WHY ADDITIVE NOISE DOES NOT HAVE SAME PB ?

$$x(t_{k+1}) - x(t_k) = \underbrace{f(\bar{x}(t_k))dt}_{O(dt)} \ll \underbrace{+ g(\bar{x}(t_k))\xi(t_k)\,dt}_{O(dt^{1/2})}$$

$\longrightarrow \langle \xi(t)\xi(t')\rangle \propto \frac{1}{dt} \Rightarrow \xi(t_k) \sim \frac{1}{dt^{1/2}}$

$\Rightarrow x(t_{k+1}) - x(t_k) \sim O(dt^{1/2})$

and $g(x) - g(\bar{x})$ ≈ → not negligeable, and so because multiplied by the noise

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REF OKSENDAL

$\frac{dx}{dt} = f(x) + g(x)\xi$ the stochastic diff. eq. is properly defined only after after the evaluation of the x_s in the r.h.s explicitely: $f(\bar{x}(t_k))$ $g(\bar{x}(t_k))$

↳ $\bar{x}(t_k) = (1-\alpha)x(t_k) + \alpha x(t_{k+1})$ $\alpha = 1/2$ stratonovich, $\alpha = 0$ ito.

→ CHAIN RULE are fixed by the value of α chosen:

$$\frac{dY(x)}{dt} = \frac{dY}{dx}\frac{dx}{dt} + D(1-2\alpha)\frac{d^2Y}{dx^2}g^2(x)$$

→ usual only if $\alpha = 1/2$

⚠ easier to build the solution with $\alpha = 0$ but beware to add all necessary terms!

additional term, given by $\langle \xi(t_k)\xi(t_l)\rangle \propto \frac{D}{dt}\delta_{kl}$ ← comes from the $\delta(t_l - t_k) \to \frac{\delta_{kl}}{dt}$.

⚠ if integrating numerically with $\neq \alpha$ ⇒ different ensemble if g is not constant (multiplicative noise)

PROOF: $g(\bar{x}_\alpha(t_k))\xi(t_k) - g(\bar{x}_{\alpha'}(t_k))\xi(t_k) = g'(x(t_k))(\alpha - \alpha')\underbrace{dx\,\xi(t_k)}_{O(dt^{1/2})O(dt^{-1/2})} = O(1)$

Fokker Planck equation

$$\frac{\partial P}{\partial t} = -\partial_x\left[\left(f(x) + 2D\alpha g(x)\partial_x g(x)\right)P(x,t)\right] + D\partial_x^2\left[g^2(x)P(x,t)\right]$$

(depends on g and α.)

↳ Now choosing wisely the force, we can have equivalence between different αs:

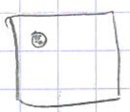
Same process for: $f + 2D\alpha gg' = f_{diff} + 2D\alpha' gg'$ (↳ to tune!)

↳ steady state:

$$P_{st}(x) = \frac{1}{Z}e^{-1/D\, V_{eff}(x)} \quad \text{with } V_{eff}(x) = -\int^x dx' \frac{f(x')}{g^2(x')} + 2D(1-\alpha)\log g(x).$$

external potential $U(x)$, with $V_{eff}(x) = U(x)$ if $f(x) = -g^2(x)U'(x) + 2D(1-\alpha)g(x)g'(x)$.

e.g. Brownian motion experiment in small volume → feels the walls



friction $\gamma_0(\bar{x})$ → $\langle \xi(t)\xi(t')\rangle = 2\gamma_0(\bar{x})k_BT\delta(t-t')$ with spatial dependence

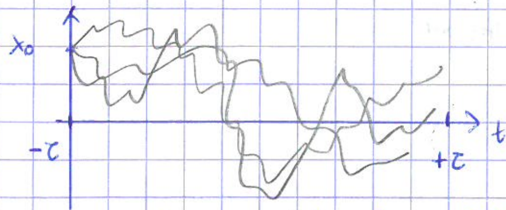
↳ MULTIPLICATIVE NOISE.

FORMALISM II

Functional formalism (Feynmann path integrals).

