Critical Soft Matter

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OUTLINE

- Introduction and motivation
- Critical states of matter
- Smectics
- Cholesterics
- Columnar phase
- Polymerized membranes
- Elastomers
- Summary and conclusions

I. INTRODUCTION

Soft condensed matter[1, 2] is a rich and vast subject of physics, that even a four-week Boulder School is insufficient to cover comprehensively. The "soft" in soft matter connotes that it is characterized by tiny cohesive energies comparable to thermal energies $k_B T_{room} \sim 1/40eV$, and are thereby conventionally distinguished from their solid-state older "sibling" characterized by 100 times larger, order of an electron-volt energetics. Because of this *quantitative* "softness", effects of thermal fluctuations are essential, with entropy playing a primary role and must be taken into account.

However, in addition to this quantitative, material-focused characterization, soft matter can also be distinguished by the *qualitative* softness of the elasticity of their ordered states. Namely, there is a large and ever-growing class of soft matter systems, (often associated with liquid crystalline order), that for symmetry reasons are distinguished by a set of *vanishing* elastic constants. This has many drastic implications, such as fluctuations-driven instability or qualitative modification of phases well-ordered at the mean-field level.

We refer to this class of phases as "critical" soft matter, because as we will illustrate below they exhibit behavior akin to a state at a critical point of a continuous phase transition, but here extending to the whole phase. Some of the prominent examples of states that fall into

this fascinating class are smectics, cholesterics, columnar and other partially periodic phases of matter[1, 2], as well as polymerized membranes[3] and liquid crystal elastomers[12]. These are illustrated in Fig.(1) below.

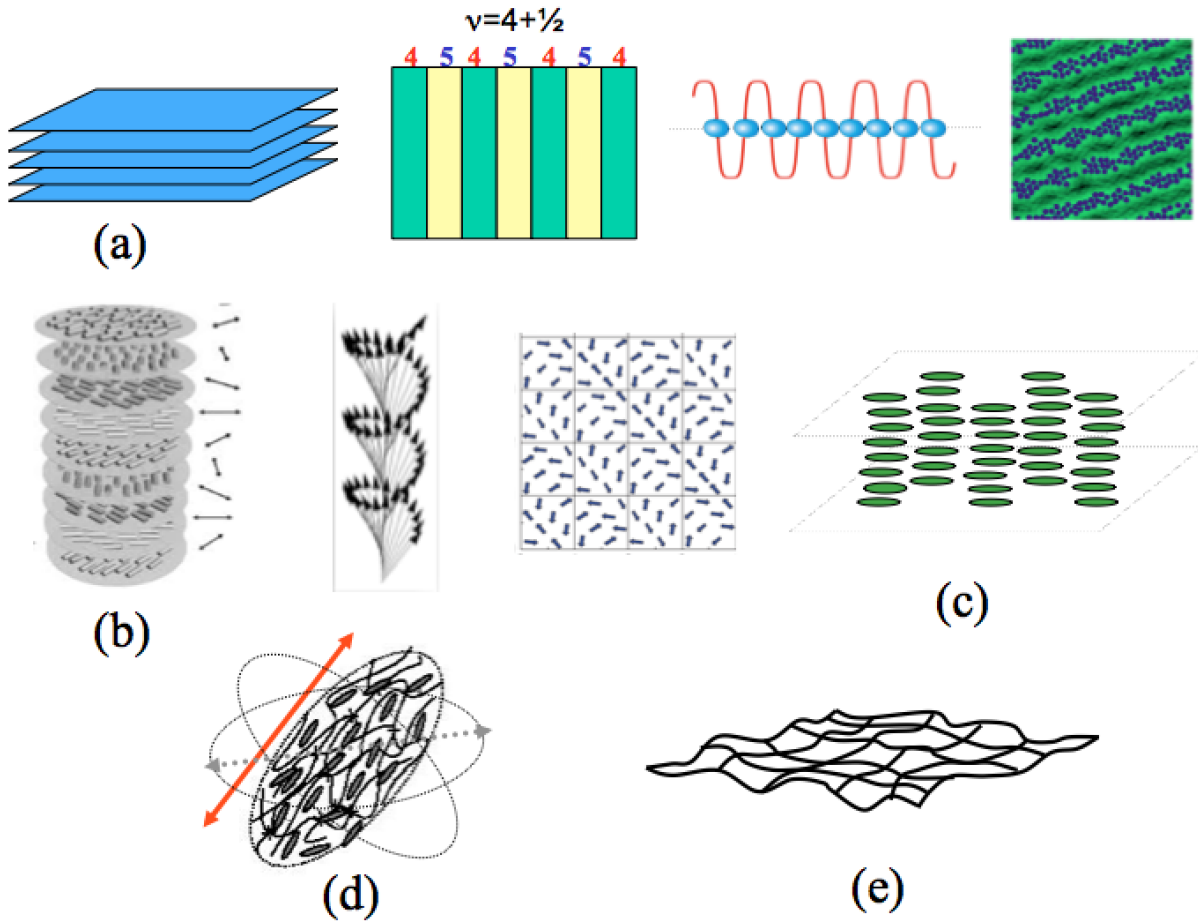

FIG. 1: Illustration of critical phases: (a) smectics realized as conventional liquid crystals[2], 2d colloids[4], quantum-Hall systems[5–7] and as striped "pair-density wave" FFLO superconducting phases in degenerate atomic gases[8], (b) cholesteric liquid crystals[2, 9] and helical state of frustrated magnets[9, 10], (c) columnar liquid crystals and spontaneous vortex lattices[11], (d) nematic elastomers[12, 13], and (e) polymerized membranes[3].

In these lectures, we will first describe in general terms the nature of such "critical matter", illustrating its distinction from conventional ordered states (whether in "soft" or "hard" matter), and then will describe in detail how above listed systems realize this quite exotic phenomenology.

II. CRITICAL STATES OF SOFT MATTER

A. Conventional phases and critical phenomena

Much of the universal low-energy equilibrium phenomena in nature can be understood in terms of symmetry-characterized disordered and ordered phases and continuous phase transitions between them. A paradigmatic example is a classical ferromagnet, that is mathematically equivalent to a large variety of physically distinct systems. Its low temperature ferromagnetic (FM) phase is characterized by a local magnetization order parameter $\vec{\phi}(\mathbf{x})$, that breaks the $O(3)$ symmetry of the high-temperature disordered paramagnetic (PM) phase down to $O(2)$, with the order parameter "living" on the coset space $M = S_2 = O(3)/O(2)$. The universal low-energy behavior of the phases and phase transition can be captured by the Landau field theory, with an effective Hamiltonian density

$$\mathcal{H}_{PM-FM} = \frac{1}{2}J(\boldsymbol{\nabla}\vec{\phi})^2 + \frac{1}{2}t|\vec{\phi}|^2 + \frac{\lambda}{4}|\vec{\phi}|^4 + \text{higher order terms}\ldots, \tag{1}$$

used to calculate the thermodynamics via $Z = \int [d\vec{\phi}] e^{-\beta H_{FM}}$ and correlation functions of $\vec{\phi}(\mathbf{x})$ and other local observables.

Ignoring fluctuations and simply minimizing $\mathcal{H}_{FM}[\vec{\phi}]$ over the order parameter, gives the mean-field description of the FM-PM phase transition, and the corresponding phases. The PM phase $\langle\vec{\phi}\rangle = 0$ is characterized by a large positive temperature $t > 0$, which allows one to neglect nonlinearities and gradients, with harmonic energy density

$$\mathcal{H}_{PM} = \frac{1}{2}t|\vec{\phi}|^2 + \text{higher order terms}\ldots.$$

The FM phase $\langle\vec{\phi}\rangle = \vec{\phi}_0$ is characterized by a large negative (reduced) temperature $t < 0$, which allows one to neglect nonlinearities (with one caveat, that in the interest of space we will neglect here), with

$$\mathcal{H}_{FM} = \frac{1}{2}|\boldsymbol{\nabla}\delta\vec{\phi}|^2 + \text{higher order terms}\ldots,$$

governing the energetics of the Goldstone mode, $\delta\vec{\phi} \perp \vec{\phi}_0$.

Problem 1: Derive the Goldstone mode Hamiltonian and show that for $d > 2$ its harmonic fluctuations are finite.

Problem 2: Examine more carefully the lowest-order nonlinearities of the Goldstone and longitudinal modes Hamiltonian. Thinking about the longitudinal (along $\vec{\phi}_0$) susceptibility,

what do you think the aforementioned caveat is? (The answer may be clearer after discussing critical phases, below).

Because of the asymptotically quadratic nature of the effective Hamiltonian of the two *phases*, their description is effectively trivial, with fluctuations only leading to corrections (above the lower-critical dimension $d_{lc} = 2$, where the ordered phase is stable) that are small at long scales.

In strong contrast, near the continuous transition itself, where the reduced temperature is small, $t \approx 0$ and therefore the quadratic term is small, the $|\vec{\phi}|^4$ nonlinearity is dominant, with universal critical behavior determined by its balance against the gradient J term, with

$$\mathcal{H}_{critical} = \frac{1}{2} J (\boldsymbol{\nabla} \vec{\phi})^2 + \frac{\lambda}{4} |\vec{\phi}|^4 + \text{higher order terms} \ldots .$$

This thereby leads to highly nontrivial effects of nonlinearities (below the upper-critical dimension, $d_{uc} = 4$), the so-called non-mean-field critical behavior, characterized by universal critical exponents.

In particular, at the critical point the order-parameter correlation function $C(\mathbf{x}, t = 0) = \langle \vec{\phi}(\mathbf{x}) \cdot \vec{\phi}(0) \rangle$ is characterized by a nontrivial power-law in real and Fourier spaces,

$$C(\mathbf{x}, t = 0) \sim 1/x^{d-2+\eta}, \quad \tilde{C}(\mathbf{q}, t = 0) \sim 1/q^{2-\eta} \equiv 1/(J(q) q^2),$$

that can be equivalently interpreted as a power-law, length-scale dependent exchange constant $J(\mathbf{q}) \sim 1/q^{\eta}$. Similarly, at the critical point the correlation function exhibit a divergent power-law response to perturbations that take the system away from the critical point, as e.g., the magnetic susceptibility $\chi(t) \equiv \tilde{C}(q = 0, t) \sim 1/t^{\gamma}$ (t should not be confused with time). Thus, as illustrated in Fig.2, in conventional systems qualitative effects of fluctuations only appear when tuned near a critical point, with phases behaving quite simply.

B. Critical phases

In stark contrast to the above conventional states, there is a class of exotic systems (e.g., smectics, cholesterics, columnar phases, elastomers, membranes,...), where the importance of fluctuations and nonlinearities extends throughout the ordered phase. Namely, as illustrated in Fig.3, in these systems the underlying symmetry of the disordered phase ("broken" in the ordered state) requires a vanishing of a set of elastic constants of the Goldstone-mode

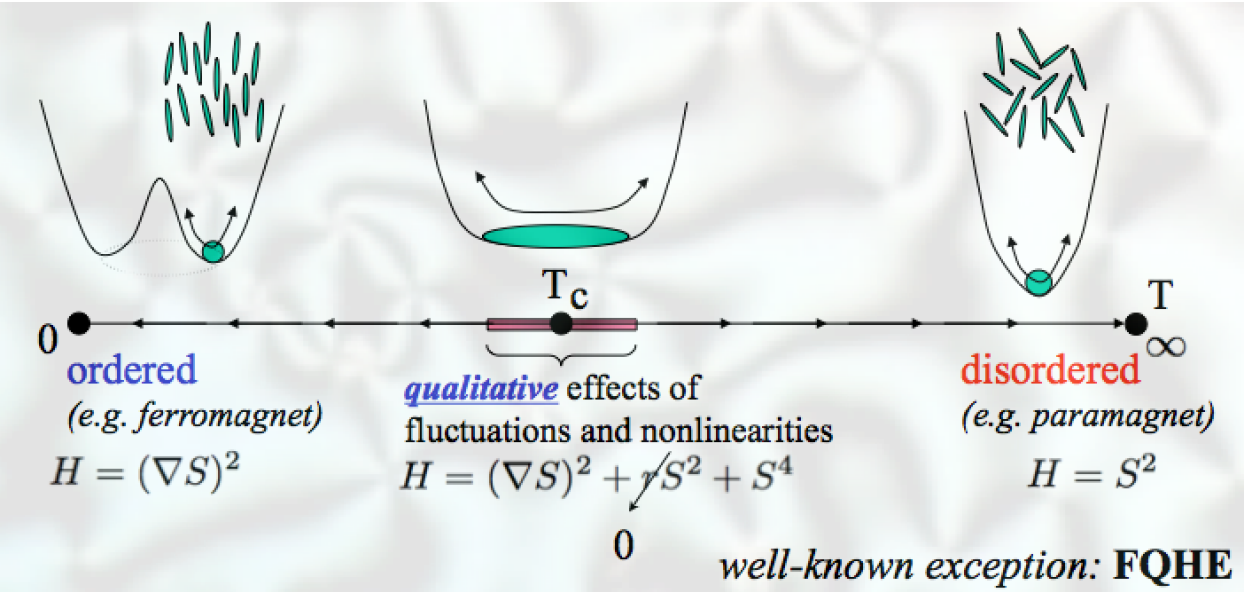

FIG. 2: Illustration of unimportance of fluctuations inside phases of conventional systems, where qualitative effects of thermal fluctuations are confined to a vicinity of a critical point.

