this chapter.

A few remarks to conclude. The interaction in (9.13) is totally non-local in time. This is a direct consequence of the fact that the disorder is time independent. In Fourier space this would mean that the frequency is conserved upon scattering on the impurities. This is very useful for the non-interacting problem since it allows to solve for a given energy, but for the interacting problem this does not help much. The fact that the effective interaction between replicas is non-local in time is a pain in the neck. A final remark for the professionals of diagrams: the role of replicas in a perturbative expansion is simply to remove the diagrams that are disconnected *before* the averaging over the disorder. Such diagrams correspond to connected diagrams when one uses the replicated action (9.13), and thus remain in the perturbative expansion. However, they are proportional to n and thus vanish in the $n \to 0$ limit, as they should since they correspond in fact to a disconnected process.

9.1.2 Collective versus single individual pinning

Let us now consider the disorder in more details. Electrons are scattered by impurities. The disorder term is simply (9.14)

$$H_{\text{dis}} = \int dx\, V(x)\rho(x) = \frac{1}{\Omega}\sum_{k,q} V_q c^{\dagger}_{k+q} c_k$$

where $V(x)$ is the random potential produced by impurities. A good approximation for it would be (9.15)

$$V(x) = \sum_i f(x - R_i)$$

where f is the potential of a single impurity, and R_i the random positions of the impurities. In standard systems the impurity potential is relatively short-range so that very often one takes $f(x) = V_0 \delta(x)$. If the impurities are uniformly distributed this is a poissonian disorder. It is characterized by *two* independent variables, the strength of one impurity V_0 and the density of impurities n_i. Varying each one of these parameters leads to quite different physical effect. There are various interesting cases. The first one is when the impurities are dense enough and weak enough so that the effect of each impurity is negligible. They can only act collectively. In that case the scale of variation of the physical quantities is much larger than the distance between the impurities (

n_i^{-1}

in one dimension). In a volume where the physical parameters are thus practically constant, there are many impurities. In that case the central limit theorem tells us that the disorder is equivalent to a gaussian disorder. Physically it means that one can replace the original disorder by a coarse grained version (9.16)

$$\frac{1}{\Xi}\int_\Xi dx\, V(x) = \tilde{V}(x)$$

(p.276) where Ξ is some volume small compared to the scale of variation of the physical quantities, but large compared to the distance between impurities. $\tilde{V}(x)$ is the coarse grained potential. Since there are many impurities in the volume Ξ the central limit theorem imposes that $\tilde{V}$ has a gaussian distribution of the form (9.17)

$$p(\tilde{V}) = e^{-\frac{1}{2}\int D^{-1}(x-x')\tilde{V}(x)\tilde{V}(x')}$$

ensuring that (9.18)

$$\overline{\tilde{V}(x)\tilde{V}(x')} = D(x-x')$$

where $D(x - x')$ is a short-range function, which is zero beyond a few impurity distances

$$n_i^{-1}$$

. Since the scale of variation of physical constants is much larger one usually approximates $D(x) \sim D\delta(x)$. In that case the disorder is gaussian and uncorrelated from point to point. The strength of the gaussian disorder can easily be estimated from (9.15) to be (9.19)

$$D = n_i V_0^2$$

Notice that the gaussian disorder now depends on *only one* parameter. The distance between impurities has completely disappeared from the problem (or only stays as the correlation length of the disorder). Mathematically, the gaussian disorder corresponds to taking infinitely dense impurities $n_i \to \infty$ but each one infinitely weak $V_0 \to 0$ so that the effect can only be collective. A single impurity gives a very weak effect. The product

$$n_i V_0^2$$

is kept constant and measures the strength of this 'collective' disorder. This is the case we will examine in Section 9.2. The totally opposite limit that we will examine in detail in Chapter 10 corresponds to the case where the density of impurities is weak, and the strength of a single impurity V_0 is getting very large. In this case each impurity is having a strong effect in its immediate vicinity, and one can essentially forget about collective effects and has to solve the problem of a single impurity.

9.2 Many impurities

9.2.1 Basics

Let us thus consider one-dimensional electrons subject to weak and dense impurities. In that case one can replace the disorder by its gaussian limit, so the disorder is (9.14) with the correlations for *V* (I drop the tilde from now on) given by (9.18). If the disorder strength *D* is much smaller than the Fermi energy, one can again consider that disorder produces effects only close to the two Fermi points. One can thus approximate (9.20)

$$H_{\text{dis}} = \frac{1}{\Omega}\sum_{q\sim o} V_q \sum_k c^{\dagger}_{k+q} c_k + \frac{1}{\Omega}\sum_{q\sim \pm 2k_F} V_q \sum_k c^{\dagger} c_k$$

There are of course also the $4k_F$, etc., harmonics. If one wants both *k* and $k + q$ to be close to a Fermi point, one obtains in the continuum limit **(p.277)**
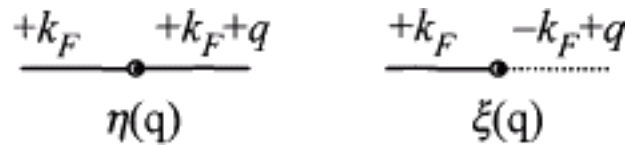

Fig. 9.1. (left) Forward scattering on impurities. In the continuum this conserves the chirality of a Fermion (that is, a right mover stays a right mover). (right) Backscattering that changes a right mover into a left mover.

(9.21)

$$H_{\text{dis}} = \int dx\, \eta(x)\left[\tilde{\psi}^{\dagger}_R(x)\tilde{\psi}_R(x) + \tilde{\psi}^{\dagger}_L(x) + \tilde{\psi}_L(x)\right]$$

$$+ \int dx [\xi(x)\tilde{\psi}^{\dagger}_L(x)\tilde{\psi}_R(x) + \xi^*(x)\tilde{\psi}^{\dagger}_R(x)\tilde{\psi}_L(x)]$$

where η(*x*) and ξ(*x*) are the continuum limits corresponding to the $q \sim 0$ and $q \sim -2k_F$ components of the random potential. They are given by (9.22)

$$\eta(x) = \frac{1}{\Omega}\sum_{q\sim o} V_q e^{iqx}$$

$$\xi(x) = \frac{1}{\Omega}\sum_{q\sim o} V_{q-2kF} e^{iqx}$$

It is easy to see that η is real since $V_q = V_{-q}$ (*V*(*x*) is real). Naively, the field ξ is complex. In fact, this depends on the precise value of $2k_F$ and whether one is on a lattice or not. I will come back to this point. Physically the field η describes forward scattering ($q \sim 0$) on the impurities whereas the fields ξ and ξ* are the backscattering where the momentum of an electron changes by $\mp 2k_F$. These processes are shown in Fig. 9.1. This is a very specific feature of one-dimensional systems. In higher dimensions, by playing with angles one can exchange an arbitrary momentum while staying close to the

Fermi surface. In one dimension, since one has to stay close to one of the Fermi points, either the particle continues in the same direction (forward scattering) or it bumps back on the impurity (backward scattering)

There are important things to notice about (9.22). Because η and ξ only contain $q \sim 0$ terms in the Fourier sum, they vary slowly at the scale of K_F^{-1}

and are thus well suited for a continuum limit. Since

$$\overline{V_q V_{q'}^*} = D\delta_{q,q'}$$

from (9.18), the $q \sim 0$ and $q \sim \pm 2k_F$ fields are essentially independent random variables. So (9.23)

$$\overline{\eta(x)\xi(x')} = 0$$

From the correlations of $V(x)$ (I take $V(x - x') = D\delta(x - x')$) one has (9.24)

$$\overline{\eta(x)\eta(x')} = D_f\delta(x - x') = D\delta(x - x')$$

$$\overline{\xi(x)\xi(x')} = 0$$

$$\overline{\xi(x)\xi^*(x')} = D_b\delta(x - x') = D\delta(x - x')$$

(p.278) The forward and backward scatterings can thus be treated independently. The fact that

$$\overline{\xi(x)\xi(x')}$$

correlations are zero, traduces the fact that one cannot have two consecutive scatterings of $2k_F$ since it would take the particle away from the Fermi surface. This is of course true if one is in the continuum. On a lattice, if $2k_F = \pi$ (that is $4k_F = 2\pi$) a $4k_F$ scattering brings the particle back to the Fermi surface. One can indeed directly see from (9.22) that if one is on a lattice and $2k_F = \pi$, then because $e^{i2k_F x} = (-1)^x$, the field ξ is now *real*. For such commensurate cases the disorder becomes instead of (9.21) (9.25)

$$H_{\mathrm{dis}} = \int dx\, \eta(x)\left[\tilde{\psi}_R^\dagger(x)\tilde{\psi}_R(x) + \tilde{\psi}_L^\dagger(x)\tilde{\psi}_L(x)\right]$$

$$+ \int dx\, \xi(x)\left[\tilde{\psi}_L^\dagger(x)\tilde{\psi}_R(x) + \tilde{\psi}_R^\dagger(x)\tilde{\psi}_L(x)\right]$$

This has drastic physical consequences as I will briefly discuss later. I will also skip the interesting case where the potential is quasi-periodic, which is intermediate between commensurate systems and disordered ones (Kolomeisky, 1993; Vidal *et al.*, 1999; Hida, 2001).