Well suited for:
- Perturbative expansion
- time dependent RG
- self consistent approximation ⇒ $p>2$ spin models, glasses
- dynamical symmetries ⇒ exact results for systems in eq (FDT), out eq FT

↳ SLIDES → REVIEW: LFC & Lecomte 1704.



↳ extract process with same I.C.: distribution $P(x,t)$ because noisy!
↳ the I.C. can also be stochastic: $P_i(x_0)$

notations: symmetric time interval, vectors: $\{x\}_{t=\tau..\tau}$, $\{\xi\}_{t=-\tau..\tau}$

$P(\{x\},\alpha) = \langle \prod_{k=1}^{N} \delta(x_k - x_k^{sol}) \rangle P_i(x_{-\tau})$

trajectory, noise, LE solution

↳ imposing $x_k = x_k^{sol}$ can be tricky (implicit equations), but the idea here will be to integrate over the noise by rewriting $0 = Eq(x_k, x_{k-1}, \alpha ..) \rightarrow \xi = L(x_k, x_{k-1}, \alpha)$ and impose this with a delta function. ⚠ JACOBIAN! $J = \det_{k,k'} \left[\frac{\delta Eq(x_k, x_{k-1}, \xi_{k-1}, \alpha)}{\delta \xi_{k'}} \right]$

$\Rightarrow P(\{x\},\alpha) = \langle \prod_{k=1}^{N} J^{-1} \delta(\xi - L(x_k, x_{k-1}, \alpha)) \rangle P_i(x_{-\tau})$

↳ then represent δ with $\int e^{i\hat{x}\xi} d\hat{x} \rightarrow P_{OL}(\{x\},\alpha) = \int \mathcal{D}x\, e^{S_{MSR}[x,\hat{x},\alpha]}$ — Martin-Siggia-Rose

"Onsager-M..."

↳ Action with nice form to identify response function $R(t,t') = \langle x(t) \frac{i\hat{x}(t')}{g(t)} \rangle$ (JANSSEN 70's)

if additive noise: $P_{MSR}(\{x_t, i\hat{x}_t\}) \rightarrow$ observables $\langle A(x_t, i\hat{x}_t) \rangle = \int \mathcal{D}x\, \mathcal{D}\hat{x}\, P_{MSR}(\{x\}, \{i\hat{x}\}, \alpha) A(x_t, \hat{x}_t)$
↳ do not depend on α. + parameters from potential.

system in equilibrium here: $P_i(x_{-\tau}) \propto e^{-\beta U(x_{-\tau})}$

↳ time reversibility invariance
$x_t \rightarrow x_{-t}$
$i\hat{x}_t \rightarrow i\hat{x}_t + \beta d_t x_{-t}$ ($1/0$)
$\alpha \rightarrow 1-\alpha$

$\Rightarrow \langle x_t i\hat{x}_{t'} \rangle = \langle x_{-t}(i\hat{x}_t + \beta d_{t'} x_{t'}) \rangle$
↳ $R(t,t') = R(t,-t') + \beta d_{t'} C(t,t')$
FDT IN EQUILIBRIUM!

Dynamics of the p-spin:

$H_J = -\sum_{i_1..i_p} J_{i_1..i_p} s_{i_1}..s_{i_p} + \sum_{i=1}^{N}(s_i^2 - N) z$ ← lagrange multiplier → not quadratic, L.E. cannot be solved directly.

↳ Use the MSR action, written from the Langevin equation: $S_{MSR}[\{i\hat{s}\},\{s\},z]$
↳ introduce auxiliary matrices
$Q_{ab}(t,t') = \frac{1}{N}\sum_{i=1}^{N} s_i^a(t) s_i^b(t)$ $a,b = \hat{s}$ or non $\hat{s}$ → 4 different matrices.