Hamiltonian characterizing the ordered state. The ordered state is thus described by "soft" (qualitatively, not just quantitatively) Goldstone-mode elasticity and is effectively critical without any fine-tuning, enforced by the underlying symmetry.

Consequently, at finite temperature, fluctuations are divergently large, limited only by strong nonlinear elasticity and lead to phenomenology akin to that of a critical point, but extending across the whole phase. Namely, as illustrated in Fig.3 such critical phase is characterized by an infrared stable, finite coupling fixed point, and as a result exhibits universal power-law correlations, length scale dependent elastic moduli, divergent response to symmetry breaking fields, etc.

For the known cases, some that we will discuss below, the necessary ingredients of critical phases are anisotropy, spontaneously broken rotational and partial translational symmetries, sufficiently low dimension such that nonlinearities are important at low temperature.

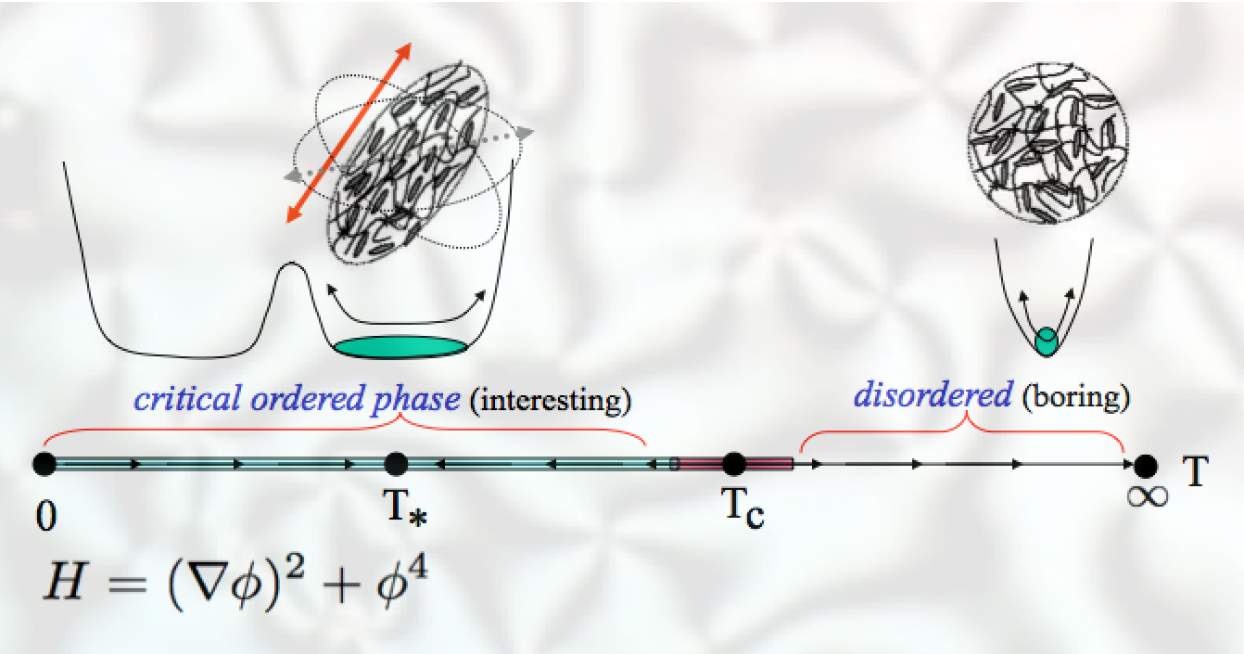

FIG. 3: Illustration of importance of fluctuations inside ordered phases of "critical" soft matter. Even deep in the ordered phase the quartic potential is missing a quadratic contribution resulting in divergent fluctuations. The bottom of the figure shows the renormalization group flow, e.g., for a smectic state in $d < 3$ dimensions, illustrating that at low T it is a "critical phase" displaying universal power-law phenomenology, controlled by a nontrivial infrared stable fixed point.

III. SMECTIC

A. Liquid crystals

Liquid crystals are states of matter that *spontaneously partially* break spatial symmetries. Defined in this symmetry- rather than materials-based way, such phases appear in systems far beyond conventional liquid crystalline materials, typically composed of highly anisotropic (rod-like or disk-like) constituents. Some of the novel systems include biopolymers like RNA and DNA[14], viruses[15], frustrated magnets[10], and strongly correlated electronic and bosonic systems (quite amazing in light of the fact that in these systems the constituents are essentially point particles), such as FFLO "striped" ("pair density wave") superconductors[7, 8, 16], finite momentum superfluids[17], and a two-dimensional electron gas in the quantum Hall regime of high filling fraction $\nu = N + 1/2$[5, 6, 18].

Illustrated in Fig.4, the most ubiquitous liquid crystal phases are the uniaxial nematic,

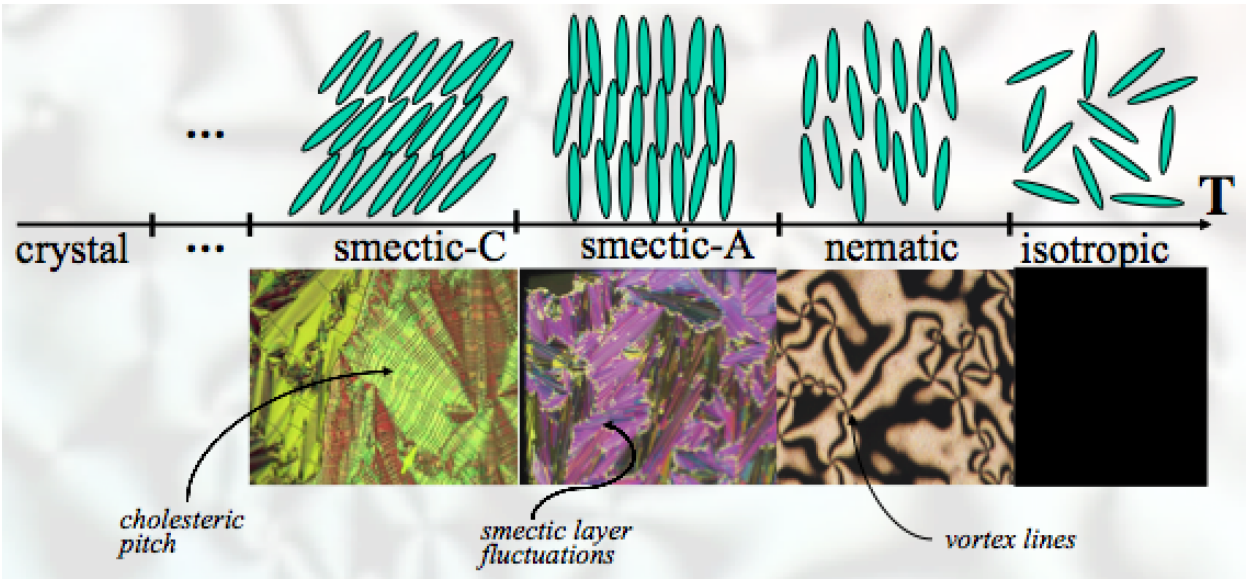

FIG. 4: Most ubiquitous nematic (orientationally ordered uniaxial fluid), smectic-A and smectic-C (one-dimensional density wave with, respectively isotropic and polar in-plane fluid orders) liquid crystal phases and their associated textures in cross-polarized microscopy (N.A. Clark laboratory).

that spontaneously breaks rotational symmetry of the parent isotropic fluid and the smectic state, a uniaxial one-dimensional density wave that further breaks translational symmetry along a single axis. The uniaxial nematic is characterized by a quadrupolar order parameter $Q_{ij} = S(\hat{n}_i\hat{n}_j - \frac{1}{3}\delta_{ij})$, with S the strength of the orientational order along the principle uniaxial axis $\hat{n}$. The smectic is a periodic array of two-dimensional fluids, characterized by a uniaxial periodic density modulation with Fourier components that are integer multiples of the smectic ordering wavevector $\mathbf{q_0} = \hat{\mathbf{n}}2\pi/a$ (a is the layer spacing) parallel to the nematic director $\hat{\mathbf{n}}$. The dominant lowest Fourier component $\psi(\mathbf{x}) = \rho_{q_0}$ can be taken as the local (complex scalar) order parameter which distinguishes the smectic-A from the nematic phase[2]. It is related to the density $\rho(\mathbf{r})$ by

$$\rho(\mathbf{x}) = \mathrm{Re}[\overline{\rho}_0 + e^{i\mathbf{q_0}\cdot\mathbf{x}}\psi(\mathbf{x})] \; , \tag{2}$$

where $\overline{\rho}_0$ is the mean density of the smectic.

As first discussed by de Gennes[2], the effective Hamiltonian functional $H_{dG}[\psi, \hat{\mathbf{n}}]$ that describes the nematic-to-smectic-A (NA) transition at long length scales, in bulk, pure liquid crystals, is:

$$H_{dG}[\psi, \hat{\mathbf{n}}] \; = \; \int d^dx \left[c|(\boldsymbol{\nabla} - iq_0\delta\hat{\mathbf{n}})\psi|^2 + t_0|\psi|^2 + \frac{1}{2}g_0|\psi|^4 \right] + H_F[\mathbf{n}] \; , \tag{3}$$

where $t_0 \propto T - T_{NA}$ is the reduced temperature for transition at T_{NA},

$$\delta\hat{\mathbf{n}}(\mathbf{x}) \equiv \hat{\mathbf{n}}(\mathbf{x}) - \hat{\mathbf{n}}_0 = \delta\hat{\mathbf{n}}_\perp + \hat{\mathbf{n}}_0(\sqrt{1-\delta\hat{\mathbf{n}}_\perp^2} - 1)$$

is the fluctuation of the local nematic director $\hat{\mathbf{n}}(\mathbf{x})$ away from its average value $\hat{\mathbf{n}}_0$, which we take to be $\hat{\mathbf{z}}$, and $H_F[\mathbf{n}]$ is the Frank effective Hamiltonian that describes the elasticity of the nematic order director:

$$H_F[\hat{\mathbf{n}}] = \int d^dx \, \frac{1}{2}\Big[K_s(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}})^2 + K_t(\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{n}})^2 + K_b(\hat{\mathbf{n}}\times\boldsymbol{\nabla}\times\hat{\mathbf{n}})^2\Big] \,, \tag{4}$$

where K_s, K_t, and K_b are the bare elastic moduli for splay, twist and bend of the nematic director field, respectively.