For the moment let me focus on the incommensurate case. For the non-interacting system, powerful techniques exist that allow to solve rather completely this problem (Berezinskii, 1974; Abrikosov and Rhyzkin, 1978; Efetov, 1983). The first important point is that forward scattering does not

affect the conductivity at all. This can be shown directly on the fermionic Hamiltonian but we will show it more generally later. Physically this can be understood easily. Since the forward scattering does not change the chirality it cannot affect the current. If one remembers the Boltzmann expression of the transport time (Mahan, 1981) it is weighted compared to the lifetime by (9.26)

$$\tau_{\text{trans}}^{-1} = \int d\theta \, \tau_{\text{imp}}^{-1} \left(1 - \cos(\theta)\right)$$

where θ is the scattering angle. In one dimension, the only 'allowed' angles are $\theta = 0$ (forward scattering) and $\theta = \pi$ (backward scattering). So we see that forward scattering will not affect transport. The backward scattering is the one giving rise to localization. One finds exponential decay of the correlation functions with a characteristic length (9.27)

$$\xi_{\text{loc}} = \frac{1}{D_b}$$

The conductivity is of the form shown in Fig. 9.2. The low frequency behavior of the conductivity can easily be understood in the fermion language in the strong pinning regime (Mott, 1990). Let us assume that each electron is pinned individually on an impurity as shown in Fig. 9.3. The power absorbed in an external field is $P = \sigma(\omega)E^2$. To absorb a photon the system should be able to make a transition from one occupied localized state of energy E_i to an empty one of energy $E_f = E_i + \hbar\omega$. If one assumes that the density of states $N(E)$ is **(p.279)**
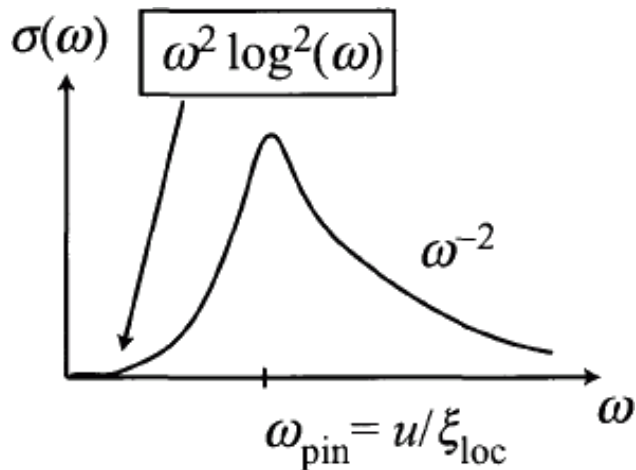

Fig. 9.2. Conductivity for the non-interacting one-dimensional electron system. The main characteristic is the existence of a pinning frequency of the order of $\omega_{\text{pin}} = v_F/\xi_{\text{loc}}$. The low-frequency behavior is $\sigma(\omega) = \omega^2 \log^2(1/\omega)$.

Fig. 9.3. Interpretation of the conductivity in the strongly localized regime. Transitions are absorption of the photons from one occupied localized state to an empty state. This can occur only within a layer ℏω around the Fermi level.

roughly unchanged by the disorder then the number of such states, which are within an energy range ℏω below the Fermi level, is ~ $N(E_F)$ℏω. Each transition absorbs ℏω), so the conductivity is σ(ω) ~ ω^2. This simple result can be refined to show that logarithmic corrections should be present. The result is confirmed by more rigorous calculations.

The non-interacting system is thus rather well understood. When interactions are included the question becomes much more complicated. However, since the bosonization form provides a natural framework to treat the interactions in one dimension, one can hope that the disorder problem in the presence of interactions will be tractable in the boson form. So, let me first rewrite the disorder Hamiltonian (9.21) in this much more convenient form. Using Appendix D one gets (9.28)

$$H_{\mathrm{dis}} = -\int dx\, \eta\!\left(x\right)\!\left[\frac{1}{\pi}\nabla\phi\right] + \int dx \frac{\xi^*(x)}{2\pi\alpha}\, e^{i2\phi(x)} + \mathrm{h.c.}$$
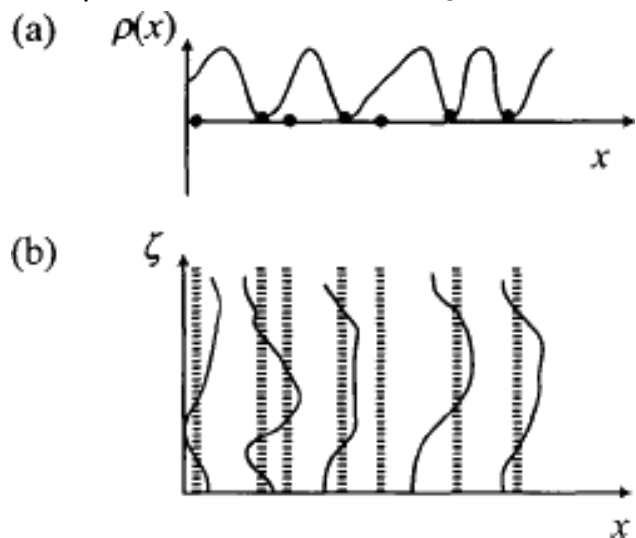
for spinless fermions and **(p.280)**

Fig. 9.4. (a) The disordered interacting electron system can be viewed as a charge density wave of periodicity $2k_F$ that tries to pin on impurities. It should distort to take advantage of the impurity potential. Quantum fluctuations of the charge density wave are measured by the LL parameter K and compete with pining, (b) It can also be viewed as a periodic array of lines that get pinned by columnar disorder, that is, by disorder that is independent of one of the coordinates z which is the imaginary time for the quantum problem. The lines are the space-time trajectories of the particles.

(9.29)

$$H_{\text{dis}} = -\int dx\, \eta\left(x\right)\left[\frac{\sqrt{2}\nabla\phi_\rho}{\pi}\right]$$

$$+ \int dx\frac{\xi^*(x)}{2\pi\alpha}e^{i\sqrt{2}\,\phi_\rho\left(x\right)}\cos(\sqrt{2}\,\phi_\sigma(x)) + \text{h.c}$$

for fermions with spin. Of course, now all interactions are included in the 'free' Luttinger liquid Hamiltonian H^0, which is essentially quadratic. This is of course the main advantage of using the boson representation. Before we move to the solution let us use the physical images explained in Chapter 3 to make two interesting analogies. We first see that we can interpret the total Hamiltonian (3.25) and (9.28) or (9.29) as one describing a charge density wave where the density is of the form (3.53), which tries to pin on the impurities, as shown in Fig. 9.4. The phase has to distort to take advantage of the impurity potential. Quantum fluctuations that are taken into account by the Π^2 term in the Hamiltonian H^0 prevent the phase ϕ to have a too well-defined value at a point and thus fight pinning. These quantum fluctuations are controlled by the LL parameter K. If $K = 0$ the problem would be purely classical. When K increases the amount of quantum fluctuations increases. Another analogy is provided by our analogy with a classical system of lines as explained in Chapter 3. We see that in that language since the disorder is independent of time the disorder would be a so-called columnar disorder as shown in Fig. 9.4. The system is thus equivalent **(p.281)** to a 'crystal' of lines pinned by columnar disorder. This allows to make contact with the body of literature devoted to such problems (Nelson and Vinokur, 1993; Blatter *et al*., 1994).

Let us now solve the problem. The forward scattering acts as a random chemical potential (compare with (2.57)). The backward scattering gives rise to a cosine-like potential not dissimilar to the one of the Mott transition. In fact if the disorder had *only* one Fourier component at $2k_F$ it would give rise exactly to a sine-Gordon term. This would be exactly similar to the Mott problem except for the fact that the most relevant periodicity is at $2k_F$

instead of $4k_F$ for the Mott problem. Since the forward scattering is like a random chemical potential it can be eliminated by the same techniques than in Chapter 2. One completes the square in H^0 by defining (9.30)

$$\tilde{\phi}(x) = \phi - \frac{K}{u}\int^x dy\, \eta(y)$$

The Hamiltonian becomes (e.g. for spinless fermions) (9.31)

$$H = H^0\left[\tilde{\phi}\right] + \int dx \frac{\xi^*(x)e^{i\frac{2k}{u}\int^x dy\, \eta(y)}}{2\pi\alpha}\, e^{i2\tilde{\phi}(x)} + \text{h.c.}$$

The only effect of the forward scattering is thus to redefine the phase of the backward scattering. Since ξ is a complex random variable with a gaussian distribution

$$\bar{\xi}^*(x) = \bar{\xi}^*(x)e^{i\frac{2k}{u}\int dx\, \eta(x)}$$

is a random variable with exactly the same distribution. Thus, the absorption of the forward scattering in (9.30) has absolutely no effect on the Hamiltonian. Another way to see this is to use the replica method. Since forward and backward scatterings are independent random variables with gaussian distributions one can perform the averages independently. If one performs the average over the backscattering only the action becomes (e.g. for spinless fermions in the incommensurate case) (9.32)

$$S \qquad\qquad = \sum_{a=1}^{n} S_a^0 - \int dx\, \eta(x)\sum_{a=1}^{n}\left[\frac{1}{\pi}\nabla\phi_a\right]$$

$$- \frac{D_b}{(2\pi\alpha)^2}\sum_{a=1,b=1}^{n}\int dx\, d\tau\, d\tau'\cos\left(2\phi(x,\tau)_a - 2\phi(x,\tau')_b\right)$$

where S^0 is the Luttinger action (3.26). One can absorb the forward scattering by using (9.30) for each replica. Since each field is shifted by the same amount and the replicated backward term is local in space, it is absolutely invariant. The forward scattering can thus be treated *exactly*, and completely eliminated from the Hamiltonian. It does not mean of course that the correlation functions are not affected by it since ϕ should be replaced by $\tilde{\phi}$. However, since (9.30) is a simple shift $\tilde{\theta} = \theta$, it means that *any* correlation function containing only θ is invariant. This is in particular the case of the current–current correlation **(p.282)** function and the superconducting correlation functions. Another way to see that is to say that $j \sim \partial_\tau\phi$ and the shift depends only on space. The conductivity is thus totally independent of the forward scattering even in the presence of interactions, in agreement with our physical arguments. The other correlation functions are easily computed. For example, the $2k_F$ density is (9.33)

$$< e^{i2\phi(r)}e^{-i2\phi(0)} >_{S[\phi]} = e^{i2\int_0^x dx\frac{k}{u}\eta(x)} < e^{i2\tilde{\phi}(r)}e^{-i2\tilde{\phi}(0)} >_{S[\tilde{\phi}]}$$

The correlation function for $\phi\#$ is the one that contains only backscattering. In the absence of backscattering it would thus just be the standard LL power law. Averaging over the disorder gives for the first term (9.34)

$$\overline{e^{i2\int_0^x dx \frac{k}{u}\eta(x)}} = e^{-\frac{2k^2}{u^2}\int_0^x dx \int_0^x dx' D_f \delta(x-x')} = e^{-\frac{2k^2 D_f}{u^2}\left|x-x'\right|}$$

The forward scattering thus causes an exponential decay of the density–density correlation functions. Let me again emphasize that this *does not* correspond to Anderson's localization since the current is unchanged.

