Dynamics of disordered systems

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Plan of Lectures

1. Introduction
2. Coarsening processes
3. Formalism
4. Dynamics of disordered spin models
References

--- Phase ordering kinetics & critical dynamics


Plan of the lecture

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. Dynamic scaling
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies
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The talk focuses on a very well-known example

Dynamics following a change of a control parameter

- If there is an equilibrium phase transition, the equilibrium phases are known on both sides of the transition.
  i.e. the asymptotic state is known.

- For a purely dynamic problem, the absorbing states are known.

- The dynamic mechanism towards equilibrium (or the absorbing states) is understood the systems try to order locally in one of the few competing states.
Interests and goals

Practical interest, *e.g.*

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Fundamental interest, *e.g.*

- A theoretical problem beyond perturbation theory.
- Are there growth phenomena in problems with yet unknown dynamic mechanisms? *e.g. glasses*
- Generic features of macroscopic systems out of equilibrium?
Our interest is to describe the dynamics of a classical (or quantum) system coupled to a classical (or quantum) environment.

The Hamiltonian of the ensemble is

$$H = H_{syst} + H_{env} + H_{int}$$

The dynamics of all variables are given by Newton (or Heisenberg) rules, depending on the variables being classical (or quantum).

$$E_{syst}(t) \neq ct,$$
and
$$e_0 \ll E_{syst} \ll E_{env}.$$
\section*{2d Ising model}

Snapshots after an instantaneous quench to $T$ at $t = 0$

At $T = T_c$ critical dynamics

At $T < T_c$ coarsening

A certain number of \textit{interfaces} or \textit{domain walls} in the last snapshots.
Membranes Proteins

Wadsten, Wöhri, Snijder, Katona, Gardiner, Cogdell, Neutze, Engström,

Lipidic Sponge Phase Crystallization of Membrane Proteins, J. Mol. Biol. 06
Phase separation in glasses

$t = 1\ \text{min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14
Phase separation in glasses

$\rho = 4$ min

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14
Phase separation in glasses

$\ t = 16 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14
Phase separation in glasses

$t = 64 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & Vandembroucq (ESPCI) 11-14
Plan of the lecture

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2. **Theoretical setting**
3. Critical and sub-critical quenches
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The Hamiltonian or energy function is

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours on the lattice.

$J > 0$ ferromagnetic coupling constant.

Critical temperature $T_c > 0$ for $d > 1$.

Equilibrium Paramagnetic & ferromagnetic states above & below $T_c$
Ginzburg-Landau
Continuous scalar statistical field theory

Coarse-grain the spin
\[ \phi(\vec{r}) = V_{\vec{r}}^{-1} \sum_{i \in V_{\vec{r}}} s_i. \]

The partition function is
\[ Z = \int D\phi \, e^{-\beta \mathcal{F}(\phi)} \]

with
\[ \mathcal{F}(\phi) = \int d^d r \left\{ \frac{1}{2} [\nabla \phi(\vec{r})]^2 + \frac{T-J}{2} \phi^2(\vec{r}) + \frac{g}{4} \phi^4(\vec{r}) \right\} \]

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around \( \phi \sim 0 \) and symmetry arguments).

Uniform saddle point in the \( V \to \infty \) limit:
\[ \phi_{sp}(\vec{r}) = \langle \phi(\vec{r}) \rangle = \phi_0. \]

The free-energy density is
\[ \lim_{V \to \infty} f_V(\beta, J, g) = \lim_{V \to \infty} V^{-1} \mathcal{F}(\phi_0). \]
2nd order phase-transition

Bi-valued equilibrium states related by symmetry

\[ \mathcal{F} \]

Ginzburg-Landau free-energy

Scalar order parameter

e.g. Ising magnets
2d Ising Model, dynamics

Archetypical example for classical magnetic systems

\[ H = -J \sum_{\langle ij \rangle} s_i s_j \]

\( s_i = \pm 1 \) Ising spins.

\( \langle ij \rangle \) sum over nearest-neighbours on the lattice.

\( J > 0 \) ferromagnetic coupling constant.

Critical temperature \( T_c > 0 \) for \( d > 1 \).

Monte Carlo rule \( s_i \rightarrow -s_i \) accepted with

\[
\begin{align*}
p &= 1 & \text{if} & \Delta E < 0 \\
p &= e^{-\beta \Delta E} & \text{if} & \Delta E > 0 \\
p &= 1/2 & \text{if} & \Delta E = 0
\end{align*}
\]

Non-conserved order parameter dynamics [\( \uparrow \downarrow \) towards \( \uparrow \uparrow \)] etc. allowed.