The "minimal coupling" between $\mathbf{n}$ and ψ is enforced by the requirement of global rotational invariance[2]. It is important to emphasize, however, that although the de Gennes Hamiltonian H_{dG} is closely analogous to that of a superconductor, there are essential differences. The physical reality of the nematic $\hat{\mathbf{n}}$ and the smectic ψ order parameters (in contrast to the gauge ambiguity in the definition of the vector potential and the superconducting order parameter), selects the liquid crystal gauge $\delta\hat{\mathbf{n}}_\perp \cdot \hat{\mathbf{n}}_0 = 0$ as the preferred physical gauge and also allows measurements of $\hat{\mathbf{n}}$ and ψ. The strict gauge invariance of H_{dG} is also explicitly broken by the splay term $K_s(\boldsymbol{\nabla}\cdot\hat{\mathbf{n}})^2$ of the Frank Hamiltonian (contrasting with the Maxwell action that involves purely gauge invariant derivatives, e.g., $(\boldsymbol{\nabla}\times\mathbf{A})^2$), that will be crucial for obtaining the smectic Goldstone-mode elasticity.

B. Smectic elasticity

1. From the nematic state via deGennes model

Within the ordered smectic phase, the fluctuations are conveniently described in terms of the magnitude and phase of ψ. It is easy to show that the fluctuations of the *magnitude* of ψ around the average value $|\psi_0| = \sqrt{t_0/g_0} = \text{const.}$ are "massive", and can therefore be safely integrated out of the partition function, leading to only *finite*, unimportant shifts in the effective elastic moduli. In contrast, the phase of ψ is a U(1) massless Goldstone mode, corresponding to spontaneously broken translational symmetry. It is the essential low energy phonon degree of freedom of the smectic phase, describing the local displacement

of the smectic layers from perfect periodic order. In accord with this discussion, deep within the smectic phase, we can represent the smectic order parameter as

$$\psi(\mathbf{x}) = |\psi_0| e^{-iq_0 u(\mathbf{x})} \,, \tag{5}$$

safely ignoring (actually integrating out the "massive") fluctuations in the magnitude $|\psi_0|$ of ψ. It is important to note that this can be done at any temperature *below* the transition, without any *qualitative* consequences for phenomena occurring on sufficiently long length scales, larger than a well-defined crossover length $\xi(T)$. The elastic model is then rigorously valid on length scales larger than $\xi(T)$ and breaks down on shorter scales, and therefore, of course can only make predictions about phenomena on scales longer than $\xi(T)$. As $T \to T_{NA}^-$, $\xi(T)$ diverges and the range of *length scales* about which the elastic model is able to make predictions shrinks, being pushed out to infinite scales.

Using this low-temperature ansatz inside the ψ-dependent part of the effective Hamiltonian, given by Eq.3 and dropping constant terms, we find

$$\begin{aligned} H[u, \delta\hat{\mathbf{n}}_\perp] &= \int d^d x \left[\frac{B}{2} (\boldsymbol{\nabla} u + \delta\hat{\mathbf{n}})^2 + \frac{K_s}{2} (\boldsymbol{\nabla} \cdot \delta\mathbf{n})^2 + \frac{K_t}{2} (\hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times \delta\mathbf{n})^2 + \frac{K_b}{2} (\hat{\mathbf{z}} \times \boldsymbol{\nabla} \times \delta\mathbf{n})^2 \right], \\ &= \int d^d x \left[\frac{B}{2} (\boldsymbol{\nabla}_\perp u + \delta\hat{\mathbf{n}}_\perp)^2 + \frac{B}{2} (\partial_z u - \frac{1}{2} \delta\hat{n}_\perp^2)^2 \right. \\ &\quad \left. + \frac{K_s}{2} (\boldsymbol{\nabla} \cdot \delta\hat{\mathbf{n}})^2 + \frac{K_t}{2} (\hat{\mathbf{z}} \cdot \boldsymbol{\nabla} \times \delta\hat{\mathbf{n}})^2 + \frac{K_b}{2} (\hat{\mathbf{z}} \times \boldsymbol{\nabla} \times \delta\hat{\mathbf{n}})^2 \right], \end{aligned} \tag{6}$$

where $B = 2c|\psi_0|^2 q_0^2$ is the smectic compression modulus. We now observe a quite fascinating phenomenon: the fluctuation mode $\boldsymbol{\nabla}_\perp u + \delta\hat{\mathbf{n}}_\perp$ is "massive" and leads to an analog of the Anderson-Higgs mechanism, a hallmark of gauge theories. As a consequence, after a simple Gaussian integration over $\delta\hat{\mathbf{n}}_\perp$, we find that at long length scales, $\delta\hat{\mathbf{n}}_\perp$ fluctuations are constrained to follow $\boldsymbol{\nabla}_\perp u$. The elastic smectic Hamiltonian is then obtained by the replacement

$$\delta\hat{\mathbf{n}}_\perp \to -\boldsymbol{\nabla}_\perp u \,, \tag{7}$$

everywhere in Eq.(6) and Frank energy Eq.(4). Valid in the long wavelength limit and provided dislocations are confined, we thus obtain an effective nonlinear elastic Hamiltonian of the smectic phase,

$$H_{sm}[u] = \int d^d x \left[\frac{K}{2} (\nabla_\perp^2 u)^2 + \frac{B}{2} (\partial_z u - \frac{1}{2} (\boldsymbol{\nabla} u)^2)^2 \right] , \tag{8}$$

where $K = K_s$ is the bend modulus. We close by noting that the aforementioned connection to the Anderson-Higgs mechanism elucidates why a smectic state is characterized by only a single Goldstone mode despite partially breaking rotational and translational symmetries. In contrast a charged superconductor is well-known to have all its Goldstone modes "eaten" by this mechanism.

Problem 3:

Fill in the details of the above derivation of smectic elasticity.

2. From the isotropic fluid

We now derive the nonlinear smectic elasticity in a more basic, complementary way, starting instead with an isotropic fluid state. We begin with a generic energy functional that captures system's tendency to develop a unidirectional wave at wavevector $\mathbf{q}_0$, with an arbitrary direction, and magnitude fixed at q_0,

$$\mathcal{H}_{sm} = \frac{1}{2} J \left[(\nabla^2 \rho)^2 - 2q_0^2 (\nabla \rho)^2 \right] + \frac{1}{2} t \rho^2 - w\rho^3 + v\rho^4 + \dots, \tag{9}$$

where J, q_0, t, w, v are parameters of the isotropic fluid phase. From the first term, clearly dominant fluctuations are on a spherical surface at a finite wavevector with a magnitude q_0. Thus let's focus on the density at a finite wavevector $\mathbf{q}$ (neglecting the inconsequential constant part of the density, $\overline{\rho}_0$), that for now we will take to be unrelated to q_0

$$\rho(\mathbf{x}) = \mathrm{Re}\left[\rho_q(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}} \right], \tag{10}$$

where $\rho_q(\mathbf{x})$ is a complex scalar and Re is a real part. Without loss of generality we take the order parameter $\rho_q(\mathbf{x})$ to have a (constant) magnitude ρ_0 and the phase $qu(\mathbf{x})$

$$\rho_q(\mathbf{x}) = \rho_0 e^{-iqu}. \tag{11}$$

Clearly $u(\mathbf{x})$ is just a phonon displacement along $\mathbf{q}$. Gradients of ρ are easy to work out

$$\nabla\rho = \rho_0 \mathrm{Re}\left[i(\mathbf{q} - q\nabla u) e^{i(\mathbf{q}\cdot\mathbf{x} - qu)} \right], \tag{12a}$$

$$\nabla^2\rho = \rho_0 \mathrm{Re}\left[\left\{ -(\mathbf{q} - q\nabla u)^2 - iq\nabla^2 u \right\} e^{i(\mathbf{q}\cdot\mathbf{x} - qu)} \right]. \tag{12b}$$

Substituting this form of ρ and its gradients into $\mathcal{H}_{sm}$ we find

$$\mathcal{H}_{sm} = \frac{1}{4}J\rho_0^2\left[(\mathbf{q}-q\nabla u)^4 + q^2(\nabla^2 u)^2 - 2q_0^2(\mathbf{q}-q\nabla u)^2\right] + \frac{1}{4}t\rho_0^2 + \frac{1}{4}v\rho_0^4 + \ldots, \tag{13a}$$

$$= \frac{1}{4}J\rho_0^2\left[\left((\mathbf{q}-q\nabla u)^2 - q^2\right)^2 + q^2(\nabla^2 u)^2 + 2(q^2-q_0^2)(\mathbf{q}-q\nabla u)^2\right] + \frac{1}{4}t\rho_0^2 + \ldots, \tag{13b}$$

$$= J\rho_0^2\left[\frac{1}{4}q^2(\nabla^2 u)^2 + \left(q\mathbf{q}\cdot\nabla u - \frac{1}{2}q^2(\nabla u)^2\right)^2 + 4(q^2-q_0^2)\left(q\mathbf{q}\cdot\nabla u - \frac{1}{2}q^2(\nabla u)^2\right)\right] + \ldots, \tag{13c}$$

where we dropped constant parts as well as fast oscillating pieces as they will average away after spatial integration of the above energy density. Note that then only even parts in ρ_0 appear. Firstly, we observe that (as discussed on general grounds above) in a harmonic part linear gradient elasticity in u only appears for gradients *along* $\mathbf{q}$, namely $\mathbf{q}\cdot\nabla$, with elasticity transverse to $\mathbf{q}$ starting with a Laplacian type. Secondly, the elastic energy density is an expansion in a rotationally-invariant strain tensor combination

$$u_{qq} = \hat{\mathbf{q}}\cdot\nabla u - \frac{1}{2}(\nabla u)^2, \tag{14}$$

whose nonlinearities in u ensure that it is fully rotationally invariant even for large rotations. To see this (picking $\hat{\mathbf{q}} = \hat{\mathbf{z}}$) note that a rigid (distortion-free) rotation of $\mathbf{q}$ ($q_0\hat{z} \to \mathbf{q} = q_0(\cos\theta\hat{z} + \sin\theta\hat{x})$), can be interpreted as $u(\mathbf{x}) = z(\cos\theta - 1) + x\sin\theta$, for which u_{qq} vanishes identically, thus, as required resulting in vanishing energy cost. Thirdly, the last term in (13c) vanishes for $|\mathbf{q}|$ picked to equal q_0.

Looking ahead, as one includes effects of fluctuations, the "bare" condition $q = q_0$ will need to be adjusted so as to eliminate the fluctuation-generated linear term in u_{qq} order by order, which amounts to an expansion in the nonlinear strain u_{qq} around the correct (fluctuation-corrected) ground state. Finally, we note that the relation between the curvature modulus K of Laplacian (first) term and the bulk modulus B gradient (second) term is not generic and can be relaxed to have distinct elastic constants, as can be seen if higher order gradient terms are included in the original energy density, Eq. (9).

Choosing the coordinate system such that $\hat{\mathbf{z}}$ is aligned along $\mathbf{q}$, we find that for $q = q_0$, Eq. (13c) reduces to a more standard smectic elastic energy density,

$$\mathcal{H}_{sm} = \frac{1}{2}K(\nabla^2 u)^2 + \frac{1}{2}B\big(\partial_z u - \frac{1}{2}(\nabla u)^2\big)^2, \tag{15}$$

consistent with Eq.(8) of previous subsection, and familiar from studies of smectic liquid crystals, with all its fascinating consequences[1, 2].

Problem 4:

Fill in the details of the above derivation of smectic elasticity.