The difficult, but physically interesting part is thus to solve the backward scattering Hamiltonian (9.35)

$$S\Big/\hbar = \int dx\, d\tau \left[\frac{1}{2\pi K}\left[\frac{1}{v}(\partial_\tau \phi)^2 + v(\partial_x \phi)^2 \right] + \frac{\xi^*(x)}{2\pi\alpha\hbar}e^{i2\phi(x)} + \text{h.c} \right]$$

I have reintroduced in the above formula $\hbar$ and other pesky constants to show explicitly the various physical limits. Note that although we are mainly concerned here with fermions (9.35) describes in fact nearly every one-dimensional disordered problem since all this problems have essentially the same boson representation. I will examine dirty bosons in Section 11.1. To understand the physical effects of the backward scattering let us go back to the interpretation of (9.35) in terms of a charge density wave that gets pinned on impurities. The elastic term in (9.35) wants the phase of the density wave to be as constant as possible, and have a nice sinusoidally modulated density, since phase and density are related by (3.53). The disorder term on the contrary wants to pin this charge density on the impurities by distorting the phase, as shown in Fig. 9.4. The problem of localization of interacting fermions is thus very similar to the one of the pinning of classical charge density waves (Fukuyama and Lee, 1978). The charge density wave is here intrinsic to the one-dimensional interacting electron gas and not due to a coupling to phonons (Phononic formation of CDW is the Peierls transition analogous to the one seen in the spin chain context in Section 6.2.2). The main features are nevertheless similar, the main difference being the fact that the effective mass of the 'CDW' is much smaller in the absence of the electron–phonon coupling (the phononic CDW has to carry the lattice distortion with it which gives a very large mass) and hence the importance of the quantum fluctuations is much higher. In the absence of quantum fluctuations, ϕ would be a classical field and we would have a good idea of what happens. This is the way Fukuyama **(p.283)** and Lee (1978) looked at this problem. Such an approximation is of course very good for 'phononic' charge density waves (Grüner, 1988) since the quantum term is $\Pi^2/(2M)$ and thus very small. For fermions this corresponds to the 'classical' limit $\hbar \to 0$, $K \to 0$ keeping $\bar{K} = K/\hbar$ fixed, and thus to very

repulsive interactions. In that case we can ignore all quantum fluctuations, and look for a *static* solution for ϕ. It is of course crucial for the existence of such solution that the disorder does not depend on time. This solution $\phi_0(x)$ describes the static distortion of the phase imposed by the random potential. In the absence of kinetic energy $(\nabla\phi)^2$, it would be easy to 'determine' ϕ_0. If we write the random field ξ^* as an amplitude $-|\xi(x)|$ and a random phase -2ζ, the disorder term is proportional to (9.36)

$$-\int dx |\xi(x)| e^{i2(\phi(x)-\varsigma(x))} + \text{h.c.}$$

The optimum is thus for $\phi_0(x)$ to follow the random phase on each point. For point-like impurities located on random positions R_i, $|\xi|$ would just be the strength of each impurity potential and $\zeta = k_F R_i$. Thus, $\phi_0(x) = \zeta(x)$ is the generalization to any type of disorder (and in particular to the gaussian disorder so dear to the theorist) of the physics expressed in Fig. 9.4: get the density minimum at each impurity. In presence of kinetic energy, following the random phase would cost too much kinetic energy. We do not know exactly how to determine the optimal $\phi_0(x)$ but we can use some scaling arguments. Let us assume that ϕ remains constant for a lengthscale ξ_{loc}. On this lengthscale ϕ takes the value that optimizes the disorder term, which now reads (9.37)

$$E_{\text{dis}} = \left[\int_0^{\xi_{\text{loc}}} \xi^*(x) \right] e^{i2\phi} + \text{h.c.}$$

Because the average of a gaussian random variable on a box of size ξ_{loc} is proportional to the square root of the size of the box, the disorder contributes as (9.38)

$$E_{\text{dis}} = -\sqrt{D_b \xi_{\text{loc}}}\, e^{i(2\phi 0 - 2\Xi)}$$

where Ξ is some phase. It clear that the optimum energy is reached if ϕ_0 adjusts to this (now unknown) phase. The global energy gain now scales as $\sqrt{\xi_{loc}}$

Between two segments of size ξ_{loc} the phase has to distort to reach the next optimal value. This is sketched in Fig. 9.5. The distortion being of the order of 2π the cost in kinetic energy reads (9.39)

$$E_{\text{kin}} \propto \frac{1}{\xi_{\text{loc}}}$$

minimizing the total cost shows that the length over which ϕ_0 remains constant is given by (9.40)

$$\xi_{\text{loc}} \propto \left(\frac{1}{D_b} \right)^{\frac{1}{3}}$$

This tells us that the system *does* pin on the impurities and that below ξ_{loc} the system looks very much like an undistorted system. Since at the scale ξ_{loc}, **(p.284)**
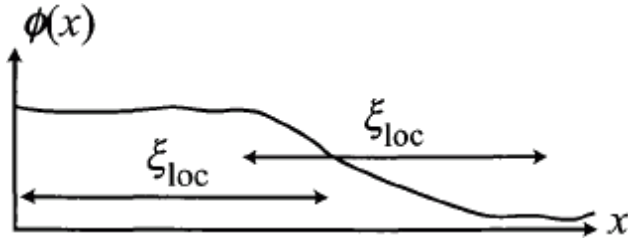

Fig. 9.5. The phase ϕ adjusts to the random phase over a length ξ_{loc}. This length results from the best compromise between the 'elastic' energy and the pinning on disorder.

ϕ_0 varies randomly one naively expects the $2k_F$ density–density correlations to decay exponentially with a characteristic size ξ_{loc}. It is thus very tempting to associate ξ_{loc} with the Anderson localization length. In the context of charge density waves this is known as the Fukuyama-Lee length (Fukuyama and Lee, 1978). The same arguments were introduced earlier in the context of classical systems, to show the existence of an analogous lengthscale for the pinning of vortices (the Larkin-Ovchinikov length, Larkin 1970; Larkin and Ovchinnikov 1979) or for random field Ising models (Imry-Ma length, Imry and Ma 1975).

Note that for the free fermion point $\xi_{\text{loc}} \propto 1/D_b$ instead of (9.40), so the above formula is clearly missing a piece of physics when K is not zero. Nevertheless, from this simple scaling argument we have obtained: (i) the fact that classical CDW or very repulsive fermions are pinned (localized) by disorder; (ii) the localization length; (iii) the fact that the ground state should contain a static distortion of the phase due to the disorder. Unfortunately, we have no other information on ϕ_0, which is certainly a drawback. Even with our limited knowledge of the statics we can nevertheless try to extract the dynamics. Let us assume that all deformations of the phase that are not contained in the static distortion are small and thus we can write (9.41)

$$\phi\left(x, \tau\right) = \phi_0\left(x\right) + \delta\phi\left(x, \tau\right)$$

with $\delta\phi(x, \tau) \ll \phi_0(x)$ in a very vague sense since we deal with random variables. One can try to expand the random term in power of $\delta\phi$ (9.42)

$$S_{\text{dis}} = -\int d\tau\, dx\, |\xi(x)| \cos\left(2\left(\phi(x, \tau) - \varsigma(x)\right)\right)$$

$$\simeq 2\int dr\, dx\, |\xi(x)| \cos\left(2\left(\phi_0(x) - \varsigma(x)\right)\right) \left(\delta\phi(x, \tau)\right)^2$$

One can thus use in principle (9.42) to compute the various physical quantities. Note that the conductivity (see Section 7.2) will not depend *directly* on the static solution ϕ_0 since $\partial_t \phi_0 = 0$, so we can hope to compute it. Of course, the dependence of the fluctuations $\delta\phi$ in ϕ_0 is hidden in (9.42). If ϕ_0 was following the **(p.285)** random phase at every point, then the cosine in (9.42) would just be a constant. Disorder would just lead to a mass term for the fluctuations and the optical conductivity would show a gap as in Fig. 7.12. In fact, this is not true at every point so (9.42) leads to a distribution of masses for the fluctuations. Unfortunately, the knowledge of ϕ_0 is too crude to compute the conductivity accurately and depending on what exactly is ξ_{loc} one can find either a gap, a non-analytic behavior or a $\sigma(\omega) \sim \omega^2$ behavior at small frequencies (Fukuyama and Lee, 1978). Based on physical intuition Fukuyama and Lee opted for the later (Fukuyama and Lee, 1978), but the method shows its limitations here and does not allow a reliable calculation of the physical quantities. More precise calculations of ϕ_0 and the conductivity can be performed in the classical limit $K \to 0$ using a transfer matrix formalism (Feigelmann and Vinokur, 1981). A self-consistent harmonic approximation, similar to the variational method of Appendix E.2, can be used to approximately take into account quantum fluctuations in the localization length (Suzumura and Fukuyama, 1983). It gives (9.43)

$$\xi_{\mathrm{loc}} \propto \left(\frac{1}{D_b}\right)^{\frac{1}{3-2K}}$$

This expression for the localization length suggests that a derealization transition is induced by the quantum fluctuations and occurs at $K = 3/2$. In the fermion language this corresponds to extremely attractive interactions.