[\( m = 0 \) to \( m = 2 \)]
Evolution

Non-conserved order parameter dynamics

\[ T(t) = T_c(1 - t/\tau_a) \]

Non-conserved order parameter \( \langle \phi \rangle(t, T) \neq ct \)

e.g. single spin flips with Glauber or Monte Carlo stochastic rules.

Development of magnetization in a ferromagnet.
**$d$-dimensional magnets**

More general ferromagnetic models

\[ H = - \sum_{\langle ij \rangle} J_{ij} \vec{s}_i \cdot \vec{s}_j \]

- $J_{ij} > 0$
- Sum over nearest-neighbours on a $d$-dim. lattice.
- $s_i = \pm 1$
- Ising spins.
- $\vec{s}_i = (s^x_i, s^y_i)$
- xy two-component spins.
- $\ell^d \phi(\vec{r}) = \sum_{i \in V} \vec{s}_i$
- Coarse-grained field over the volume $V = \ell^d$
- Linear size of the system $L \gg \ell \gg a$

for $d > 1$ and $L \to \infty$.

Coupling to the bath mimicked by Monte Carlo updates
Stochastic dynamics

Open systems

- **Microscopic**: identify the ‘smallest’ relevant variables in the problem (e.g., the spins) and propose stochastic updates for them, as the Monte Carlo or Glauber rules.

- **Coarse-grained**: write down a stochastic differential equation for the field, such as the effective (Markov) Langevin equation

\[
\underbrace{m\ddot{\vec{\phi}}(\vec{r}, t)}_{\text{Inertia}} + \underbrace{\gamma_0 \dot{\vec{\phi}}(\vec{r}, t)}_{\text{Dissipation}} = \underbrace{\vec{F}(\vec{\phi})}_{\text{Deterministic}} + \underbrace{\vec{\xi}(\vec{r}, t)}_{\text{Noise}}
\]

with \( \vec{F}(\vec{\phi}) = -\delta \mathcal{F}(\vec{\phi}) / \delta \vec{\phi} \) (with the double-well \( f \))

*e.g.*, time-dependent stochastic scalar Ginzburg-Landau equation or the stochastic Gross-Pitaevskii equation.
### Models

**Discrete vs. continuous**

<table>
<thead>
<tr>
<th>Ising spin models</th>
<th>Field theories</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H = -\sum_{ij} J_{ij} s_i s_j$</td>
<td>$\mathcal{F}[\phi] = \int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 - \mu \phi^2 + \frac{g}{4} \phi^4 \right]$</td>
</tr>
<tr>
<td>NCOP $[\uparrow\downarrow \mapsto \uparrow\uparrow]$</td>
<td>$\partial_t \phi(\vec{r}, t) = \delta_{\phi(\vec{r}, t)} \mathcal{F}[\phi] + \xi(\vec{r}, t)$</td>
</tr>
<tr>
<td>COP $[\uparrow\downarrow \mapsto \downarrow\uparrow]$</td>
<td>$\partial_t \phi(\vec{r}, t) = \nabla^2 \delta_{\phi(\vec{r}, t)} \mathcal{F}[\phi] + \eta(\vec{r}, t)$</td>
</tr>
</tbody>
</table>

Overdamped limit is fine

In the COP case $\langle \eta(\vec{x}, t) \eta(\vec{y}, t') \rangle = 2k_B T \nabla^2 \delta(\vec{x} - \vec{y}) \delta(t - t')$

And generalisations for vector cases. **Quenched disorder** can be introduced by taking the $J_{ij}$ or the parameters in the field theory from a pdf.
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Equilibrium configurations

Up & down spins in a $2d$ Ising model

- $T \rightarrow \infty$
  - $\langle s_i \rangle_{eq} = 0$
  - $\phi(\vec{r}) = 0$

- $T = T_c$
  - $\langle s_i \rangle_{eq} = 0$
  - $\phi(\vec{r}) = 0$

- $T < T_c$
  - $\langle s_i \rangle_{eq^+} > 0$
  - $\phi(\vec{r}) > 0$

Coarse-grained scalar field $\phi(\vec{r}) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i$
The problem

Up & down spins in a $2d$ Ising model

Equilibrium configurations
e.g. up & down spins in a $2d$ Ising model (IM)

$\langle \phi \rangle = 0$
$\langle \phi \rangle = 0$
$\langle \phi \rangle \neq 0$

$g \rightarrow \infty$
$g = g_c$
$g < g_c$

In a canonical setting the control parameter is $g = \frac{T}{J}$.