C. Finite T Gaussian fluctuation in a harmonic smectic

To assess the extent of thermal fluctuations of the smectic Goldstone mode u we first analyze them within a harmonic approximation, neglecting elastic nonlinearities in H_{sm}. [1, 19]. In terms of the Fourier modes $u_{\mathbf{k}}$, the Hamiltonian decouples, reducing to

$$H_{sm} = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}\left(Kk_\perp^4 + Bk_z^2\right)|u_{\mathbf{k}}|^2, \tag{16}$$

thus allowing a straightforward computation of phonon correlation functions via standard Gaussian integrals or equivalently invoking the equipartition theorem ($k_B T/2$ of energy per mode). In particular, either analysis gives mean-squared fluctuations

$$\langle u^2\rangle_0^T = \int_{L_\perp^{-1}}^{\Lambda_\perp} \frac{d^d k}{(2\pi)^d}\frac{T}{Bk_z^2 + Kk_\perp^4}, \tag{17a}$$

$$\approx \begin{cases} \frac{T}{2\sqrt{BK}} C_{d-1} L_\perp^{3-d}, & d<3, \\ \frac{T}{4\pi\sqrt{BK}} \ln q_0 L_\perp, & d=3, \end{cases} \tag{17b}$$

where we defined a constant $C_d = S_d/(2\pi)^d = 2\pi^{d/2}/[(2\pi)^d\Gamma(d/2)]$, with S_d a surface area of a d-dimensional sphere, and introduced an infrared cutoff by considering a system of finite extent $L_\perp \times L_z$, with L_z the length of the system along the ordering (z) axis and $L_\perp$ transverse to z. Unless it has a huge aspect ratio, such that $L_z \sim L_\perp^2/\lambda >> L_\perp$, any large system ($L_\perp, L_z >> \lambda$) will have $\lambda L_z \ll L_\perp^2$.

The key observation here is that the smectic phonons exhibit fluctuations that diverge, growing logarithmically in 3d and linearly in 2d with system size $L_\perp$; for $d > 3$ fluctuations are bounded. Thus 3d smectics are akin to a 2d xy systems, such as superfluid films and two-dimensional crystals, [20–23], also expected to exhibit a power-law order.

The expression for the mean-squared phonon fluctuations in (17b) leads the emergence of important crossover length scales $\xi_\perp, \xi_z$, related by

$$\xi_\perp = (\xi_z\sqrt{K/B})^{1/2}, \tag{18a}$$

$$\equiv \sqrt{\xi_z \lambda}, \tag{18b}$$

that characterize the finite-temperature smectic state. These are defined as scales $L_\perp, L_z$ at which phonon fluctuations are large, comparable to the smectic period $a = 2\pi/q_0$. Namely, setting

$$\langle u^2 \rangle_0^T \approx a^2 \tag{19}$$

in Eq. (17b) we find

$$\xi_\perp \approx \begin{cases} \frac{a^2\sqrt{BK}}{T} \sim \frac{K}{Tq_0}, & d = 2, \\ a e^{4\pi a^2\sqrt{BK}/T} \sim a e^{\frac{cK}{Tq_0}}, & d = 3, \end{cases} \tag{20a}$$

where in the second form of the above expressions we took the simplest approximation for the smectic anisotropy length $\lambda = \sqrt{K/B}$ to be $\lambda = a \sim 1/q_0$, and introduced an order 1 Lindemann constant c[24], that depends on the somewhat arbitrary definition of "large" phonon root-mean-squared fluctuations.

The smectic connected correlation function

$$C_u(\mathbf{x}_\perp, z) = \langle [u(\mathbf{x}_\perp, z) - u(\mathbf{0}, 0)]^2 \rangle_0 \, . \tag{21}$$

is also straightforwardly worked out, in 3d giving the logarithmic Caillé form[19]

$$\begin{aligned} C_u^{3d}(\mathbf{x}_\perp, z) &= 2T \int \frac{d^2k_\perp dk_z}{(2\pi)^3} \frac{1 - e^{i\mathbf{q}\cdot\mathbf{x}}}{Kk_\perp^4 + Bk_z^2} \, , \\ &= \frac{T}{2\pi\sqrt{KB}} \, g_T^{3d}\left(\frac{z\lambda}{x_\perp^2}, \frac{x_\perp}{a}\right) \, , \\ &= \frac{T}{2\pi\sqrt{KB}} \left[\ln\left(\frac{x_\perp}{a}\right) - \frac{1}{2}\mathrm{Ei}\left(\frac{-x_\perp^2}{4\lambda|z|}\right)\right], \qquad (22a) \\ &\approx \frac{T}{2\pi\sqrt{KB}} \begin{cases} \ln\left(\frac{x_\perp}{a}\right), & x_\perp \gg \sqrt{\lambda|z|} \, , \\ \ln\left(\frac{4\lambda z}{a^2}\right), & x_\perp \ll \sqrt{\lambda|z|} \, , \end{cases} \qquad (22b) \end{aligned}$$

where $\mathrm{Ei}(x)$ is the exponential-integral function. As indicated in the last form, in the asymptotic limits of $x_\perp \gg \sqrt{\lambda z}$ and $x_\perp \ll \sqrt{\lambda z}$ above 3d correlation function reduces to a logarithmic growth with $x_\perp$ and z, respectively.

In 2d we instead have[25]

$$
\begin{aligned}
C_u^{2d}(x,z) &= 2T\int \frac{dk_x dk_z}{(2\pi)^2}\frac{1-e^{i\mathbf{k}\cdot\mathbf{x}}}{Kk_x^4+Bk_z^2}\,, \\
&= \frac{T}{2\pi\sqrt{KB}}\, g_T^{2d}\left(\frac{z\lambda}{x^2},\frac{x}{a}\right)\,, \\
&= \frac{2T}{B}\left[\left(\frac{|z|}{4\pi\lambda}\right)^{1/2} e^{-x^2/(4\lambda|z|)} + \frac{|x|}{4\lambda}\mathrm{erf}\big(\frac{|x|}{\sqrt{4\lambda|z|}}\big)\right] \qquad (23a)\\
&\approx \frac{2T}{B}\begin{cases} \left(\frac{|z|}{4\pi\lambda}\right)^{1/2}, & x \ll \sqrt{\lambda|z|}\,, \\ \frac{|x|}{4\lambda}, & x \gg \sqrt{\lambda|z|}\,, \end{cases} \qquad (23b)
\end{aligned}
$$

where $\mathrm{erf}(x)$ is the Error function.

As a consequence of above divergent phonon fluctuations, the smectic density wave order parameter (11) vanishes in thermodynamic limit

$$
\begin{aligned}
\langle\rho_q(\mathbf{x})\rangle_0 &= 2\rho_0\langle\cos\left[\mathbf{q}_0\cdot\mathbf{x}-qu(\mathbf{x})\right]\rangle_0, \\
&= 2\rho_0 e^{-\frac{1}{2}q_0^2\langle u^2\rangle_0}\cos\left(\mathbf{q}_0\cdot\mathbf{x}\right), \\
&= 2\tilde{\rho}_0(L_\perp)\cos\left(\mathbf{q}_0\cdot\mathbf{x}\right), \qquad (24a)
\end{aligned}
$$

with the thermally suppressed order parameter amplitude given by

$$
\begin{aligned}
\tilde{\rho}_0(L_\perp) &= \rho_0\begin{cases} e^{-L_\perp/\xi_\perp}, & d=2, \\ \left(\frac{a}{L_\perp}\right)^{\eta/2}, & d=3, \end{cases} \qquad (25a)\\
&\to 0, \quad \text{for } L_\perp\to\infty, \qquad (25b)
\end{aligned}
$$

where we used results for the phonon and phase fluctuations, (17b), and defined the Caillé exponent

$$
\eta = \frac{q_0^2 T}{8\pi\sqrt{BK}}. \qquad (26)
$$

Thus, in qualitative contrast to its mean-field cartoon, at long scales (longer than $\xi_{\perp,z}$) the smectic state is characterized by a *uniform* mass density.

Since the average density is actually uniform, a better characterization of the smectic state is through the structure function, $S(\mathbf{q})$, a Fourier transform of the density correlation

function, that in 3d is given by

$$S(\mathbf{q}) = \int d^3x \langle \delta\rho(\mathbf{x})\delta\rho(0)\rangle e^{-i\mathbf{q}\cdot\mathbf{x}}, \tag{27a}$$

$$\approx \frac{1}{2}\sum_{q_n} |\rho_{q_n}|^2 \int_{\mathbf{x}} \langle e^{-iq_n(u(\mathbf{x})-u(0))}\rangle_0 e^{-i(\mathbf{q}-q_n\hat{z})\cdot\mathbf{x}}, \tag{27b}$$

$$\approx \frac{1}{2}\sum_{n} \frac{|\rho_{q_n}|^2}{|q_z - nq_0|^{2-n^2\eta}}, \quad \text{for } d=3, \tag{27c}$$

where we approximated phase and phonon fluctuations by Gaussian statistics (in 3d valid up to weak logarithmic corrections[26]). Thus as anticipated we find that the logarithmically divergent 3d phonon fluctuations lead to a structure function, with highly anisotropic ($q_z \sim q_\perp^2/\lambda$) *quasi*-Bragg peaks replacing the true (δ-function) Bragg peaks characteristic of a true long-range periodic order.[1, 2, 27]

In two dimensions, smectic order is even more strongly suppressed by thermal fluctuations. The linear growth of the 2d phonon fluctuations leads to exponentially short-ranged correlations of the density, expected to result in dislocation unbinding at any finite temperature, thereby completely destroying smectic state in 2d.[25]

D. Nonlinear elasticity: beyond Gaussian fluctuations

1. *Perturbation theory*

As is clear from the derivation of the previous subsection, the restoration of the translational symmetry (a vanishing $\langle\rho_q(\mathbf{x})\rangle$, etc.) by thermal fluctuations is a robust prediction of the quadratic theory, that cannot be overturned by the left-out nonlinearities. However, the asymptotic long-scale form of the correlation functions computed within the harmonic approximation only extends out to the nonlinear length scales $\xi_{\perp,z}^{NL}$, beyond which the divergently large smectic phonon fluctuations invalidate the neglect of the nonlinear phonon operators

$$\mathcal{H}_{\text{nonlinear}} = -\frac{1}{2}B(\partial_z u)(\nabla u)^2 + \frac{1}{8}B(\nabla u)^4. \tag{28}$$

These will necessarily qualitatively modify predictions (22b), (23b), and (27) on scales longer than the crossover scales $\xi_{\perp,z}^{NL}$, that we compute next.

To see this, we use a perturbative expansion in the nonlinear operators (28) to assess the size of their contribution to e.g., the free energy. Following a standard field-theoretic

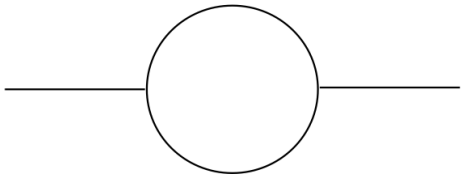

FIG. 5: Feynman graph that renormalizes the elastic moduli K, B of the smectic state.

analysis these can be accounted for as corrections to the compressional B and bend K elastic moduli, with the leading contribution to δB, summarized graphically in Fig.5, and given by

$$\delta B = -\frac{1}{2}TB^2\int_{\mathbf{q}} q_\perp^4 G_u(\mathbf{q})^2 \,, \tag{29a}$$

$$\approx -\frac{1}{2}TB^2\int_{-\infty}^{\infty}\frac{dq_z}{2\pi}\int_{L_\perp^{-1}}\frac{d^{d-1}q_\perp}{(2\pi)^{d-1}}\frac{q_\perp^4}{(Kq_\perp^4+Bq_z^2)^2} \,,$$

$$\approx -\frac{1}{8}\frac{C_{d-1}T}{3-d}\left(\frac{B}{K^3}\right)^{1/2} L_\perp^{3-d}B \,. \tag{29b}$$

In above, we used smectic correlator, $G_u(\mathbf{q})$, focused on $d \leq 3$ (which allowed us to drop the uv-cutoff (Λ) dependent part that vanishes for $\Lambda \to \infty$), and cutoff the divergent contribution of the long wavelength modes via the infra-red cutoff $q_\perp > 1/L_\perp$ by considering a system of a finite extent $L_\perp$. Clearly the anharmonicity become important when the fluctuation corrections to the elastic constants (e.g., δB above) become comparable to its bare microscopic value. As we will see in more detail below, the divergence of this correction as $L_\perp \to \infty$ signals the breakdown of the conventional harmonic elastic theory on length scales longer than a crossover scale $\xi_\perp^{NL}$

$$\xi_\perp^{NL} \approx \begin{cases} \frac{1}{T}\left(\frac{K^3}{B}\right)^{1/2}, & d=2, \\ a e^{\frac{c}{T}\left(\frac{K^3}{B}\right)^{1/2}}, & d=3, \end{cases} \tag{30}$$

which we define here as the value of $L_\perp$ at which $|\delta B(\xi_\perp^{NL})| = B$. Within the approximation of the smectic screening length $\lambda = a$, these nonlinear crossover lengths reduce to the phonon disordering lengths (20),(18b), defined by a Lindemann-like criterion. Clearly, on scales longer than $\xi_{\perp,z}^{NL}$ the perturbative contributions of nonlinearities diverge and therefore cannot be neglected. Their contribution are thus expected to qualitatively modify the harmonic predictions of the previous subsection.