$\Rightarrow \int \mathcal{D} Q_{a,b}(t,t')\, e^{-N S_{new}(Q_{ab}(t,t'))} = e^{-N S_{new}(Q^{S.P}_{a,b}(t,t'))}$ ($S_{new} = O(1)$, saddle point)

$+ \; Q^{S.P.}_{a,b}(t,t') = \langle Q_{ab}(t,t') \rangle \longrightarrow$ average observables: $t \geq t'$
$C(t,t')$
$R(t,t')$
$R(t,t) = 0$ } by causality
$Q_4(t,t') = 0$ } $\langle i\hat{s}\, i\hat{s} \rangle$

where the two remaining observables must verify:

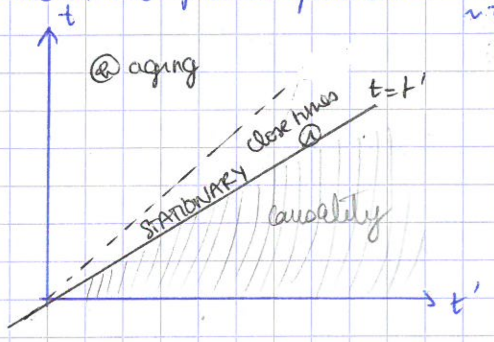
$\lim N \to +\infty \begin{cases} (\partial_t - z(t)) C(t,t') = \int_0^t dt'' \Sigma(t,t'') C(t'',t') + \int_0^{t'} dt'' D(t,t'') R(t',t'') \\ (\partial_t - z(t)) R(t,t') = \int_{t'}^{t} dt'' \Sigma(t,t'') R(t'',t') \end{cases}$

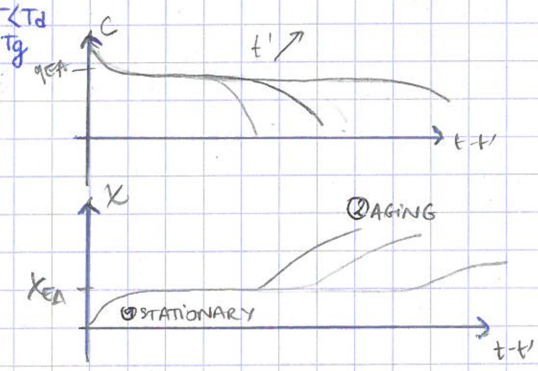
SCHWINGER-DYSON EQS. with random I.C. (quench high T → low T)

with $\begin{cases} \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t') R(t,t') \text{ "self energy"} \\ D(t,t') = \frac{p}{2} C^{p-1}(t,t') \text{ "vertex"} \end{cases}$

$\Rightarrow$ Forces derive from a potential $\Sigma = \frac{dD}{dC} R(t,t')$

Rk. if imposing equilibrium I.C. extra terms in SCHWINGER-DYSON equations.
↳ Now random I.C. from a quench.





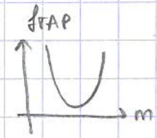
$\chi(t,t') = \int_{t'}^{t} dt'' \, R(t,t'')$

→ NON TRIVIAL RESPONSE

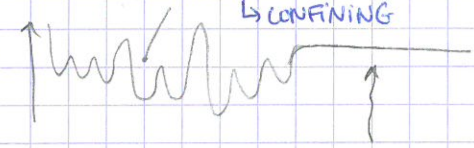
We sketched the results for numerical solutions. Possible to write analytical solutions for the different regimes (stationary and aging) separately.

Free energy landscape
↳ $f_{TAP}(\{m_i\}, T)$

$T \gg T_d$



$T < T_d$



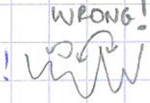
Threshold ≡ attractor for the quenched dynamics.

can check that the q_{EA} we found is related to width of the non flat directions!

flat direction, a bit tilted, drift.
↳ DIFFUSION

⇒ In the confining states, there is no second stage of relaxation! $C(t,t') \xrightarrow[t-t' \to \infty]{} q_{state}$

⇒ So the fact that we see the two relaxations → goes to the threshold!
↳ And escaping from the threshold to a confining ≡ $t(N) \sim e^{N}$ → NEVER! WRONG!



NOTE: Try to see how threshold is coming from equations!