Question: starting from equilibrium at $T_0 \rightarrow \infty$ or $T_0 = T_c$ how is equilibrium at $T = T_c$ or $T < T_c$ attained?
2d Ising model

Snapshots after an instantaneous quench at $t = 0$

At $T = T_c$ critical dynamics
At $T < T_c$ coarsening

A certain number of interfaces or domain walls in the last snapshots.
Domain growth

- At $T = T_c$ the system needs to grow structures of all sizes.

  Critical coarsening.

- At $T < T_c$: the system tries to order locally in one of the two competing equilibrium states at the new conditions.

  Sub-critical coarsening.

  The **linear size of the equilibrated patches increases in time**.

  - The relaxation time $t_r$ needed to reach $\langle \phi \rangle_{eq}(T/J)$ diverges with the size of the system, $t_r(T/J, L) \to \infty$ when $L \to \infty$ for $T \leq T_c$.

  - Dissipative dynamics $d\langle \mathcal{E} \rangle/dt < 0$. Energy density is reduced by diminishing the density of **domain walls**.
In both cases one sees the growth of ‘red and white’ patches and interfaces surrounding such geometric domains.

More precisely, spatial regions of local equilibrium (with vanishing or non-vanishing order parameter) grow in time and a growing length $\mathcal{R}(t, T/J)$ can be computed with the help of dynamic scaling.
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• Black curve: equilibrium relaxation, $r^{2-d-\eta}$.

• Coloured curves are for different times after the quench and they slowly approach the equilibrium one.

• From $C(R_c(t), t) = 1/e$ one gets $R_c(t) \simeq t^{1/z_{eq}}$

(Other prescriptions give equivalent results.)
After quenches (set $J = 1$ for notational simplicity)

At late times there is a single length-scale, the typical radius of the equilibrium structures (domains below $T_c$) $R(t, T)$, such that the structure is (in statistical sense) independent of time when lengths are scaled by $R(t, T)$, e.g.

$$C(r, t) \equiv \langle s_i(t)s_j(t) \rangle_{|\vec{x}_i - \vec{x}_j|=r} \sim f \left( \frac{r}{R(t, T)} \right),$$

$$C(t, t_w) \equiv \langle s_i(t)s_i(t_w) \rangle \sim f_c \left( \frac{R(t, T)}{R(t_w, T)} \right),$$

etc. when $L \gg r \gg \xi(T)$, $t, t_w \gg t_0$ and $C$ small enough (see below).

Suggested by experiments and numerical simulations. Proved for a few cases.

Dynamic scaling

in phase ordering kinetics

Growing length $\ell(t)$ and equilibrium reached for $\ell(t_{eq}) \sim L$

Typically $\ell(t) \sim t^{1/z}$ and $t_{eq} \sim L^z$

Excess energy w.r.t. the equilibrium one stored in the domain walls; $\Delta \mathcal{E}(t) \sim \ell^{-a}(t)$
Dynamic scaling

Quench of the $2d$IM with NCOP from $T_0 \rightarrow \infty$ to $T = 0$

$\langle \phi(t) \rangle = 0 \quad C_{eq}^c(r) \simeq e^{-r/\xi_{eq}}$

- Coloured curves are $C(r, t)$ for different times after the quench.
- The growing length is $R(t, T) \simeq t^{1/z_d}$ with $z_d = 2$
- $R(t, T)$ is the averaged linear size of the domains.

Review Bray 94
Dynamic scaling

Quench of the 2dIM with NCOP from $T_0 \to \infty$ to $T < T_c$

Scaling regime \( a \ll r \ll L, \quad r \simeq \mathcal{R}(t,T) \simeq t^{1/z_d} \)

\[
C(r,t) \simeq m_{eq}^2(T) f_c \left( \frac{r}{\mathcal{R}(t,T)} \right)
\]

Scaling looks perfect
Space-time correlation
Separation of time-scales & dynamic scaling

Critical quench

\[ C(r, t) \simeq C_{eq}(r) f_c \left( \frac{r}{R_c(t)} \right) \]

\[ C_{eq}(r) \simeq r^{2-d-\eta}, \lim_{x \to 0} f_c(x) = 1 \text{ and } \lim_{x \to \infty} f_c(x) = 0. \]

Sub-critical

\[ C(r, t) \simeq [C_{eq}(r) - m_{eq}^2] + m_{eq}^2 f \left( \frac{r}{R(t, T)} \right) \]

\[ C(0, t) = 1 \ \forall t, \lim_{r \to 0} C_{eq}(r) = 1, \lim_{r \to \infty} C_{eq}(r) \propto \langle s_i \rangle_{eq}^2 = m_{eq}^2, \]
\[ \lim_{x \to 0} f(x) = 1 \text{ (long times) and } \lim_{x \to \infty} f(x) = 0 \text{ (short distances)}. \]
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Use the growing length $\mathcal{R}(t, T)$ to identify dynamic universality classes.