Similar analysis gives a *positive* fluctuation correction to the bend modulus K, stiffening the undulation mode.

Problem 5:

In the process of computing a correction to the bend modulus K, you will discover that indeed a lower-order term proportional to $(\boldsymbol{\nabla} u)^2$ is generated, seemingly leading to violation of underlying rotational invariance discussed above. This seeming paradox is resolved by noting that in addition a term $\partial_z u$ is also generated with a coefficient that is exactly -2 that of the $(\boldsymbol{\nabla} u)^2$ piece. Thus the generated fluctuation correction neatly assembles into a term linear in the rotationally invariant nonlinear strain tensor u_{qq}, (14).

Verify the above claim by explicit perturbative calculation and demonstrate (following the derivation of smectic elasticity) that such linear term simply corresponds to a fluctuation-generated shift in the smectic wave-vector q away from its lowest-order value q_0.

2. Renormalization group analysis in $d = 3 - \epsilon$ dimensions

To describe the physics beyond the crossover scales, $\xi_{\perp,z}^{NL}$ – i.e., to make sense of the infra-red divergent perturbation theory found in Eq.29b – requires a renormalization group analysis. This was first performed in the context of conventional 3d liquid crystals and Lifshitz points in a seminal work by Grinstein and Pelcovits (GP)[26]. For completeness, we complement GP's treatment with Wilson's momentum-shell renormalization group (RG) analysis, extending it to an arbitrary dimension d, so as to connect to the behavior in 2d, that has an exact solution[28].

To this end we integrate (perturbatively in $\mathcal{H}_{\rm nonlinear}$) short-scale Goldstone modes in an infinitesimal cylindrical shell of wavevectors, $\Lambda e^{-\delta\ell} < q_\perp < \Lambda$ and $-\infty < q_z < \infty$ ($\delta\ell \ll 1$ is infinitesimal). The leading perturbative momentum-shell coarse-graining contributions come from terms found in direct perturbation theory above, but with the system size divergences controlled by the infinitesimal momentum shell. The thermodynamic averages can then be equivalently carried out with an effective coarse-grained Hamiltonian of the same form (15), but with all the couplings infinitesimally corrected by the momentum shell. For smectic moduli B and K this gives

$$\delta B \approx -\frac{1}{8} g B \delta\ell, \quad \delta K \approx \frac{1}{16} g K \delta\ell, \tag{31}$$

where dimensionless coupling is given by

$$g \;=\; C_{d-1}\Lambda_\perp^{3-d}T\left(\frac{B}{K^3}\right)^{1/2} \approx \frac{T}{2\pi}\left(\frac{B}{K^3}\right)^{1/2} \;, \tag{32}$$

and in the second form we approximated g by its value in 3d. Eqs.(31) show that B is softened and K is stiffened by the nonlinearities in the presence of thermal fluctuations, making the system effectively more isotropic, as one may expect on general physical grounds.

For convenience we then rescale the lengths and the remaining long wavelength part of the fields $u^<(\mathbf{r})$ according to $r_\perp = r'_\perp e^{\delta\ell}$, $z = z'e^{\omega\delta\ell}$ and $u^<(\mathbf{r}) = e^{\phi\delta\ell}u'(\mathbf{r'})$, so as to restore the ultraviolet cutoff $\Lambda_\perp e^{-\delta\ell}$ back up to $\Lambda_\perp$. The underlying rotational invariance ensures that the nonlinear fluctuation corrections preserve the rotationally invariant strain operator $\left(\partial_z u - \frac{1}{2}(\nabla_\perp u)^2\right)$, renormalizing it as a whole. It is therefore convenient (but not necessary) to choose the dimensional rescaling that also preserves this form. It is easy to see that this choice leads to

$$\phi = 2 - \omega \;. \tag{33}$$

The leading (one-loop) changes to the effective coarse-grained and rescaled free energy functional can then be summarized by differential RG flows

$$\frac{dB(\ell)}{d\ell} \;=\; (d+3-3\omega - \frac{1}{8}g(\ell))B(\ell) \;, \tag{34a}$$

$$\frac{dK(\ell)}{d\ell} \;=\; (d-1-\omega + \frac{1}{16}g(\ell))K(\ell) \;. \tag{34b}$$

From these we readily obtain the flow of the dimensionless coupling $g(\ell)$

$$\frac{dg(\ell)}{d\ell} \;=\; (3-d)g - \frac{5}{32}g^2 \;, \tag{35}$$

whose flow for $d < 3$ away from the $g = 0$ Gaussian fixed point encodes the long-scale divergences found in the direct perturbation theory above. As summarized in the bottom of Fig.3 for $d < 3$ the flow terminates at a nonzero fixed-point coupling $g_* = \frac{32}{5}\epsilon$ (with $\epsilon \equiv 3 - d$), that determines the nontrivial long-scale behavior of the system (see below). As with treatments of critical points[1], but here extending over the whole smectic phase, the RG procedure is quantitatively justified by the proximity to $d = 3$, i.e., smallness of ϵ.

We can now use a standard matching calculation to determine the long-scale asymptotic form of the correlation functions on scales beyond $\xi^{NL}_{\perp,z}$. Namely, applying above coarse-graining RG analysis to a computation of correlation functions allows us to relate a correlation function at long length scales of interest to us (that, because of infrared divergences

is impossible to compute via a direct perturbation theory) to that at short scales, evaluated with coarse-grained couplings, $B(\ell)$, $K(\ell)$,.... In contrast to the former, the latter is readily computed via a perturbation theory, that, because of shortness of the length scale is convergent. The result of this matching calculation to lowest order gives correlation functions from an effective Gaussian theory

$$G_u(\mathbf{k}) \approx \frac{T}{B(\mathbf{k})k_z^2 + K(\mathbf{k})k_\perp^4}, \tag{36}$$

with moduli $B(\mathbf{k})$ and $K(\mathbf{k})$ that are singularly wavevector-dependent, determined by the solutions $B(\ell)$ and $K(\ell)$ of the RG flow equations (34a) and (34b) with initial conditions the microscopic values B and K.

2d analysis: In $d = 2$, at long scales $g(\ell)$ flows to a nontrivial infrared stable fixed point $g_* = 32/5$, and the matching analysis predicts correlation functions characterized by anisotropic wavevector-dependent moduli

$$\begin{aligned} K(\mathbf{k}) &= K\left(k_\perp \xi_\perp^{NL}\right)^{-\eta_K} f_K(k_z\xi_z^{NL}/(k_\perp\xi_\perp^{NL})^\zeta) \,, & \text{(37a)}\\ &\sim k_\perp^{-\eta_K}, \\ B(\mathbf{k}) &= B\left(k_\perp \xi_\perp^{NL}\right)^{\eta_B} f_B(k_z\xi_z^{NL}/(k_\perp\xi_\perp^{NL})^\zeta) \,, & \text{(37b)}\\ &\sim k_\perp^{\eta_B}. \end{aligned}$$

Thus, on scales longer than $\xi_{\perp,z}^{NL}$ these qualitatively modify the real-space correlation function asymptotics of the harmonic analysis in the previous subsection. In Eqs.(37) the universal anomalous exponents are given by

$$\begin{aligned} \eta_B &= \frac{1}{8}g_* = \frac{4}{5}\,\epsilon \,, \\ &\approx \frac{4}{5} \,, \quad \text{for } d = 2 \,, & \text{(38a)}\\ \eta_K &= \frac{1}{16}g_* = \frac{2}{5}\,\epsilon \,, \\ &\approx \frac{2}{5} \,, \quad \text{for } d = 2 \,, & \text{(38b)} \end{aligned}$$

determining the $z - \mathbf{x}_\perp$ anisotropy exponent via (36) to be

$$\begin{aligned} \zeta &\equiv 2 - (\eta_B + \eta_K)/2 \,, & \text{(39a)}\\ &= \frac{7}{5}, \text{ for } d = 2 \,, & \text{(39b)} \end{aligned}$$

as expected reduced by thermal fluctuations down from its harmonic value of 2. The $\mathbf{k}_\perp - k_z$ dependence of $B(\mathbf{k}), K(\mathbf{k})$ is determined by universal scaling functions, $f_B(x), f_K(x)$ that we will not compute here. The underlying rotational invariance gives an *exact* relation between the two anomalous $\eta_{B,K}$ exponents

$$3 - d = \frac{\eta_B}{2} + \frac{3}{2}\eta_K \ , \tag{40a}$$

$$1 = \frac{\eta_B}{2} + \frac{3}{2}\eta_K \ , \quad \text{for } d = 2, \tag{40b}$$

which is obviously satisfied by the anomalous exponents, Eqs.(38b),(38a), computed here to first order in $\epsilon = 3 - d$.

Thus, as advertised, we find that a finite temperature, a 2d smectic state is highly nontrivial and qualitatively distinct from its mean-field perfectly periodic form. In addition to a vanishing density modulation and associated fluctuation-restored translational symmetry, it is characterized by a universal nonlocal length-scale dependent moduli, Eq. (37). Consequently its Goldstone mode theory and the associated correlations are not describable by a local field theory, that is an analytic expansion in local field operators. Instead, in 2d, on length scales beyond $\xi^{NL}_{\perp,z}$ thermal fluctuations and correlations of this smectic *critical* phase are controlled by a nontrivial fixed point, characterized by *universal* anomalous exponents $\eta_{K,B}$ and scaling functions $f_{B,K}(x)$ defined above.

Above we obtained this nontrivial structure from an RG analysis and estimated these exponents within a controlled but approximate ϵ-expansion. Remarkably, in 2d an exact solution of this problem was discovered by Golubovic and Wang[28]. It predicts an anomalous phenomenology in a qualitatively agreement with the RG predictions above, and gives exact exponents

$$\eta_B^{2d} = 1/2, \quad \eta_K^{2d} = 1/2, \quad \zeta^{2d} = 3/2. \tag{41}$$

3d analysis: In $d = 3$, the nonlinear coupling $g(\ell)$ is marginally irrelevant, flowing to 0 at long scales. Despite this, the marginal flow to the Gaussian fixed point is sufficiently slow (logarithmic in lengths) that (as usual at a marginal dimension[1]) its power-law in ℓ dependence leads to a universal, asymptotically *exact* logarithmic wavevector dependence[26]

$$K(\mathbf{k}_\perp, k_z = 0) \sim K|1 + \frac{5g}{64\pi}\ln(1/k_\perp a)|^{2/5} \ , \tag{42a}$$

$$B(\mathbf{k}_\perp = 0, k_z) \sim B|1 + \frac{5g}{128\pi}\ln(\lambda/k_z a^2)|^{-4/5}. \tag{42b}$$

This translates into the smectic order parameter correlations given by

$$\langle \rho_q^*(\mathbf{x})\rho_q(0)\rangle \;\sim\; e^{-c_1(\ln z)^{6/5}}\cos(q_0 z), \tag{43}$$

(c_1 a nonuniversal constant) as discovered by Grinstein and Pelcovits[26]. Although these 3d anomalous effects are less dramatic and likely to be difficult to observe in practice, theoretically they are quite significant as they represent a qualitative breakdown of the mean-field and harmonic descriptions, that respectively ignore interactions and thermal fluctuations.