9.2.2 Physical properties

The previous method starts directly from the localized phase. It provides some limited information about this phase, but suffers from serious limitations. An alternative approach is to start from the pure Luttinger liquid and investigate the effects of disorder perturbatively, and build a renormalization group analysis. The RG provides us with the best possible description of the delocalized phase and the critical properties of the transition. It also gives a very accurate description of the localized phase *up to* lengthscales of the order of the localization length ξ_{loc}. To build an RG we use the same techniques that were described in Section 2.3.2 It is even simpler since the replicated disorder term already contains a double integral over time and we can thus expand to first-order only. Note that in fact this corresponds to an expansion to second order in the random potential V

since $D_b \sim VV$. Before we start with the RG itself one has to notice that it is necessary to introduce a cutoff in the double time integral. This cutoff is not there in the original disorder term, since the double integral over time should be unrestricted to give the elastic scattering on the impurities. We thus replace (9.44)

$$D_b \iint d\tau \, d\tau' \to D_b \iint_{|\tau-\tau'|>\alpha} d\tau \, d\tau'$$

Introducing this cutoff has consequences that I will examine in detail in Section 9.2.3, but for the moment we will ignore this little difference.

(p.286) Using the methods of Section 2.3.2 or Appendix E.1 one can easily get the renormalization equations for the spinless case. The renormalization of D_b is the easiest since it is only the dimension of the operator (9.32). At this order only terms diagonal in replica index survive upon average since the quadratic part of the action is diagonal in replicas: (9.45)

$$\frac{\partial D_b}{dl} = \left(3 - 2K\right)D_b$$

The 3 comes from the two integrations in time and the integration over space, and the $-2K$ from the correlation $\# e^{i2\phi}e^{-i2\phi}\# \sim (1/\Upsilon)^{2K}$. In a similar way to that for the Mott transition (9.32) gives also a renormalization of the quadratic part of the action (9.46)

$$- D_b \int dx \int_{\alpha<|\tau-\tau'|<\alpha'} d\tau \, d\tau' \cos\left(2\phi_a(x, \tau) - 2\phi_a(x, \tau')\right)$$

$$\propto \int dx \, d\tau \left(\partial_\tau \phi(x, \tau)\right)^2$$

Such a term renormalizes the coefficient $1/(uK)$ in the action (3.26). Since the disorder generates only $(\partial_\tau\phi(x, \tau))^2$ terms and no $(\partial_x\phi(x, \tau))^2$, the coefficient u/K is not renormalized. Disorder thus gives both a renormalization of K and the velocity, since it breaks the space–time invariance. The RG equations are (9.47)

$$\frac{dK}{dl} = -\frac{K^2}{2}\tilde{D}_b$$

$$\frac{d\tilde{D}_b}{dl} = \left(3 - 2K\right)\tilde{D}_b$$

$$\frac{du}{dl} = -\frac{uK}{2}\tilde{D}_b$$

where (9.48)

$$\tilde{D}_b = \frac{2D_b\alpha}{\pi u^2}$$

We now briefly analyze these equations. The flow is shown in Fig. 9.6. In the variables

$K, \sqrt{\widetilde{D}_b}$

the flow is identical to the one obtained in Section 2.3.2 The equation for the disorder indicates that two fixed points are possible. If $K > K_c$ the disorder flows to zero, whereas it increases under renormalization if $K < K_c$. Note that the critical value K_c depends on the disorder. However, at the transition the whole boundary renormalizes to the *universal* value $K* = 3/2$, thus in particular $K_c(D_b \rightarrow 0) = 3/2$. The phase where the disorder flows to zero is a Luttinger liquid, with renormalized parameters $K*$ and $u*$. Note that since $K* > 3/2$ this phase has strong superconducting fluctuations. From Section 7.2, it is a perfect conductor, with a finite charge stiffness # = $u* K*$. The naive interpretation of this phase is thus that it is a delocalized phase in **(p.287)**
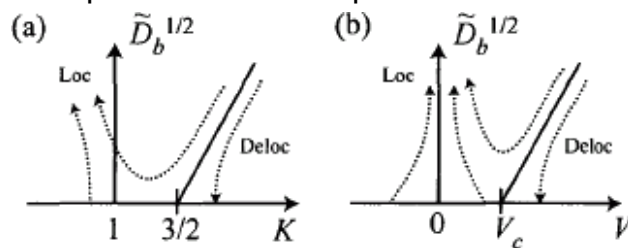

Fig. 9.6. Phase diagram and flow for spinless fermions in presence of disorder: (a) flow in $D\~_b$ and K variables; (b) flow in $D\~_b$ and interactions V. Disorder kills inelastic interactions.

which the disorder has been screened by the strongly attractive interactions. Below the transition line the disorder flows to strong coupling. Since the flow is going to strong coupling one cannot use the perturbative RG equations above the lengthscale for which $\~D_b \sim 1$. To understand the physics of this phase we can proceed in two ways. The simplest is to notice that this phase contains the free electron line $K = 1$ for which we know that electron localization takes place. This phase is thus a localized phase. The other method is to use the fact that the disorder becomes large so one can try to tackle the strong coupling action. I will come back to this in Section 9.2.3

Although one cannot use the RG equations above a certain lengthscale one can still use them to extract many physical quantities, in a way similar to what we did for the Mott transition. For example, one can extract the localization length. Let us renormalize up to a point where $\~D_b(l*) \sim 1$. The true localization length of the system is given by (9.49)

$$\xi_{\text{loc}} = e^{-l*} \xi_{\text{loc}}\left(l*\right)$$

but if $\tilde{D}_b(l^*) \sim 1$ the localization length of such a problem is of the order of the lattice spacing (this is the same trick than the one used to compute the gap for a Mott phase). Thus, (9.50)

$$\xi_{\mathrm{loc}} \sim \alpha e^{-l^*}$$

One can then integrate the flow to get l^*. As for the Mott phase this depends on the position in the phase diagram. When one is deep in the localized phase (far from the transition) one can consider K as constant (see Section 2.3.2) and thus (9.51)

$$\tilde{D}_b(l) = \tilde{D}_b(l=0)e^{(3-2K)l}$$

Thus, (9.52)

$$\xi_{\mathrm{loc}} \sim \alpha\left(\frac{1}{\tilde{D}_b}\right)^{\frac{1}{3-2K}}$$

One recovers the form (9.43). Thus, we see that the self-consistent harmonic calculation corresponds in fact to the limit of infinitesimal disorder, which is a **(p.288)** fact that we already noticed in Section 2.3.2 for such variational methods. For $K = 1$ one does recover that $\xi_{\mathrm{loc}} \sim 1/D_b$, in agreement with the exact solution for free fermions, a sign that we have taken the right physics into account. Close to the transition the localization length diverges in a different way (setting $K = 3/2 + \eta$) (see Section 2.3.2) (9.53)

$$\xi_{\mathrm{loc}} \sim e^{2\pi/\sqrt{9\tilde{D}_b-\eta^2}}$$

As for the Mott transition, one can extract the frequency and temperature dependence of the conductivity. Let us here look at the temperature dependence by a very simple technique (Giamarchi and Schulz, 1988*b*). The idea is simply to renormalize until the cutoff is of the order of the thermal length $l_T \sim u/T$ corresponding to $e^{l^*} \sim l_T/\alpha$. At this lengthscale the disorder can be treated in the Born approximation. As the conductivity is a physical quantity it is not changed under renormalization and we have (9.54)

$$\sigma\left(n,(0),\tilde{D}_b(0),0\right) = \sigma\left(n(l),\tilde{D}(l),l\right) = \sigma_0\frac{n(l)\tilde{D}_b(0)}{n(0)\tilde{D}_b(l)} = \sigma_0\frac{e^l\tilde{D}_b(0)}{\tilde{D}_b(l)}$$

where $\sigma(n(l),\tilde{D}_b(l), l) = \sigma(l)$ and $n(l)$ are, respectively, the conductivity and the electronic density at the scale l.