They depend on the dimension of the order parameter and the dynamic mechanism of growth (intimately related to the conservation laws).
Growing length

Dynamic universality classes at the critical point

At $T_c$, dynamic RG techniques work very well.


One finds dynamic scaling, with the growing length

$$\mathcal{R}_c(t) \sim t^{1/z_c}$$

$z_c$ can be computed with methods that are very similar to critical exponents in static phase transitions.

Dynamic universality classes classified by the $z_c$ values.

The scaling functions can be estimated as well
Growing length

Dynamic universality classes below the critical point

No systematic method

Focus on the dynamic mechanisms
Scalar field w/NCOP dynamics

- Curvature driven \((T = 0)\): \(\vec{v} \equiv \frac{d\vec{n}}{dt} \propto K \hat{n}\) with \(K = \nabla \cdot \hat{n}\)

- Domain wall roughening \((T > 0)\)

- Domain wall roughening and pinning by quenched disorder

\[t=10^5, 10^4, 10^3, 10^2, 10^1, 10^0\]

e.g. elastic line in random media. Kolton et al 05
MC dynamics 2dIM

The typical length-scale \( \Leftrightarrow \) a typical area

\[ R(t, T) \sim \lambda(T) t^{1/2} \quad \Leftrightarrow \quad A(t, T) \sim \lambda^2(T) t \]

NB the exponent \( \frac{1}{2} \) is independent of \( T \) and the details of the dynamics, lattice, etc. as long as the order parameter is non-conserved & there is no disorder.

The \( T \)-dependence in \( \lambda(T) \) is due to the roughening of the domain walls.
Phase separation

Demixing transitions

Two species ● and ●, repulsive interactions between them.

Sketch

Experimental phase diagram

Binary alloy, Hansen & Anderko, 54
Scalar field w/COP

Phase separation

via spinodal process
with deep quench (unstable region)

Phase separation
via nucleation process
with shallow quench (metastable region)

Matter diffusion
Phase separation

Spinodal decomposition in binary mixtures

$A$ species ≡ spin up; $B$ species ≡ spin down

$2d$ Ising model with Kawasaki dynamics at $T$

locally conserved order parameter

$$\mathcal{R}(t, T) \sim \lambda(T) t^{1/3}$$

Huse 93

50 : 50 composition Rounder boundaries
Weak disorder

\textit{e.g., random ferromagnets}

At short time scales the dynamics is relatively fast and independent of the quenched disorder; thus

\[ \mathcal{R}(t, T) \simeq \lambda(T)t^{1/z_d} \]

At longer time scales domain-wall pinning by disorder dominates.

Assume there is a length-dependent barrier \( B(\mathcal{R}) \sim \Upsilon\mathcal{R}^\psi \) to overcome

The \textit{Arrhenius time} needed to go over such a barrier is \( t_A \sim t_0 e^{\frac{B(\mathcal{R})}{k_BT}} \)

This implies

\[ \mathcal{R}(t, T) \simeq \left( \frac{k_BT}{\Upsilon} \ln \frac{t}{t_0} \right)^{1/\psi} \]
Weak disorder

Still two ferromagnetic states related by symmetry

\[ \mathcal{R}(t, T) \simeq \begin{cases} 
\lambda(T)t^{1/z_d} & \mathcal{R} \ll L_c(T) \text{ curvature-driven} \\
L_c(T)(\ln t/t_0)^{1/\psi} & \mathcal{R} \gg L_c(T) \text{ activated}
\end{cases} \]

with \( L_c(T) \) a growing function of \( T \).