We conclude this section by noting that all of the above analysis is predicated the validity of the purely elastic model, Eq. (15), that neglects topological defects, such as dislocations. If these unbind (as they undoubtedly do in 2d at any nonzero temperature[25]), then our above prediction only hold on scales shorter than the separation ξ_{disl} between these defects.

IV. CHOLESTERIC AND HELICAL STATES

A. Background

Another beautiful example of a system that realizes a critical phase is a cholesteric phase of chiral liquid crystals, illustrated in Fig.1(b), actually the first liquid crystal phase discovered in cholesterol benzoate by Reinitzer in 1886. This orientationally periodic state is ubiquitous in nature and, as is obvious from its structure is equivalent to the helical state that appears in non-centrosymmetric magnets like MnSi, FeSi and many others[29].

In this section we will derive the low-energy, long wavelength (longer than its period, $a = 2\pi/q_0$) elasticity of a cholesteric. The latter is expected from symmetry and explicit *harmonic* derivation by Lubensky, et al.[2, 30] to be identical to that of a smectic, i.e., that of a one-dimensional crystalline order, that spontaneously breaks underlying translational and rotational symmetry, with homogeneous and isotropic layers transverse to the ordering wavevector, $\mathbf{q}_0$. In fact as we will see there are subtle but important distinctions, that particularly affect the nature of phase transitions in/out of this state.

In a chiral nematic liquid crystal, an additional chiral q_0 term is allowed, giving:

$$\mathcal{H}_F^* \;=\; \frac{1}{2}K_s(\nabla\cdot\hat{n})^2 + \frac{1}{2}K_b(\hat{n}\times\nabla\times\hat{n})^2 + \frac{1}{2}K_t(\hat{n}\cdot\nabla\times\hat{n}+q_0)^2 + \mathcal{H}_{SS}, \tag{44}$$

that tends to twist the nematic structure into a helix with a pitch $2\pi/q_0$ along a spontaneously chosen axis.

Lack of inversion symmetry also allows the so-called saddle-splay boundary term

$$\mathcal{H}_{SS} \;=\; \frac{1}{2}K_{24}\nabla\cdot\big[(\hat{n}\cdot\nabla)\hat{n}-\hat{n}\nabla\cdot\hat{n}\big], \tag{45}$$

that is crucial for understanding the stability of Blue phases as it drives a nonplanar, non-collinear expression of chirality. To see this in more detail we note that for a planar state characterized by a phase $\phi(\mathbf{r})$ and fixed reference frame,

$$\hat{n}(\mathbf{r}) \;=\; [\cos\phi, \sin\phi, 0], \tag{46}$$

the saddle-splay density is given by:

$$\mathcal{H}_{SS} \;=\; \frac{1}{2}K_{24}\boldsymbol{\nabla}\cdot\big[-\partial_y\phi, \partial_x\phi\big] = -\frac{1}{2}K_{24}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi. \tag{47}$$

which is nonzero only for a singular $\phi(\mathbf{r})$, e.g., with screw dislocations of the cholesteric state. Otherwise it vanishes for an ordered cholesteric state characterized by a single-valued phase $\phi(\mathbf{r}) = \mathbf{q}\cdot\mathbf{r} + \chi(\mathbf{r})$.

To simply the analysis we focus on Frank elasticity with independent space and "spin" rotational invariance that appears in the isotropic limit of $K_s = K_b = K_t = K_{24}$. The spin-space coupling still enters through the chiral term. The resulting chiral Frank energy density is then given by

$$\mathcal{H}_F^* \;=\; \frac{1}{2}K\left[(\partial_i\hat{n}_j)^2 + 2q_0\hat{n}\cdot\nabla\times\hat{n} + q_0^2\right], \tag{48}$$

$$\;=\; \frac{1}{2}K\left(\partial_i\hat{n}_j + q_0\epsilon_{ijk}\hat{n}_k\right)^2 - \frac{1}{2}Kq_0^2, \tag{49}$$

where after integration by parts we utilized the identity

$$(\partial_i\hat{n}_j)^2 \;=\; (\nabla\cdot\hat{n})^2 + (\nabla\times\hat{n})^2 + \nabla\cdot\big[(\hat{n}\cdot\nabla)\hat{n} - \hat{n}\nabla\cdot\hat{n}\big] \tag{50}$$

In the absence of K_{24}, $\mathcal{H}_F^*$ is a sum of squares and is therefore clearly minimized by a twist-only cholesteric state

$$\hat{n}(\mathbf{r}) \;=\; \hat{\mathbf{e}}_{10}\cos(\mathbf{q}\cdot\mathbf{r}) + \hat{\mathbf{e}}_{20}\sin(\mathbf{q}\cdot\mathbf{r}), \tag{51}$$

($\hat{\mathbf{e}}_{10}, \hat{\mathbf{e}}_{20}, \hat{\mathbf{e}}_{30} \equiv \hat{\mathbf{e}}_{10}\times\hat{\mathbf{e}}_{20}$ form a constant orthonormal triad) with a constant twist

$$\hat{n}\cdot\nabla\times\hat{n} = -q_0. \tag{52}$$

We therefore use the above form to derive the Goldstone-mode elasticity of the cholesteric state.

Problem 6:

Verify that in the cholesteric state:

$$\hat{n}\cdot\nabla\times\hat{n} = -\mathbf{q}\cdot\hat{\mathbf{e}}_3, \tag{53a}$$

$$\partial_i\hat{n}_j = q_i[-\hat{\mathbf{e}}_{1j}\sin(\mathbf{q}\cdot\mathbf{r})+\hat{\mathbf{e}}_{2j}\cos(\mathbf{q}\cdot\mathbf{r})], \tag{53b}$$

$$\epsilon_{ijk}\hat{n}_k = (\hat{\mathbf{e}}_{2i}\hat{\mathbf{e}}_{3j}-\hat{\mathbf{e}}_{3i}\hat{\mathbf{e}}_{2j})\cos(\mathbf{q}\cdot\mathbf{r})+(\hat{\mathbf{e}}_{3i}\hat{\mathbf{e}}_{1j}-\hat{\mathbf{e}}_{1i}\hat{\mathbf{e}}_{3j})\sin(\mathbf{q}\cdot\mathbf{r}), \tag{53c}$$

$$(\partial_i\hat{n}_j)^2 = q^2, \tag{53d}$$

$$(\epsilon_{ijk}\hat{n}_k)^2 = 2. \tag{53e}$$

We note, that because of the identity Eq. (53e), unsymmetrized strain $\partial_i\hat{n}_j$ cannot saturate the complete square as it cannot equal to the fully antisymmetric $q_0\epsilon_{ijk}\hat{n}_k$. Thus energy density in (49) is frustrated in the helical state, ultimately due to the saddle-splay K_{24} term.

B. Low-energy Goldstone mode theory of a cholesteric/helical state

We now derive the theory of low energy Goldstone modes about the cholesteric/helical ground state. A general state close to the latter can be parameterized according to

$$\hat{n}(\mathbf{r}) = \hat{\mathbf{e}}_1(\mathbf{r})\cos(\mathbf{q}\cdot\mathbf{r}+\chi(\mathbf{r}))+\hat{\mathbf{e}}_2(\mathbf{r})\sin(\mathbf{q}\cdot\mathbf{r}+\chi(\mathbf{r})), \tag{54}$$

where fluctuations are captured through a spatially dependent orthonormal triad $\hat{\mathbf{e}}_1(\mathbf{r}),\hat{\mathbf{e}}_2(\mathbf{r}),\hat{\mathbf{e}}_3(\mathbf{r})$ and the helical phase

$$\chi(\mathbf{r}) = -q_0u(\mathbf{r}) \tag{55}$$

that corresponds to the phonon field $u(\mathbf{r})$ of the chiral layers, The helical state breaks a group of three dimensional translations and rotations $G = T\times O(3)$ of the isotropic fluid down to $H = T_x\times T_y\times U(1)$ (latter $U(1)=\text{diagonal}[T_z,O_z(2)]$). Thus, since $\dim[G/H=O(3)]=3$ *three independent* Goldstone modes $\chi(\mathbf{r})$ (one), and $\hat{\mathbf{e}}_3(\mathbf{r})$ (two) are expected; the azimuthal angle $\phi(\mathbf{r})$ defining the orientation of the $\hat{\mathbf{e}}_{1,2}$ around $\hat{\mathbf{e}}_3$ is not independent of $\chi(\mathbf{r})$ as it can be absorbed into it. The low-energy coset space is isomorphic to $S^1\times S^2$, a ball (radius π) of the group space of SO(3).

Substituting the form for $\hat{n}(\mathbf{r})$ from Eq. (54) into the Frank energy of the chiral nematic, Eq. (49), and using

$$\begin{aligned}\partial_i \hat{n}_j &= (q_i + \partial_i\chi)[-\hat{\mathbf{e}}_{1j}\sin(\mathbf{q}\cdot\mathbf{r}+\chi) + \hat{\mathbf{e}}_{2j}\cos(\mathbf{q}\cdot\mathbf{r}+\chi)] \\ &\quad +\partial_i\hat{\mathbf{e}}_{1j}\cos(\mathbf{q}\cdot\mathbf{r}+\chi) + \partial_i\hat{\mathbf{e}}_{2j}\sin(\mathbf{q}\cdot\mathbf{r}+\chi),\end{aligned} \tag{56}$$

together with

$$\partial_i\hat{\mathbf{e}}_{1j} = a_i\hat{\mathbf{e}}_{2j} + c_{1i}\hat{\mathbf{e}}_{3j}, \tag{57a}$$

$$\partial_i\hat{\mathbf{e}}_{2j} = -a_i\hat{\mathbf{e}}_{2j} + c_{2i}\hat{\mathbf{e}}_{3j}, \tag{57b}$$

$$a_i = \hat{\mathbf{e}}_2\cdot\partial_i\hat{\mathbf{e}}_1, \quad c_{1i} = -\hat{\mathbf{e}}_1\cdot\partial_i\hat{\mathbf{e}}_3, \quad c_{2i} = -\hat{\mathbf{e}}_2\cdot\partial_i\hat{\mathbf{e}}_3, \tag{57c}$$

where we introduced spin-connections $\mathbf{a}, \mathbf{c_1}, \mathbf{c_2}$, we find (taking $K = 1$ for simplicity)

$$\begin{aligned}\mathcal{H}_F^* &= \frac{1}{2}\Big[(a_i\hat{\mathbf{e}}_{2j} + c_i\hat{\mathbf{e}}_{3j})\cos(\mathbf{q}\cdot\mathbf{r}+\chi) + (-a_i\hat{\mathbf{e}}_{1j} + c_{2i}\hat{\mathbf{e}}_{3j})\sin(\mathbf{q}\cdot\mathbf{r}+\chi) \\ &\quad +(q_i+\partial_i\chi)\big[-\hat{\mathbf{e}}_{1j}\sin(\mathbf{q}\cdot\mathbf{r}+\chi) + \hat{\mathbf{e}}_{2j}\cos(\mathbf{q}\cdot\mathbf{r}+\chi)\big] \\ &\quad +q_0(\hat{\mathbf{e}}_{2i}\hat{\mathbf{e}}_{3j} - \hat{\mathbf{e}}_{3i}\hat{\mathbf{e}}_{2j})\cos(\mathbf{q}\cdot\mathbf{r}+\chi) + q_0(\hat{\mathbf{e}}_{3i}\hat{\mathbf{e}}_{1j} - \hat{\mathbf{e}}_{1i}\hat{\mathbf{e}}_{3j})\sin(\mathbf{q}\cdot\mathbf{r}+\chi)\Big]^2 - \frac{1}{2}q_0^2, \\ &= \frac{1}{2}(\nabla\chi + \mathbf{a} + \mathbf{q} - q_0\hat{\mathbf{e}}_3)^2 + \frac{1}{4}(\mathbf{c}_1 + q_0\hat{\mathbf{e}}_2)^2 + \frac{1}{4}(\mathbf{c}_2 - q_0\hat{\mathbf{e}}_1)^2 - \frac{1}{2}q_0^2.\end{aligned} \tag{58}$$