$$\sigma_0 = e^2 v_F^2 / 2\pi h D_b$$

is the conductivity in the Born approximation, expressed with the initial parameters. This is the simplified version of coupling RG with the memory function calculation as explained in Section 7.2. If one is deep in the localized phase, one can again retain only the RG equation for the disorder and consider K as constant and one has (9.55)

$$\sigma\left(T\right) \sim \frac{1}{\tilde{D}_b} T^{2-2K}$$

This result is schematized in Fig. 9.7. It poses a paradox since (9.45) gives a localized-delocalized boundary at $K = 3/2$ whereas (9.55) gives perfect conductivity above $K = 1$ (that is, the non-interacting point). This shows that the RG equation for the disorder *cannot* be considered alone. Indeed, the RG equation for the disorder traduces in fact the dressing of the scattering on the disorder by the interactions. Such an effect has been derived long ago using either diagrams or RG (Gorkov and Dzyaloshinski, 1973; Mattis, 1974; Luther and Peschel, 1974; Apel, 1982; Apel and Rice, 1982*a*; Apel and Rice, 1982*b*; Giamarchi and Schulz, 1987; Giamarchi and Schulz, 1988*b*). In fact (9.45) would not allow in itself to really determine the metal-insulator transition point. One can immediately see that if one introduces a new variable such as (9.56)

$$\overline{D_b} = e^{-al}D_b$$

the dimension of such a variable would be $(3-a-2K)$, leaving the location of the transition point as determined from (9.45) *alone* quite arbitrary. For example, **(p.289)**
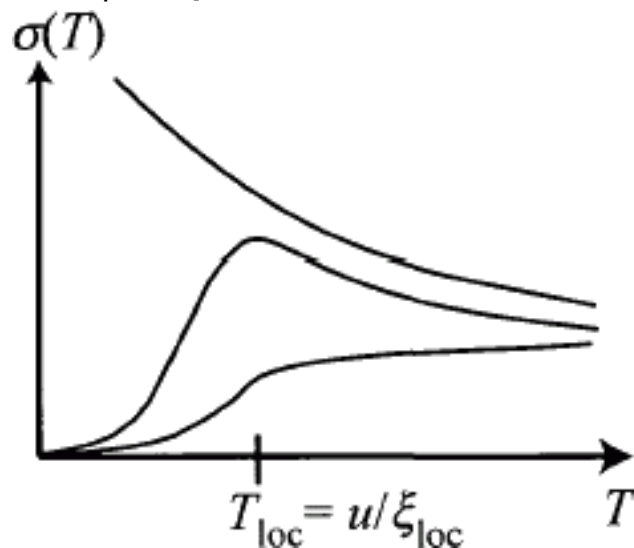

Fig. 9.7. Temperature dependence of the conductivity. For $K > 3/2$ (top) the system is delocalized and the conductivity increases with decreasing temperature. For $1 < K < 3/2$ (middle) the system is localized but the conductivity starts increasing with decreasing temperature. The renormalization of K due to disorder pushes the system to the localized side forcing the conductivity to decrease with decreasing temperature. For $K < 1$ (bottom) the conductivity decreases with temperature even at

high temperatures. Below temperatures of the order of u/ξ_{loc}, the system is strongly localized and the conductivity decreases exponentially.
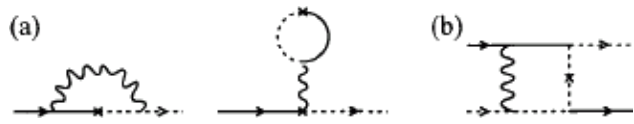

Fig. 9.8. Diagrams describing the renormalization of the disorder by the interactions (a) and the renormalization of the interactions by the disorder (b). Solid and dotted lines are fermions with $\pm k_F$, the wiggly line is the interaction and the cross is the impurity scattering.

remembering that

$$n_i V_0^2$$

one would get for the renormalization of the strength of one impurity (n_i scales as e^l) (9.57)

$$\frac{dV_0}{dl} = \left(1 - K\right)V_0$$

One could naively conclude (incorrectly) from this result that the metal-insulator transition takes place at $K = 1$. The answer to this simple paradox is of course that (9.45) is complemented by the other RG equation that describes the renormalization of the interactions by the disorder. Both RG equations (9.45) and (9.47) have a diagrammatic representation shown in Fig. 9.8. In the presence of (9.47) one cannot rescale arbitrarily the variable~D_b, and the position of the **(p.290)** transition is now unambiguously fixed to $K = 3/2$. One easily checks that if one starts with $1 < K < 3/2$ the resistivity starts by *decreasing* with decreasing temperature but then the renormalization of K takes over and when $K(l) < 1$ starts shooting up again as shown in Fig. 9.7. Note that the advantage of the bosonization derivation is to allow to reach the non-perturbative point in interactions $K = 3/2$ where such a metal-insulator transition would take place. These predictions are directly relevant for spin chains, using the equivalence between spins and spinless fermions. The predictions for the phase diagram have been confirmed by numerical calculations (Schmitteckert *et al.*, 1998).

Let us stop here for spinless fermions and go to the case of fermions with spin, since this presents some twists compared to the much simpler spinless case. I will not give here the full derivation of the equations. The methodology to derive them is identical to the one presented in Chapter 2 and the derivation is well detailed in the literature (Giamarchi and Schulz, 1988b). I will rather focus on the salient points of this problem. To obtain the

RG equations one starts with (9.29) and uses the methods of Chapter 2 and also sketched in this chapter for spinless fermions. One obtains (9.58)

$$\frac{dK_\rho}{dl} = -\frac{u_\rho}{2u_\sigma} K_\rho^2 \tilde{D}_b(l)$$

$$\frac{dK_\sigma}{dl} = -\left[\frac{\tilde{D}_b(l)}{2} + \frac{y^2(l)}{2}\right] K_\sigma^2$$

$$\frac{dy}{dl} = \left(2 - 2K_\sigma(l)\right) y(l)$$

$$\frac{d\tilde{D}_b}{dl} = \left(3 - K_\rho(l) - K_\sigma(l)\right) \tilde{D}_b(l)$$

$$\frac{du_\rho}{dl} = -\frac{u_\rho^2}{2u_\sigma} K_\rho \tilde{D}_b(l)$$

$$\frac{du_\sigma}{dl} = -\frac{u_\sigma K_\sigma}{2} \tilde{D}_b(l)$$

where now (9.59)

$$\tilde{D}_b = \frac{2D_b \alpha}{\pi u_\sigma^2} \left(\frac{u_\sigma}{u_\rho}\right)^{K_\rho}$$

and $y = g_{1\perp}/(\pi v_F)$. The equations look formidable but are in fact quite simple. The equation for $\tilde{D}_b$ is as before the dimension of the disorder operator. There is now a renormalization of the charge and spin LL parameters K_ρ and K_σ due to the disorder. Of course, there is still the standard renormalization of K_σ due to the backscattering $g_{1\perp}$. Note that one recovers the spinless fermions case by setting $y = 0$ and $K_\rho = K_\sigma$ as one should (all couplings between different spin species are zero). Unfortunately, these equations are incomplete since it is easy to see that they do not obey spin rotation symmetry. The disorder $\tilde{D}_b$ appears in the renormalization of K_σ but the equivalent term is not present in **(p.291)** the renormalization of y. So the flow will not stay on the separatrix $g_{1\parallel} = g_{1\perp}$ as it should. This is due to the fact that we have not collected all the contributions of order two. The precise details are given in Giamarchi and Schulz 1988*b* but let me present here the basic idea. This is again due to the fact that one can bring two operators at the same point as explained in Chapter 3. For example, if one combines one backscattering operator $g_{1\perp}$ with a disorder term (9.29) one gets (9.60)

$$y\tilde{D}_b \int dx \cos\left(\sqrt{8}\,\phi_\sigma(x)\right) \int dx'\, e^{i\sqrt{2}\,\phi_\rho(x)} \cos\left(\sqrt{2}\,\phi_\sigma(x')\right)$$

If the two points are within a cutoff distance of each other $|x - x'| < \alpha$ one can combine the two cosines to get (forgetting less relevant operators) (9.61)

$$\sim y\tilde{D}_b \int dx\, e^{i\sqrt{2}\,\phi_\rho(x)} \cos\left(\sqrt{2}\,\phi_\sigma(x)\right)$$

which is exactly a disorder term. In the same way combining two disorder terms at the same point cancels the ϕ_ρ contributions and gives (9.62)

$$\tilde{D}_b^2 \int dx \cos\left(\sqrt{2}\,\phi_\sigma(x)\right)^2 \sim \frac{\tilde{D}_b^2}{2} \int dx \cos\left(\sqrt{8}\,\phi_\sigma(x)\right)$$

which is a backscattering term. When varying the cutoff there is thus a renormalization of the disorder by terms of the order of $\tilde{y}D_b$ and a renormalization of the backscattering by terms of order

$$\tilde{D}_b^2$$

. The total equations are thus (9.63)

$$\frac{dK_\rho}{dl} = -\frac{u_\rho}{2u_\sigma} K_\rho^2 \tilde{D}_b(l)$$

$$\frac{dK_\sigma}{dl} = -\left[\frac{\tilde{D}_b(l)}{2} + \frac{y^2(l)}{2}\right] K_\sigma^2$$

$$\frac{dy}{dl} = \left(2 - 2K_\sigma\right)y - \tilde{D}_b$$

$$\frac{d\tilde{D}_b}{dl} = \left(3 - K_\rho - K_\sigma - y\right)\tilde{D}_b$$

where I have not written again the equations for the velocities. In particular, the renormalization of the disorder is changed, and the coefficient is not the simple dimension of the operator any more. This has serious consequences that I will examine in Section 9.2.3

By now we are old hands at looking at fixed points so we can analyze the flow rapidly. For systems invariant by spin rotation ($g_{1\parallel} = g_{1\perp}$) one is forced to stay on the separatrix so there are only two possibilities: $K_\sigma \to 1$ and $y \to 0$ or $y \to -\infty$. If one is in the first case, then the disorder is irrelevant for $K_\rho > 2$. This regime is thus delocalized with a LL fixed point. By comparing with Fig. 2.9 we see that this region is dominated by triplet superconducting correlations. If **(p.292)**
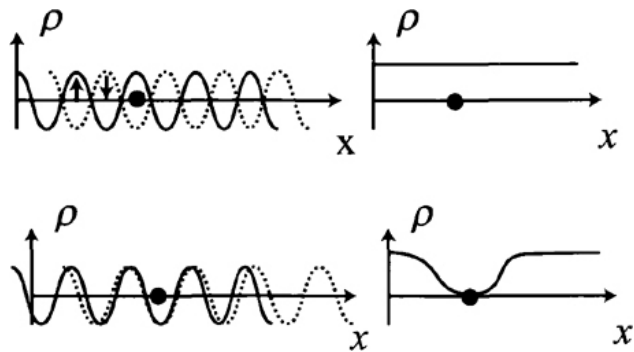
Fig. 9.9. Distortion of the charge order imposed by the impurity. In a totally rigid spin density wave state the density is uniform, so naively the disorder would not couple at lowest order. In fact, the disorder shifts locally the positions of the two density waves of spin up and down electrons to create locally a charge density wave modulation and pins it.