Inverting times as a function of length \( t \simeq [\mathcal{R}/\lambda(T)]z_d e^{\mathcal{R}/L_c(T)} \)

At short times this equation can be approximated by an effective power law with a \( T \)-dependent exponent:

\[ t \simeq \mathcal{R}^{\tilde{z}_d(T)} \quad \tilde{z}_d(T) \simeq z_d [1 + ct/L_c(T)] \]

Fisher & Huse, Paul, Rieger & Schehr, etc. Crossover in Bustingorry et al. 09
Planar magnets

**Schrielen pattern**: gray scale according to $\sin^2 2\theta_i(t)$

Spin-waves Vortices (planar spins turn around these points)

After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t, T) \sim \lambda(T) \left[ t / \ln \left( t / t_0(T) \right) \right]^{1/2}$$

Yurke *et al* 93, Bray & Rutenberg 94, Jelic & LFC 11
Frustrated magnets

e.g., 2d spin ice or vertex models

Stripe growth in the FM phase

Anisotropic growth, $\mathcal{R}_\perp(t, T)$ and $\mathcal{R}_\parallel(t, T)$

Levis & LFC 11
Universality classes

as classified by the growing length

\[ \mathcal{R}(t, T) \simeq \begin{cases} 
\lambda(T) \, t^{1/2} & \text{scalar NCOP} \quad z_d = 2 \\
\lambda(T) \, t^{1/3} & \text{scalar COP} \quad z_d = 3 \\
\lambda(T) \left( \frac{t}{\ln t} \right)^{1/2} & \text{planar NCOP in} \quad d = 2 \\
n\text{etc.} 
\end{cases} \]

Temperature and other microscopic parameters appear in the prefactor.
Universality classes

as classified by the growing length

\[ R(t, T) \approx \begin{cases} 
\lambda(T) \, t^{1/2} & \text{scalar NCOP} \\
\lambda(T) \, t^{1/3} & \text{scalar COP} \\
\lambda(T) \left( \frac{t}{\ln t} \right)^{1/2} & \text{planar NCOP in} \\
e tc.
\end{cases} \]

\[ z_d = 2 \]

\[ z_d = 3 \]

\[ d = 2 \]

Super-universality?

Are scaling functions independent of temperature and other parameters?

Review Bray 94, Corberi, Lippiello, Mukherjee, Puri & Zannetti 11
Plan of the lecture

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. Dynamic scaling (again)
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies
Dynamic scaling

Scaling functions

very early MC simulations Lebowitz et al 70s & experiments

One identifies a growing linear size of equilibrated patches

\[ R(t, T) \]

If this is the only length governing the dynamics, the space-time correlation functions should scale with \( R(t, T) \) according to

At \( T = T_c \)

\[ C(r, t) \sim C_{eq}(r) f_c\left( \frac{r}{R_c(t)} \right) \]

Scaling fct \( f_c \)

At \( T < T_c \)

\[ C(r, t) \sim C_{eq}^c(r) + m_{eq}^2 f\left( \frac{r}{R(t, T)} \right) \]

Scaling fct \( f ? \)

Reviews Hohenberg & Halperin 77 (critical) Bray 94 (sub-critical)
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Two-time self-correlation

e.g., MC simulation of the 2dIM at $T < T_c$

$$C(t, t_w) = N^{-1} \sum_{i=1}^{N} \langle s_i(t) s_i(t_w) \rangle$$

Stationary relaxation

Aging decay

Separation of time-scales: stationary – aging

$$C(t, t_w) = C_{st}(t - t_w) + m_{eq}^2 f \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right)$$

$$C_{st}(0) = 1 - m_{eq}^2, \lim_{x \to \infty} C_{st}(x) = 0, f(1) = 1, \lim_{x \to \infty} f(x) = 0.$$
Two-time self-correlation

Comparison

Critical coarsening \((T = T_c)\)  
Sub-critical coarsening \((T < T_c)\)

Separation of time-scales

Multiplicative  
Additive
Aging

Older samples relax more slowly

Older samples need more time to relax

spontaneously (correlation functions)

after a change in conditions (response functions)

$t_w$ is the time that measures the age of the system

Huge literature on this phenomenology. Some reviews of experimental measurements were written by Struick on polymer glasses, Vincent et al. & Nordblad et al. on spin-glasses, McKenna et al. on all kinds of glasses.
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The perturbation couples linearly to the observable $B[\{s_i\}]$

$$H \rightarrow H - hB[\{s_i\}]$$

The linear instantaneous response of another observable $A[\{s_i\}]$ is

$$R_{AB}(t, t_w) \equiv \left\langle \frac{\delta A[\{s_i\]}(t)}{\delta h(t_w)} \right|_{h=0} \right)$$

The linear integrated response or dc susceptibility is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' R_{AB}(t, t')$$
Linear response

Critical and sub-critical coarsening

Critical coarsening

\[ \chi(t, t_w) = \beta - \chi_{eq}(t - t_w) g \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right) \]

Sub-critical coarsening

\[ \chi(t, t_w) = \chi_{eq}(t - t_w) + \left[ \mathcal{R}(t_w, T) \right]^{-a \chi} g \left( \frac{\mathcal{R}(t, T)}{\mathcal{R}(t_w, T)} \right) \]

In both cases: \( \chi_{eq}(t - t_w) = -(k_B T)^{-1} dC_{eq}(t - t_w)/d(t - t_w) \).