Taking $\mathbf{q} = q_0\hat{z}$, with $\hat{z}$ defining the helical axis (distinct from the normal to the helical plane, $\hat{\mathbf{e}}_3$) and noting the compatibility condition on effective flux or its equivalent Pontryagin density

$$\nabla\times\mathbf{a} = \epsilon_{ij}\hat{\mathbf{e}}_3\cdot\partial_i\hat{\mathbf{e}}_3\times\partial_j\hat{\mathbf{e}}_3 = 0, \tag{59}$$

required by well-defined cholesteric layers, i.e., in the absence of dislocations and disclinations in the layer structure, allows us to take

$$\mathbf{a} = \boldsymbol{\nabla}\phi. \tag{60}$$

Under this condition ϕ can be eliminated in favor of χ, i.e., $\chi + \phi \to \chi$ and in the laboratory coordinate system $\hat{x}, \hat{y}, \hat{z}$, the fluctuations are characterized by the local helical frame described by χ and $\hat{\mathbf{e}}_3$, with

$$\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_{3\perp} + \hat{z}\sqrt{1-\hat{\mathbf{e}}_{3\perp}^2} \approx \hat{\mathbf{e}}_{3\perp} + \hat{z}(1 - \frac{1}{2}\hat{\mathbf{e}}_{3\perp}^2). \tag{61}$$

We thus obtain

$$\mathcal{H}_F^* \;=\; \frac{1}{2}(\nabla_\perp\chi - q_0\hat{\mathbf{e}}_{3\perp})^2 + \frac{1}{2}(\partial_z\chi + \frac{1}{2}q_0\hat{\mathbf{e}}_{3\perp}^2)^2 + \frac{1}{4}(\mathbf{c}_1^2+\mathbf{c}_2^2) + \frac{q_0}{2}(\mathbf{c}_1\cdot\hat{\mathbf{e}}_2 - \mathbf{c}_2\cdot\hat{\mathbf{e}}_1). \quad (62)$$

From the minimization of the first term (or equivalently integrating out the independent $\hat{\mathbf{e}}_{3\perp}$ degree of freedom, I obtain an effective constraint

$$\nabla_\perp\chi = q_0\hat{\mathbf{e}}_{3\perp}, \quad (63)$$

that is an example of an emergent Higgs mechanism (akin to smectic liquid crystals discussed above) locking the cholesteric layers orientation with the molecular frame orientation. With this constraint (valid at low energies) the effective Hamiltonian reduces to

$$\begin{aligned}
\mathcal{H}_F^* \;&=\; \frac{1}{2}\big[\partial_z\chi + \frac{1}{2q_0}(\nabla_\perp\chi)^2\big]^2 + \frac{1}{4q_0^2}(\hat{\mathbf{e}}_{\alpha\gamma}\cdot\partial_\beta^\perp\partial_\gamma^\perp\chi)^2 + \frac{q_0}{2}(\hat{\mathbf{e}}_{1\alpha}\hat{\mathbf{e}}_{2\beta}\partial_\beta\partial_\alpha\chi - \hat{\mathbf{e}}_{2\alpha}\hat{\mathbf{e}}_{1\beta}\partial_\beta\partial_\alpha\chi), && (64)\\
&=\; \frac{1}{2}\big[\partial_z\chi + \frac{1}{2q_0}(\nabla_\perp\chi)^2\big]^2 + \frac{1}{4q_0^2}(\hat{\mathbf{e}}_{\alpha\gamma}\cdot\partial_\beta^\perp\partial_\gamma^\perp\chi)^2 + q_0\hat{z}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi), && (65)\\
&=\; \frac{1}{2}B\big[\partial_z u - \frac{1}{2}(\nabla u)^2\big]^2 + \frac{1}{2}\overline{K}(\nabla^2 u)^2 && (66)
\end{aligned}$$

where $\hat{\mathbf{e}}_{\alpha,i}\hat{\mathbf{e}}_{\alpha,j} = \delta_{ij}$ and $\hat{\mathbf{e}}_{1,i}\hat{\mathbf{e}}_{2,j} - \hat{\mathbf{e}}_{2,i}\hat{\mathbf{e}}_{i,j} = \hat{\mathbf{e}}_{3,k}\epsilon_{ijk}$ were used, and to eliminate the last term in the penultimate expression above we used the condition of the single-valuedness of the phase field χ, i.e., well defined cholesteric layers with no dislocations. Above we defined effective moduli

$$B \;=\; Kq_0^2, \quad \overline{K} = K. \quad (67)$$

Thus, as advertised and expected on symmetry grounds we indeed demonstrated that cholesteric Goldstone modes elasticity, even at the nonlinear level is identical to that of a conventional smectic. We thus immediately conclude that a cholesteric must exhibit all novel smectic properties, most importantly the universal criticality of its ordered state, worked out in the previous section.

We note, however, that although within the cholesteric phase, the two are identical, because of the presence of chirality and in particular the $q_0\hat{z}\cdot(\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi)$ term above, the cholesteric differs from a smectic in terms of the nature of its expected transition and phases that are adjacent to it. The latter disorders into a nematic state, while the former (precluded from this by chirality) can transition into Blue phases and the Twist-Grain-Boundary phases[1, 2]. In fact the situation is quite analogous to a superconductor with and without

an external magnetic field, with the former and latter cases corresponding to the smectic and cholesteric, respectively. This explicitly-demonstrated result contrasts qualitatively with the original conjecture by Toner and Nelson[25]).

Problem 7:

Using the explicit form of $\mathbf{a} = \hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla}\hat{\mathbf{e}}_1$ demonstrate

$$\boldsymbol{\nabla} \times \mathbf{a} \;=\; \hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla}\hat{\mathbf{e}}_3 \times \boldsymbol{\nabla}\hat{\mathbf{e}}_3 \tag{68}$$

Hint: You may need to use the identity $(\hat{\mathbf{e}}_{1\ell}\hat{\mathbf{e}}_{2m} - \hat{\mathbf{e}}_{1m}\hat{\mathbf{e}}_{2\ell}) = 2\hat{\mathbf{e}}_n\epsilon_{\ell mn}$ (can you prove it?).

V. OTHER EXAMPLES

A. Columnar phase

A fluid composed of a high aspect ratio disk-shaped constituents in addition to the nematic phase, also prominently exhibits a "columnar phase".[2] In it disks stack into one-dimensional fluid columns, that in turn freeze into a triangular lattice (retaining fluid order along the columns). This discotic liquid crystal phase, forming a 2d crystal of 1d fluid columns is in a sense dual to the smectic state of 1d periodic array of 2d fluids.

As we will argue below, for $d < 5/2$ it is also a critical phase, though given the dimensional constraint not a very practical one from the experimental realization point of view. The state is characterized by a two-component phonon field $\mathbf{u} = (u_x, u_y)$, that are two Goldstone modes associated with translational symmetry breaking in the plane transverse to the columns. Analogously to a smectic, the remaining naively expected Goldstone modes associated with translational and rotational symmetry are gapped out by the emergent Higgs mechanism.

Although we can derive the Goldstone mode elasticity for the columnar phase by following the approach analogous to that of a smectic, here we will simply write it down based on symmetry and experience with the smectic. To this end we note that the columnar state exhibits two types of rigidities, the in-plane (transverse to the columnar axis $\hat{z}$) crystal and bending of the columns. The former is quite clearly characterized by elasticity of a 2d triangular lattice. The latter is captured by curvature filament elasticity. Together these give energy density

$$\mathcal{H}_{col} = \frac{1}{2}\kappa(\partial_z^2\mathbf{u})^2 + \frac{\lambda}{2}\,u_{\alpha\alpha}^2 + \mu\,u_{\alpha\beta}^2 \tag{69}$$

where κ is the curvature modulus and μ and λ are the Lamé elastic moduli[1], and (with the summation convention over repeated indices) the strain tensor is given by

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \partial_\gamma u_\alpha \partial_\gamma u_\beta\right) \approx \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \partial_z u_\alpha \partial_z u_\beta\right), \tag{70}$$

where phonons $\mathbf{u}(x, y, z) = (u_x, u_y)$ are two-dimensional but the space is three-dimensional, and in the second approximate form we only kept the most important nonlinearity.

As in the discussion of the smectic elasticity, the columnar elasticity (69) is constrained by symmetry. The linear z derivative in the harmonic terms is forbidden by the rotational invariance about the x axes (broken spontaneously), at infinitesimal level corresponding to $\mathbf{u} \to \mathbf{u} + \theta z \hat{\mathbf{y}}$. This "softness" with respect to $\partial_z \mathbf{u}$ modes then requires us to keep corresponding nonlinearities in the in-plane strain tensor, $u_{\alpha\beta}$ above, the form of which are dictated by the in-plane rotational invariance.

As in a smectic, here too the analysis of quadratic phonon fluctuations leads to power-law divergence for $d < 5/2$, requiring inclusion of elastic nonlinearities. Account of these via an RG analysis leads to a nontrivial infrared stable fixed point that characterizes the universal properties of the resulting columnar *critical phase*.

B. Polymerized membrane

A fluctuating, *tensionless* and therefore curvature-controlled membrane is another fascinating system, stimulated in part by ubiquitous biophysical realizations (e.g., lyposomes, cellular membrane, etc)[3]. With our focus on critical phases, of particular interest are polymerized membranes (realized for example as red-blood cell cytoskeleton and graphene sheets[31]), that exhibit a finite in-plane shear rigidity.

As was first argued by Nelson and Peliti[32], in addition to the rotationally invariant "crumpled" phase of linear polymers and liquid membranes, polymerized membranes exhibit a finite temperature "flat" phase. As we will examine below, this too is a *critical* phase characterized by universal power-law correlations (e.g., membrane roughness) and anomalous elasticity, with the state's very existence in two-dimensional membranes a beautiful illustration of a phenomena of order-from-disorder.

A detailed derivation of the flat-phase Goldstone-modes elasticity is available[3, 32] starting from the Landau theory of the crumpled phase in terms of the tangent vectors $\boldsymbol{\nabla}\vec{r}(\mathbf{x})$,

where $\vec{r}(\mathbf{x})$ is the embedding of the D-dimensional membrane (parameterized by $\mathbf{x}$) in the d-dimensional space, $\vec{r}$ ($D = 2$ and $d = 3$ is the physical case). The flat phase parameterized by

$$\vec{r}(\mathbf{x}) = (\zeta\mathbf{x} + \mathbf{u}(\mathbf{x}), \vec{h}(\mathbf{x})),$$

spontaneously breaks the $O(d)$ symmetry of the crumpled phase down to $O(D)$, where ζ scale factor is the effective order parameter of the flat phase, $\mathbf{u}(\mathbf{x})$ is the D-component in-plane phonon, and $\vec{h}(\mathbf{x})$ is the "height function" describing the membrane's transverse undulation into the embedding space; we have implicitly generalized to arbitrary co-dimension $d_c = d - D$.