$K_\rho < 2$ then the disorder is relevant and the system localizes. Note that in our equations if $\tilde{D}_b \to \infty$ it seems that it always forces $y \to -\infty$. This is obviously an artefact of the lowest-order RG equations. $y \to -\infty$ means a CDW ground state that is very efficiently pinned on impurities since the density varies spatially. On the other hand, $y \to 0$ gives a SDW type phase. In such a phase the density is uniform while the spin density is modulated. Thus, at lowest order the average of the coupling (9.29) vanishes. Of course, what happens in fact is that locally the disorder shifts the SDW order to make a distortion of density to which it can couple as shown in Fig. 9.9. This process occurs at next order, this is why in the lowest order RG when the disorder is relevant it always favors the CDW phase over the SDW. If the repulsion between opposite spins is very strong the electrons localize individually, as shown in Fig. 9.10, giving rise to a random SDW. Because the electrons are localized, only the spin degrees of freedom exist and since the distance between the electrons is random, one has a random exchange antiferromagnet as shown in Fig. 9.10.

The localization length can be computed by the method explained above. Far from the transition one can replace in (9.63) $K^*_\sigma = 1$ and $y = 0$ in the RG equation for the disorder and one gets (9.64)

$$\xi_{\text{loc}} \sim \left(\frac{1}{\tilde{D}_b}\right)^{\frac{1}{2-K_\rho}}$$

Close to the transition extracting the localization length from the RG is more tricky because all equations have to be taken into account. One finds an exponential divergence of the localization length (Giamarchi and Schulz, 1988b).
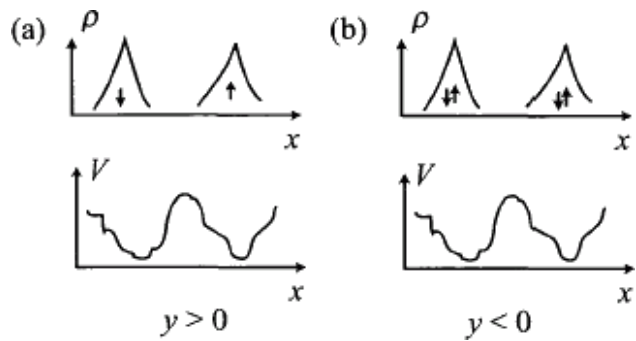
(p.293)

Fig. 9.10. For repulsive and strong enough interactions it is more favorable for the electrons to localize individually. The charge degrees of freedom are frozen but spin degrees of freedom remain and the resulting phase is thus a random antiferromagnet (a). If the interactions are not repulsive enough or attractive the electrons want to localize by pair and form singlet states. The spin degrees of freedom are gapped and the ground state is a pinned charge density wave (b).

If y is not large enough, then the disorder term pushes y to negative values. When y is negative it renormalizes to $-\infty$. As we saw in Section 2.3.2 this means that the system opens a spin gap. The electrons thus tend to pair. Because of the spin gap

$$\left(\sqrt{2}\phi_\sigma\right)$$

acquires a finite average value. Thus, at energies smaller than the spin gap the effective disorder is (9.65)

$$H_{\mathrm{eff}} = \int dx \frac{C\xi^*(x)}{(2\pi\alpha)} e^{i\sqrt{2}\phi_\rho(x)}$$

where (9.66)

$$C = \langle\cos\left(\sqrt{2}\,\phi_\sigma\right)\rangle$$

Since the fluctuations due to the ϕ_σ field are suppressed the disorder is more efficient. I will come back to this paradoxical result in the next section. The system is now consisting (at low-energy) of pairs of electrons bound together. These 'molecules' act as a hard core bosons able to hop from site to site. If the spin gap was large the hopping element would be $t_{\mathrm{eff}} = t^2/\Delta_\sigma$. This part of the phase diagram has thus strong connections with the boson problem that I will examine in Section 11.1. The RG equations can be derived (Giamarchi and Schulz, 1988b) for the effective disorder (9.65). Since only the charge part remains the derivation is very similar to the one we showed for spinless bosons. One obtains **(p.294)**
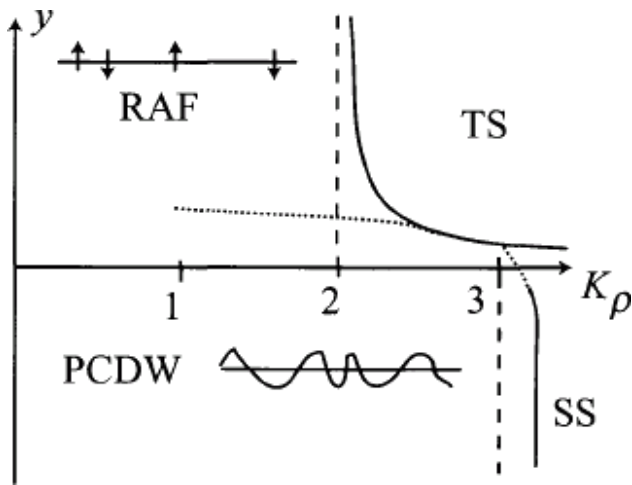
Fig. 9.11. Phase diagram for a one-dimensional disordered system. Solid lines denote the parts that can be extracted from the RG approach (see text). The dashed lines are the parts that cannot be extracted reliably from the weak coupling RG. TS and SS are, respectively, triplet and singlet superconductivity. RAF is a random antiferromagnet. PCDW is a pinned charge density wave. (After Giamarchi and Schulz, 1988b.)

(9.67)

$$\frac{dK_\rho}{dl} = -\frac{K_\rho^2}{2}\tilde{D}_b$$

$$\frac{d\tilde{D}_b}{dl} = \left(3 - K_\rho\right)\tilde{D}_b$$

$$\frac{du_\rho}{dl} = -\frac{u_\rho K_\rho}{2}\tilde{D}_b$$

where

$$\tilde{D}_b = 2CD_b\alpha/\left(\pi u_\rho^2\right)$$

. Notice the change in dimension for the disorder from $(3 - K_\rho - K_\sigma - y)$ to $(3 - K_\rho)$, due to the fact that all spin fluctuations are now frozen. As a result the transition now occurs for $K^*_\rho = 3$, that is, for even more attractive interactions. The disorder pins much more effectively the system with a spin gap. The delocalized regime is a Luther-Emery liquid (Luttinger liquid with a spin gap). It corresponds to dominant singlet superconducting fluctuations. The charge stiffness is finite $\# = 2u^*_\rho K^*_\rho$. In the localized regime since the spins are paired the system has only charge fluctuations. The localized phase is thus a pinned charge density wave. The generic phase diagram is shown in Fig. 9.11. The localization length is now given (deep in the localized phase) by (9.68)

$$\xi_{\text{loc}} \propto \left(\frac{1}{\tilde{D}_{\text{b}}}\right)^{\frac{1}{3-K\rho}}$$
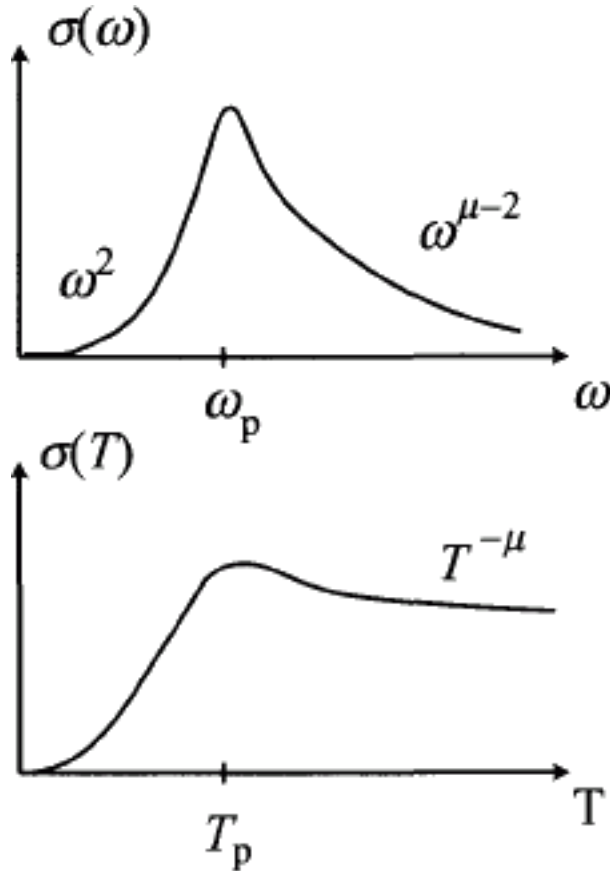
(p.295)

Fig. 9.12. Frequency and temperature dependence of the conductivity in a LL liquid. $T_p = \omega_p = u_\rho/\xi_{\text{loc}}$ is the pinning frequency or temperature. $\mu = K_\rho - 1$ for a system with spin and spin isotropic interactions ($K*\sigma = 1$) provided that $K_\rho > 1/3$. For $K_\rho < 1/3$ the disorder pins the $4k_F$ density fluctuations and $\mu = 4K_\rho - 2$. For a spinless fermion system $\mu = 2K - 2$.

while it has a Berezinskii–Kosterlitz–Thouless form similar to (9.53), close to the transition.