To be proven in the 3rd Lecture

Reviews

Crisanti & Ritort 03, Calabrese & Gambassi 05, Corberi et al. 07, LFC 11
Linear response

Sub-critical coarsening in the MC dynamics of 2dIM

Lippiello, Corberi & Zannetti 05
Linear response

Coarsening vs glassy

There is no (weak) long-term memory in the coarsening problem. Just the stationary part will remain asymptotically, contrary to the sketch on the right for glasses & spin-glasses.
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8. **Mean-field models**
9. Modern studies
The spherical $\rho = 2$ model

\[
H = -\sum_{ij} J_{ij} s_is_j + z \left( \sum_i s_i^2 - N \right)
\]

- Fully connected interactions
- Gaussian distributed interaction strengths $J_{ij}$
- Spherical spins $\sum_i s_i^2 = N$
- $z$ is a Lagrange multiplier

\[
\rho(\lambda_\mu) \propto \sqrt{(2J)^2 - \lambda_\mu^2}
\]

Key: the largest eigenvalue becomes \textbf{diffusive},

\[
\lambda_{\text{max}} - z_\infty = 0
\]

Same scaling laws for two-time corr. and resp. but no space dependence
The O(N) model

Upgrade the field to a vector $\phi \mapsto \vec{\phi}$ with $a = 1, \ldots, N$ components

$$\vec{\phi} = (\phi_1, \ldots, \phi_N)$$

The (over-damped) Ginzburg-Landau equation is now

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = -\frac{\delta F[\vec{\phi}]}{\delta \phi_a(\vec{r}, t)} + \xi_a(\vec{r}, t)$$

The $N \to \infty$ limit allows one to decouple the vector components :

$$\phi_a(\vec{r}, t)[\mu - \frac{1}{N} \sum_{b=1}^{N} \phi_b^2(\vec{r}, t)] \mapsto \phi_a(\vec{r}, t) z(t)$$

and the equations are now linear with a global constraint.

Coarsening is linked to the growth of the diffusive $\vec{k} = 0$ mode.
The O(N) model

Upgrade the field to a vector $\phi \mapsto \vec{\phi}$ with $a = 1, \ldots, N$ components

$$\vec{\phi} = (\phi_1, \ldots, \phi_N)$$

The equations are now linear with a global constraint

$$\gamma_0 \partial_t \phi_a(\vec{r}, t) = \nabla^2 \phi_a(\vec{r}, t) + z(t) \phi_a(\vec{r}, t) + \xi_a(\vec{r}, t)$$

and

$$z(t) = \mu - N^{-1} \sum_a \phi_a^2(\vec{r}, t)$$

Solve for $\phi_a(\vec{r}, t)$ as a function of $z(t)$ and then impose the constraint to fix $z(t)$.

Coarsening is linked to the growth of the $\vec{k} = 0$ mode, i.e. tendency to homogeneous order.
Summary

- At and below $T_c$ growth of equilibrium structures.

- The linear size of the equilibrium patches is measured by $\mathcal{R}(t, T)$.

- At $T_c$ vanishing order parameter
  
  Multiplicative scaling
  
  \[ C \approx C_{eq} C_{ag}; \chi \approx \chi_{eq} \chi_{ag} \]

- Below $T_c$ non-vanishing order parameter
  
  Additive scaling
  
  \[ C \approx C_{eq} + C_{ag}; \chi \approx \chi_{eq} + \chi_{ag} \]

- In both cases $C_{ag}$ is finite while $\chi_{ag}$ vanishes asymptotically.

We shall discuss $\chi$ and how it compares to $C$ later.
Phase ordering kinetics

The lecture was about

- Growth of equilibrium patches at $T_c$ and below $T_c$.
- Divergence of $t_{eq}(L)$ with the system size.
- Existence of a single growing length $R(t, T)$
- Separation of time-scales and dynamic scaling, e.g. $C = C_{eq} + C_{ag}$.
- Two kinds of correlations: Space-time and two-time ones.
- Dynamic universality classes at and below $T_c$.
- The more tricky/rich linear susceptibility.