The resulting energy functional is a sum of a bending and an in-plane elastic contributions:

$$H_{flat}[\vec{h}, \mathbf{u}] = \int d^D x \left[\frac{\kappa}{2} (\nabla^2 \vec{h})^2 + \mu u_{\alpha\beta}^2 + \frac{\lambda}{2} u_{\alpha\alpha}^2 \right] , \tag{71}$$

where the strain tensor is

$$u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \vec{r} \cdot \partial_\beta \vec{r} - \delta_{\alpha\beta}) \approx \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha \vec{h} \cdot \partial_\beta \vec{h}), \tag{72}$$

where we defined the strain tensor in terms of deviation of the embedding-induced metric $g_{\alpha\beta}$ from the flat metric, and in the second form neglected elastic nonlinearities that are subdominant at long scales. We can also integrate out the noncritical in-plane phonon field $\mathbf{u}$, that appears only harmonically, to obtain a convenient equivalent form, purely in terms of $\vec{h}$

$$H_{flat}[\vec{h}] = \int d^D x [\frac{\kappa}{2} (\nabla^2 \vec{h})^2 + \frac{1}{4d_c} (\partial_\alpha \vec{h} \cdot \partial_\beta \vec{h}) R_{\alpha\beta,\gamma\delta} (\partial_\gamma \vec{h} \cdot \partial_\delta \vec{h})] \tag{73}$$

where for convenience, we rescaled Lamé coefficients so that the quartic coupling is of order $1/d_c$. The four-point coupling fourth-rank tensor is given by

$$R_{\alpha\beta,\gamma\delta} = \frac{K - 2\mu}{2(D-1)} P^T_{\alpha\beta} P^T_{\gamma\delta} + \frac{\mu}{2} \left(P^T_{\alpha\gamma} P^T_{\beta\delta} + P^T_{\alpha\delta} P^T_{\beta\gamma} \right) , \tag{74}$$

where $P^T_{\alpha\beta} = \delta_{\alpha\beta} - q_\alpha q_\beta / q^2$ is a transverse (to $\mathbf{q}$) projection operator. The convenience of this decomposition is that $K = 2\mu(2\mu + D\lambda)/(2\mu + \lambda)$ and μ moduli renormalize independently and multiplicatively.[35]

We note that this last h-only elastic form indeed reflects our general discussion about critical phases in Sec.II, namely that they are described by an energy functional (of ϕ^4 form with $\vec{\phi}_\alpha \equiv \partial_\alpha \vec{h}$) enforced to be critical (i.e., missing the ϕ^2 term) by the underlying rotational symmetry, as illustrated in Fig.3.

Following by now standard analysis[32–35] we learn that for $D < 4$ (and arbitrary d) the membrane exhibits height undulations that diverge with its extent L. As for smectics and other critical phases, we can make sense of the associated divergent perturbation theory in elastic nonlinearities by performing an RG analysis controlled by $\epsilon = 4 - D$ or equivalently $1/d_c$ (large embedding dimension).

The resulting infrared stable fixed point controls the properties of the highly nontrivial "flat" phase, that in fact is critical and power-law rough. It is characterized by root-mean-squared height undulations

$$h_{rms} \;=\; \sqrt{\langle h^2(\mathbf{x})\rangle} \sim L^{\zeta}, \tag{75}$$

with the roughness exponent $\zeta < 1$ (i.e., despite divergent undulations the symmetry remains broken as $L^{\zeta} \ll L \to \infty$), whose best estimate for $D = 2, d = 3$ [35] is

$$\zeta = 0.59,$$

and has also been computed within ϵ and $1/d_c$ expansions[33–35].

The "flat" critical phase is also characterized by universal anomalous elasticity with length scale-dependent elastic moduli

$$\kappa(q) \sim q^{-\eta}, \quad \mu(q) \sim \lambda(q) \sim q^{\eta_u}, \tag{76}$$

where for physical membranes $\eta \approx 0.82$ and the underlying rotational invariance imposes exact relations

$$\eta_u = 4 - D - 2\eta_\kappa, \quad \zeta = (4 - D - \eta)/2. \tag{77}$$

These indicate that at long scales thermal fluctuations stiffen the bending of the membrane and soften its in-plane moduli. We note that as a result $h_{rms}(L)/L \sim L^{-\eta/2} \to 0$ indicates that the "flat" phase is stabilized (for finite positive $\eta > 0$ generated by thermal fluctuations) by the very fluctuations that attempt to destabilize it, a phenomenon known as order-from-disorder.

The theory also predicts a *universal* and *negative* Poisson ratio

$$\lim_{q\to 0} \;\; \sigma \equiv \frac{\lambda(q)}{2\mu(q) + (D-1)\lambda(q)} = -\frac{1}{3}\;, \text{ for } D = 2, \tag{78}$$

that measures the ratio of compression of the membranes along an axis transverse to its strained direction. Its negative value indicates that the membrane actually expands transversely as it is being stretched. This value of $-1/3$ compares extremely well with the most recent and largest simulations.[36] This amazing fluctuation-driven anomalous elasticity phenomenology can be understood qualitatively by playing around with a roughened piece of paper.

C. Nematic elastomer

A final example that we will discuss briefly is the nematic elastomer, namely rubber composed of mesogenic groups (see Fig.6), that exhibit a spontaneous transition to a nematic phase, thereby driving an accompanying spontaneous uniaxial distortion of the elastic matrix, illustrated in Fig.7.[12]

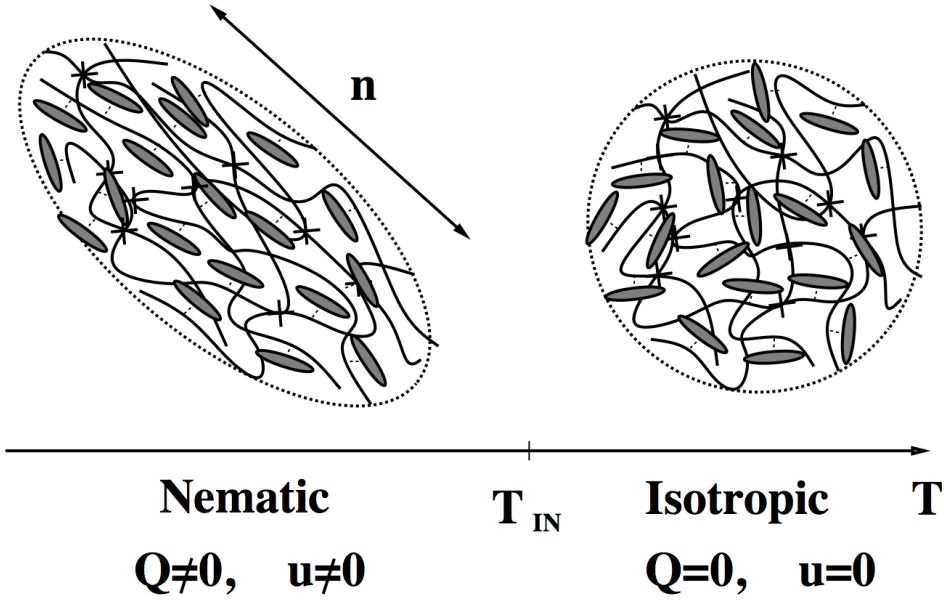

FIG. 6: Spontaneous uniaxial distortion of nematic elastomer driven by isotropic-nematic transition.

Even in the absence of fluctuations, bulk nematic elastomers were predicted[37] and later observed to display an array of fascinating phenomena[12, 38], the most striking of which is the vanishing of stress for a range of strain, applied transversely to the spontaneous nematic direction. This striking softness is generic, stemming from the spontaneous orientational

symmetry breaking by the nematic state, accompanied by a Goldstone mode, that leads to the observed soft distortion and strain-induced director reorientation[37, 39], illustrated in Fig.7. This unique elastic phenomenon is captured by a harmonic version of the uniaxial Hamiltonian in Eq.(79), with w_{ij} taken as a linear strain tensor. The hidden rotational symmetry guarantees a vanishing of one of the five elastic constants[37, 39], that usually characterize harmonic deformations of a three-dimensional uniaxial solid[41], here with the uniaxial axis $\hat{z}$ chosen spontaneously.

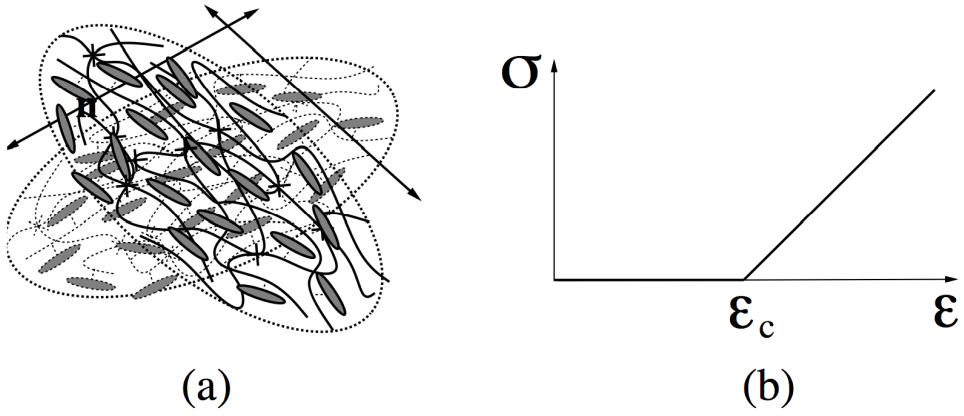

FIG. 7: (a) Simultaneous reorientation of the nematic director and of the uniaxial distortion is a low-energy nemato-elastic Goldstone mode of an ideal elastomer, that is responsible for its softness and (b) its idealized flat (vanishing stress) stress-strain curve for a range of strains $\varepsilon < \varepsilon_c$.

Given this softness of the harmonic elasticity associated with symmetry-imposed vanishing of the $\mu_{z\perp}$ modulus, thermal and heterogeneity-driven fluctuations are divergent in 3d.[13, 40]. Thus, as in other "soft" matter discussed in earlier sections, we expect a qualitative importance of elastic nonlinearities in the presence of thermal fluctuations and network heterogeneity.

The resulting minimal elastic Hamiltonian density has the following form:

$$\mathcal{H}_{elast} \;=\; \frac{1}{2}B_z\, w_{zz}^2 + \lambda_{z\perp}\, w_{zz}\, w_{\alpha\alpha} + \frac{1}{2}\,\lambda\, w_{\alpha\alpha}^2 + \mu\, w_{\alpha\beta}^2 + \frac{K}{2}\,(\nabla_\perp^2 u_z)^2, \tag{79}$$

where akin to earlier examples the components of the (rescaled) effective nonlinear strain tensor $\underline{\underline{w}}$ are given by

$$w_{zz} \;=\; \partial_z u_z + \frac{1}{2}\,(\nabla_\perp u_z)^2, \tag{80}$$

$$w_{\alpha\beta} \;=\; \frac{1}{2}\,(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \,\partial_i u_z\,\partial_j u_z)\,. \tag{81}$$

Similar to their effects in smectic, columnar liquid crystals and other soft matter discussed earlier, in bulk elastomers thermal fluctuations (and network heterogeneity) lead to anomalous elasticity, with universally length-scale dependent elastic moduli,

$$K_{eff} \sim L^{\eta}, \quad \mu_{eff} \sim L^{-\eta_u} \tag{82}$$

controlled by an infrared stable fixed point[13, 40]. The resulting critical state is characterized by a universal non-Hookean stress-strain relation

$$\sigma_{zz} \sim (\varepsilon_{zz})^{\delta}$$

and a *negative* Poisson ratio for extension $\varepsilon_{xx} > 0$ transverse to the nematic axis

$$\varepsilon_{yy} = \frac{5}{7}\varepsilon_{xx}, \quad \varepsilon_{zz} = -\frac{12}{7}\varepsilon_{xx}. \tag{83}$$

While considerable progress has been made in understanding these fascinating materials, many open questions, particularly associated with network heterogeneity remain open.

VI. SUMMARY AND CONCLUSIONS

In these lectures we have discussed a novel class of soft matter, where "softness" is *qualitative* (rather than just quantitative, with moduli comparable to thermal energy). Namely, as a result of the underlying symmetry broken in the ordered state, certain elastic moduli of the associated Goldstone modes vanish identically. As a result, such system exhibit fluctuations that are divergent in the thermodynamic limit, and require treatment of Goldstone modes' nonlinearities in the ordered phase (not just near a critical point). Treating these within renormalization group analysis, leads to an infrared stable fixed point that controls the resulting highly nontrivial and universal ordered state. The latter exhibits properties akin to that of a critical point, but extending over the entire phase, which we therefore naturally refer to as a "critical phase".

As prominent examples of such systems, we have discussed smectics, cholesterics, columnar phases, polymerized membranes and nematic elastomers. While significant progress has been made in characterizing these systems, discovery of new systems (e.g., as quantum examples), understanding of effects of random heterogeneity and dynamics remain challenging open problems.

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