For the system with spin the temperature dependence or frequency dependence of the conductivity can be extracted from the RG, down to energy scales where the disorder is of order one (see Section 7.2) and that correspond to $\omega \sim u_\rho/\xi_{\text{loc}}$ or $T \sim u_\rho/\xi_{\text{loc}}$. Deep in the localized phase one can neglect the renormalization of the LL parameter K in the initial steps of

the RG, since the RG trajectories are nearly vertical (see Section 7.2). The temperature and frequency dependence of the conductivity is thus given by (9.69)

$$\rho(T) \sim D_b T^{K\rho - 1}$$

$$\sigma(\omega) \sim \left(\tfrac{1}{\omega}\right)^{3 - K\rho}$$

This is summarized in Fig. 9.12. Of course, more generally the LL parameters are renormalized by the disorder and all equations should be used (see Section 7.2), leading to a more complicated functional dependence than a simple power law. **(p.296)** Below the pinning frequency and temperatures, the RG cannot be used. For the frequency dependence one can use the method explained in the next section to show that the conductivity behaves as ω^2 (up to logarithmic terms) in good agreement with the solution for free fermions. For the temperature dependence calculations are more subtle (Lee and Larkin, 1978; Nattermann *et al*., 2003). One finds a variable range hopping law of the form $\rho(T) \sim e(1/T)^{1/2}$.

A final word. For systems with spin, the above values of the exponents apply when the interactions are not too repulsive. Indeed, we have retained the coupling of disorder to the $2k_F$ component of the density. This is correct as long as this component is the most relevant one. We have seen in Chapter 3 that for $K_\rho < 1/3$ the $4k_F$ component of the density is in fact the one that has the slowest decay. This component should thus be kept for the coupling to disorder, which becomes (9.70)

$$H = \int dx\, \xi^*(x) e^{i\sqrt{8}\,\phi_\rho(x)}$$

The RG equation for the disorder becomes (9.71)

$$\frac{d\tilde{D}_b}{dl} = \left(3 - 4K_\rho\right)\tilde{D}_b$$

Thus, $\sigma(T) \sim T^{2 - 4K\rho}$ and $\sigma(\omega) \sim (1/\omega)^{4 - 4K\rho}$. Note that for the case of Coulomb interactions $K_\rho \to 0$ and thus one recovers universal exponents (Maurey and Giamarchi, 1995) for the transport properties (up to log corrections).

9.2.3 Extensions and pitfalls

Let us examine some further consequences of the RG equations derived in the previous section and discuss some of the finer points. Since it is a relatively specialized section it can be safely skipped unless you are interested in the dirty detail of the disordered systems.

9.2.3.1 Inelastic scattering

The first point to notice is that quite remarkably (9.47) seems wrong. Indeed, *K* naively depends on the (inelastic) interactions. Perturbatively, for the pure system $K = 1 - V/(2\pi\upsilon_F)$. If one starts for $K = 1$, that is, for the non-interacting system, it would thus seem from (9.47) that the *elastic* scattering on the impurities can generate *inelastic* fermion-fermion interactions. The solution of this paradox is hidden in the precise way the RG procedure is build. In order to have the elastic nature of the scattering on impurities, the time integrations in (9.28) should be done independently for τ and τ′. When we have performed the RG we have introduced a cutoff and imposed |τ − τ′| > α. Thus, a part is left out of (9.44) which is (9.72)

$$D_b \int dx \int_{|\tau - \tau'| < \alpha} d\tau \, d\tau' \rho(x, \tau) \rho(x, \tau') \simeq 2D_b \alpha \int dx \int d\tau \, \rho(x, \tau) \rho(x, \tau)$$

This is exactly an inelastic interaction term. Thus, *K* contains not only the original inelastic interactions *V* but also a small correction coming from the disorder itself. In order to determine the flow for *V* it is thus necessary to take this **(p.297)** small correction into account (Giamarchi and Schulz, 1988*b*), which gives the flow of Fig. 9.6(-b). One thus sees that the elastic case $V = 0$ indeed remains elastic and also that for spinless fermions, the perturbative flow indicates that the inelastic interactions are reduced by the disorder. This is compatible with the physical image that one would get at strong disorder: fermions localize individually and since the overlap of wavefunctions is exponentially small, so is the effect of interactions. One could thus naively expect that below ξ_{loc} the effect of interactions are strong but disappear above ξ_{loc}. For fermions with spin the exchange interaction between the localized spins remains even in the localized phase.

9.2.3.2 RG equations

The second important point is the consequences of getting the proper RG equation (9.63) for the disorder. This is in particular crucial if one tries to relate the variation of some physical quantity to microscopic parameters. Let us, for example, see how the localization length varies with the interaction *U* for a Hubbard model. Indeed, for a Hubbard interaction the relation between the LL parameters and *U* is given by (7.9) at small *U*. Substituting in the RG equation gives for the initial steps of the flow (9.73)

$$\frac{d\tilde{D}_b}{dl} = \left(1 - \frac{U}{\pi v_F}\right)\tilde{D}_b$$

whereas the substitution in the *incorrect* equation ([9.58](#)) or at the fixed point $K_\sigma = 1$, $y = 0$ would lead to

$$\left(1 + \frac{U}{\pi \nu F}\right)$$

, leading to quite different physics. Equation ([9.73](#)) implies that for Hubbard type interactions repulsive interactions make the system *less* localized (Giamarchi and Shastry, [1995](#)) than for attractive interactions, that is,

$$\xi_{loc}^{U>0} > \xi_{loc}^{U<0}$$

. Similar effects exist for the charge stiffness and the persistent currents, that is, for a system with spin the persistent currents are in fact enhanced by repulsive interactions. Of course, for the Hubbard model both the repulsive and the attractive sides are always localized since $K_\rho < 2$ (see Chapter [7](#)) and for the attractive side one needs $K_\rho > 3$ to delocalize. The counter-intuitive increase of the localization length by a positive U can be explained physically: interactions have two effects: (i) They tend to reinforce, when attractive, the superconducting fluctuations in the system. This screens disorder and makes it less effective. This is the only effect occurring for spinless fermions. (ii) When spin degrees of freedom exist, repulsive interactions also tend to make the density more uniform by spreading the charge (see Fig. 9.9). This makes it more difficult to couple to disorder. These two effects compete and for the Hubbard model (purely local interaction) the second effect wins, hence the decrease of localization length when the interactions become more attractive. Note that for the Hubbard model, since $K_\rho < 2$ both the repulsive and the attractive side are always localized. To reach derealization one needs attractive interactions with a range of at least nearest neighbors. For more on this problem see Bouzerar *et al.* ([1994](#)), Giamarchi and Shastry ([1995](#)), Berkovits and Avishai ([1995](#)) and Gambetti-Cesare *et al.* ([2002](#)).

(p.298) 9.2.3.3 Strong coupling

Can we analyze the strong coupling in the same way than what we did for the Mott phase. Indeed, if the coefficient of the cosine becomes large one would naively think that one can expand the cosine, leading to the disorder term (9.74)

$$D_b \int dx \iint d\tau \, d\tau' \sum_{a,b} \left(\phi_a(x, \tau) - \phi_b(x, \tau') \right)^2$$

Unfortunately, it is easy to see that this is essentially wrong. In particular, this only gives $\phi(w_n = 0)$ contributions to the action that does not modify the current. Thus, no trace of localization will be found in the conductivity, which

is obviously incorrect. Even a better approximation for the cosine such as the variational approach of Appendix E.2 does not improve the result.

This peculiarity is due to the fact that we have ultimately to take the limit $n \to 0$ to get the physics of the disordered system. There are various ways to take this limit. One is the so-called replica symmetric way, which is the one the expansion of the cosine would give. The correlation $\#\phi_a\phi_b\#$ has a value for equal replica indices and another one for different replica indices. This is the natural choice since it corresponds to the structure of the action. Unfortunately, this solution is unstable, as it often happens in disordered problems, and one has to break the replica symmetry. It would take us too far to explain the process here so I refer the reader to Giamarchi and Le Doussal (1996) and Giamarchi and Orignac (2003) for more details. Taking into account replica symmetry breaking it is possible to use a variational method along the lines of Appendix E.2 to analyze the strong coupling regime. It is a very nice approach since it can reach the energy regimes that the RG cannot reach.