Is there a static growing length in all systems with slow dynamics?  Which one?
Plan of the lecture

1. The phenomenon
2. Theoretical setting
3. Critical and sub-critical quenches
4. Dynamic scaling
5. Dynamic universality classes
6. Two-time correlations and ageing
7. Two-time responses and loss of memory
8. Mean-field models
9. Modern studies
Multiplicative noise

Numerical integration of the scalar field equations

NCOP

\[ R(t, T) \simeq t^{1/2} \]

COP

\[ R(t, T) \simeq t^{1/3} \]

Ibañes, García-Ojalvo, Toral & Sancho 00
Voter model
Similar questions can be asked in very well-known problems in math, e.g.

**Dynamics of a voter model starting from a random initial condition**

- Purely dynamic, violation of detailed balance, no phase transition
- Two absorbing states
- The *dynamic mechanism* towards absorption is understood:
  - Domain growth is driven by interfacial noise
2d Voter Model (VM)

Archetypical example of opinion dynamics

\( H \) does not exist - kinetic model

\[ s_i = \pm 1 \] Ising spins that sit on the vertices of a lattice.

**Voter update rule**

- choose a spin at random, say \( s_i \)
- choose one of its 2d neighbours at random, say \( s_j \)
- set \( s_i = s_j \)

In two dimensions full consensus, *i.e.* \( m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1 \) is reached in a timescale \( t_C \approx L^2 \) (with \( \ln L \) corrections)

Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86
Phase ordering kinetics

$s_i = \pm 1$ at $t = 0$ MCs, snapshots at $t = 4, 64, 512, 4096$ MCs

Ising

$T = 0$

$T_c$

Voter
Active matter dynamics

2d dumbbell system

$|\psi_{6i}|$ at $\phi = 0.74$ and $\text{Pe} = 10$ (coarsening towards co-existence)
Percolation issues
2d square IM at T=0

t=0.0
2d square IM at T=0

\[ t = 0.57533 \]
2d square IM at T=0

\[ t = 0.94844 \]
2d square IM at $T=0$

$t=2.00847$
2d square IM at T=0

t=2.57898
2d square IM at T=0

t=6.58423
2d square IM at T=0

t=7.46144
2d square IM at T=0

The percolating structure was decided at $t_p \simeq 8$ MCs

Arenzoni, Bray, LFC & Sicilia 07  Blanchard, Corberi, LFC & Picco 14
Complex field & cold atoms
Complex field theory in $3d$

Relativistic bosons; $^4$He, type II superconductors, cosmology, etc.

\[-c^{-2} \ddot{\psi} + \nabla^2 \psi + 2i\mu \dot{\psi} = g(\psi^2 - \rho)\psi\]

$c$ is the velocity of light, $\rho$ and $g$ parameters in (Mexican hat) potential.

Limits

\[\mu \to 0 : \quad -c^{-2} \ddot{\psi} + \nabla^2 \psi = g(|\psi|^2 - \rho)\psi\]  
Goldstone

\[c \to \infty : \quad 2i\mu \dot{\psi} + \nabla^2 \psi = g(|\psi|^2 - \rho)\psi\]  
Gross-Pitaevskii

models
Complex field theory in $3d$

Relativistic bosons; $^4$He, type II superconductors, cosmology, etc.

$$-c^{-2}\ddot{\psi} + \nabla^2 \psi + 2i\mu\dot{\psi} = g(\psi^2 - \rho)\psi$$

The energy functional

$$E = \int d^3x \left( c^{-2}|\dot{\psi}|^2 + |\vec{\nabla}\psi|^2 - g\rho\psi^2 + g\psi^4 \right)$$

is conserved under the dynamics.

The energy is minimised by the static configuration $\psi = \sqrt{\rho} e^{i\chi}$ with $\chi = ct$

There are static vortex solutions, e.g. $\psi(x) = f(r) e^{in\theta}$ with $f(0) = 0$ and $f(r \to \infty) = \sqrt{\rho}$, and $n \in \mathbb{Z}$ (thin tubes at the centre of which the field vanishes and the phase turns around).

Tsubota, Kasamatsu & Kobayashi 13, Kobayashi & Nitta 15, etc.
Complex field theory in \textit{3d}

Stochastic noise and dissipation added

\[-c^{-2} \ddot{\psi} + \nabla^2 \psi + 2i\mu \dot{\psi} - \gamma \dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T} \xi\]

Langevin-like dynamics

\[-\gamma \text{ viscosity, } \xi \text{ complex Gaussian white noise in normal form} \]

\[
\langle \xi_i(x, t) \rangle = 0 \text{ and } \langle \xi_i(x, t_1) \xi_j(y, t_2) \rangle = \delta_{ij} \delta^{(3)}(x - y)\delta(t_1 - t_2)
\]

Passage to Fokker-Planck formalism allows to show that the dynamics takes the system to

\[
\lim_{t \to \infty} P(\psi, t) = P_{GB}(\psi) \propto e^{-\beta E}
\]

Kobayashi & LFC 16
Complex field theory in $3d$

Relativistic bosons; $^4\text{He}$, type II superconductors, cosmology, etc.

\[
-c^{-2} \ddot{\psi} + \nabla^2 \psi + 2i\mu \dot{\psi} - \gamma \dot{\psi} = g(\psi^2 - \rho)\psi - \sqrt{\gamma T} \xi
\]

Langevin-like dynamics

$-\gamma$ viscosity, $\xi$ Gaussian white noise in normal form

In the limit $c \to \infty$, the stochastic Gross-Pitaevskii equation

\[
(2i\mu - \gamma) \dot{\psi} = -\nabla^2 \psi + g(\psi^2 - \rho)\psi + \sqrt{\gamma T} \xi
\]

Gardiner et al 00s
$3d$ **XY lattice model**  

**Archetypical classical magnetic example**

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$J > 0$ ferromagnetic coupling constant.

$\langle ij \rangle$ sum over nearest-neighbours on a $3d$ lattice  

$\vec{s}_i$ planar spins: two components with constant modulus $\Rightarrow$ angle $\theta_i$.

Second order phase transition with spontaneous symm breaking at $T_c > 0$.

Order parameter: spin-alignment, $\vec{m} \equiv N^{-1} \sum_i \langle \vec{s}_i \rangle$.

No intrinsic spin dynamics, **Monte Carlo rules** mimic coupling to thermal bath.

Non-conserved order parameter dynamics $[\uparrow \downarrow \text{towards} \uparrow \uparrow]$ etc. allowed.
Statics

Phase transition and order parameter in the field equation

\[ L^3 \, m = \left| \sum_{ijk} \langle \psi_{ijk} \rangle \right| \]

critical temperature

\[ T_c = 2.26 \]

critical exponent

\[ \beta = 0.347 \]

Kobayashi & LFC 16

\( T_c \) and critical exponents from kurtosis (Binder parameter), susceptibility, specific heat, etc. Values compatible w/results from simulations Ballesteros et al. 96, Hasenbusch & Török 99 and \( \epsilon \) expansion Guida & Zinn-Justin 98, Täuber & Diehl 14 for models in the same universality class.
Periodic boundary conditions (torus) implies that the vortex lines are closed, i.e. loops.

Stochastic reconnection rule.

All vortex loops in blue, the longest one in red.
Dynamics after a quench

with $g$ the control parameter

In the picture: annealing with finite rate.

Infinitely fast quench: $T \gg T_c$ for $t < 0$ and $T = 0$ for $t > 0$
Complex field theory in $3d$

Progressive elimination of vortex loops after a quench

$T \gg T_c$

$T = 0$

$t = 0$

$t = 3$

$t = 5$

As $\rho_{\text{vortex}} \downarrow$ the reconnection rule loses importance

Kobayashi & LFC 16
Slow cooling & Kibble-Zurek
Finite rate quenching protocol

How is the scaling modified for a very slow quenching rate?

\[ \Delta g \equiv g(t) - g_c = -t/\tau_Q \]

with \( \tau_{Q_1} < \tau_{Q_2} < \tau_{Q_3} < \tau_{Q_4} \)

Standard time parametrization

\[ g(t) = g_c - t/\tau_Q \]

Simplicity argument: linear cooling could be thought of as an approximation of any cooling procedure close to \( g_c \).
They should affect the Cosmic Microwave Background, double quasars, etc.

Picture from M. Kunz’s group (Université de Genève)
Topological defects

instantaneous configurations

Domain walls in the $2d$IM

Vortices in the $3d$ xy model

One can give a precise mathematical definition but the visual one is enough
Density of topological defects

Kibble-Zurek mechanics for 2nd order phase transitions

The three basic assumptions

- Defects are created close to the critical point.
- Their density in the ordered phase is inherited from the value it takes when the system falls out of equilibrium on the symmetric side of the critical point. It is determined by Critical scaling above $T_c$
- The dynamics in the ordered phase is so slow that it can be neglected.

and one claim

- results are universal.

that we critically revisited within ‘thermal’ phase transitions
Topological defects

after an instantaneous quench : dynamic scaling

\[ \Delta n(t) \simeq [R(t, T)]^{-d} \simeq [\lambda(T(t))]^{-d} t^{-d/z_d} \]

Remember the initial \((g \to \infty)\) configuration: already there!