9.2.3.4 Commensurate disorder

How much of this physics is modified if one is at commensurate filling and the disorder Hamiltonian is (9.25) instead of (9.21)? We see immediately two important differences. First, ξ is real since there is no random phase any more to which the phase φ will try to adjust. The physics of the disordered phase is thus very different. The backward scattering term is (9.75)

$$H = \int dx\, \xi(x)\cos(2\phi(x))$$

Thus, the system has only two different minima $2\phi = 0$ or $2\phi = \pi$ depending on whether ξ is positive or negative. The physics of the commensurate systems is not about φ trying to adjust to a random phase, as for the incommensurate one, but about how to make kinks between these two minima at favorable places as indicated in Fig. 9.13. This leads to quite different properties. In particular, the commensurate system has a state at zero energy that is localized only as (9.76)

$$\psi \sim e^{-\sqrt{|x|}}$$

so formally the localization length diverges at zero energy. There are other interesting properties and I refer the reader to Fisher (1994), Monthus *et al*. (1998) and Damle and Huse (2002) for more details.
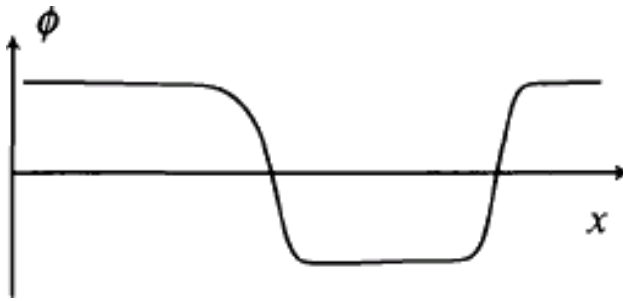
(p.299)

Fig. 9.13. Commensurate disorder. The phase now has only two values that lead to degenerate energy of the ground state. The physics of the system with such a disorder is thus controlled by the excitations that bring the phase from one value to the other. This is very different from the standard case where the phase has to adjust to a random value.

One last word about commensurate systems. Because of the form of disorder (9.75) one cannot eliminate the forward scattering any more by transformation (9.30). Since the elimination of the forward scattering was giving a random phase to ξ it thus replaces the commensurate backward scattering by a 'normal' one when present. For this reason the commensurate case is not very relevant for the case of fermions. For the case of spins, on the other hand, it is easy to find a form of the disorder that does not generate forward scattering such as random exchange. Indeed, random exchange does not change the spin density by spin rotation symmetry, it thus does not contain any component that would be for the fermions the equivalent of a chemical potential (that is, the forward scattering). Its effect has been worked out in (6.63) for the spin-Peierls distortion. The coupling to a staggered modulation of the exchange is $\sin(2\phi)$. A random exchange thus produces a term (9.77)

$$\int dx\, \delta J(x)\sin(2\phi(x))$$

Random spin chains are thus prime candidates to study the effects of commensurate disorder.

9.3 Quantum wires

An experimental realization of Luttinger liquids is provided by the so-called quantum wires. The idea is to confine the electron gas in a narrow enough channel, so that only one direction of motion matters. For example, let us assume that one starts with a two-dimensional electron gas. Such systems are routinely made for quantum hall devices and semiconducting industry. One can then either by lithography or by applying a potential repelling

the electrons from a gate confine the electrons in a channel of width L_y as explained in Fig. 9.14. The **(p.300)**
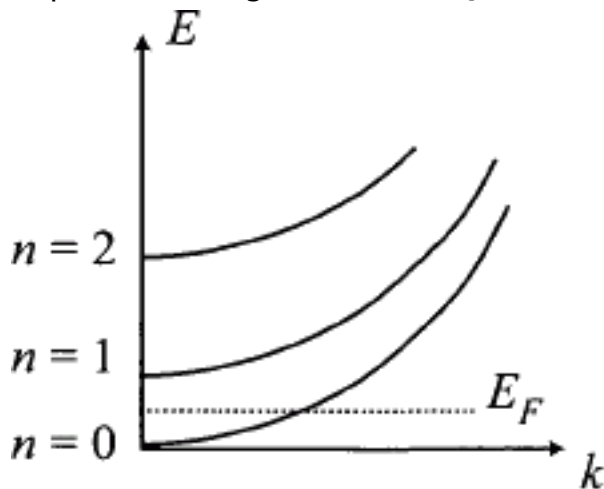

Fig. 9.14. Confinement of the electron gas in a one-dimensional wire. Only half of the dispersion relation $E(k)$ is shown for clarity. k is the momentum parallel to the wire direction. The degrees of freedom transverse to the wire direction lead to the formation of minibands, labelled by a quantum number n. If the channel in which the electrons are confined is narrow enough the distance between minibands can be made large compared to the temperature. One can be in a situation where only one miniband is occupied and the quantum wire is equivalent to a one-dimensional system. If two (or more) minibands are occupied the wire is equivalent to a ladder system.

wavefunction of the system is thus of the form (9.78)

$$\psi(x, y) = e^{ikx}\phi(y)$$

where ϕ depends on the precise form of the confining potential (for an infinite well ϕ is also a combination of plane waves). The energy is of the form (9.79)

$$E = \frac{k^2}{2m} + \frac{k_y^2}{2m}$$

where for simplicity I have taken ϕ to be a plane wave. The important point is the fact that due to the narrowness of the transverse channel l, the quantization of k_y is sizeable. Indeed, the change in energy by changing the transverse quantum number n_y is at least (e.g. $n_y = 0$ to $n_y = 1$) (9.80)

$$\Delta E = \frac{(2\pi)^2}{2ml^2}$$

This leads to minibands as shown in Fig. 9.14. If the distance between the mini-bands is larger than the temperature one can by changing the chemical

potential (with an external gate) be in a situation where only one miniband is occupied. The transverse degrees of freedom are thus frozen and only *k* matters. The system is a one-dimensional electron gas.
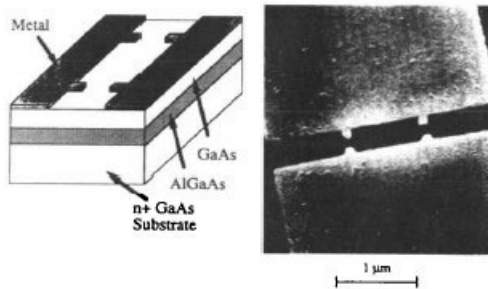
(p.301)

Fig. 9.15. A two-dimensional electron gas formed at the interface between GaAs and the insulating AlGaAs. The carrier density is controlled by the application of a voltage V_g to the substrate. To transform the 2DEG into a one-dimensional wire, metal electrodes on the surface confine the electron gas into a one-dimensional channel. The voltage v_b on the electrodes allows to control the width of the channel. (From Meirav *et al.*, 1990 (Copyright (1990) by the American Physical Society).)

An actual example is shown in Fig. 9.15 (Meirav *et al.*, 1990). Various variants can be found (Thornton *et al.*, 1986; Scott-Thomas *et al.*, 1989; Calleja *et al.*, 1991; Goñi *et al.*, 1991; Tarucha *et al.*, 1993; Hwang *et al.*, 1994). Typically, so far the possible length of a wire using these techniques is about 10 μm. The Fermi energy in these systems is quite small ~ 100K, which imposes to work at very low temperatures. This has the advantage of allowing to forget most of the phonons, but has the drawback that the length of the wire is becoming a real limitation since the thermal length $L_T \sim u/T$ can become comparable to the size of the wire. Although in practice one could imagine getting a wire as thin as desired, it is much less easy to do so in practice, without cutting the wire due to some potential fluctuations.

We already saw some spectroscopy data in Fig. 4.2, which was in good agreement with the expectations for a one-dimensional electron gas. Let us examine here the transport properties in such wires. An example of conductance is shown in Fig. 9.16. One clearly sees the conductance quantization at the value e^2/h (per spin). This is in agreement with the theoretical expectation (see Section 7.2) and clearly shows that there is only one miniband at the Fermi level. From these data one sees that there are deviations to the perfect conductance quantization when the temperature is

lowered. It is tempting to interpret these deviations as due to the disorder present in the wire following the analysis presented in this chapter. One indeed observes a decrease of conductance with temperature as shown in Fig. 9.16.
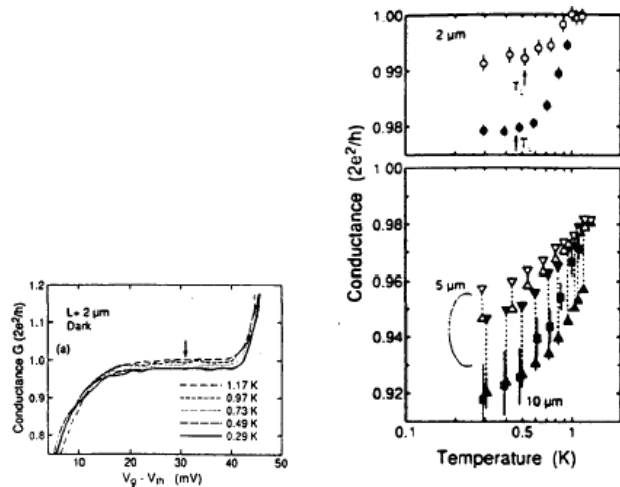
(p.302)

Fig. 9.16. Conductance of a quantum wire. One observes quantization of the conductance at a value of e^2/h per spin. This is an indication that there is only one channel of conduction at the Fermi level. The deviations from the perfect quantization can be interpreted as being due to the disorder in the wire. The data are consistent with the expected behavior for a LL but given the short-range of temperatures it is difficult to draw definite conclusions. (From Tarucha *et al.*, 1995 (Copyright (1995), with permission by Elsevier).)

Another very interesting technique (Yacoby *et al.*, 1996; de Picciotto *et al.*, 2000) consists in making the wire at the edge of a two-dimensional electron gas. This technique allows to have a very uniform width for the wire. The interpretation of the data in these systems is however more complex due to large contacts between the one-dimensional electron gas and the large 2D–1D scattering (de Picciotto *et al.*, 2001). These systems allow however for beautiful experiments both in transport and in tunneling between wires (Auslaender *et al.*, 2002; Carpentier *et al.*, 2002; Tserkovnyak *et al.*, 2002).

It would be impossible to review here all possible measurements that can be made in these systems. This is a field in constant expansion given the progress in the experimental realization of these systems. Quantum wires are clearly one of the ultimate weapons to study individual one-dimensional systems.

Notes:

(35) And also because these are essentially the only integrals one knows how to do!