Infinite U(1) Symmetry of the Quasi-Linear Approximation

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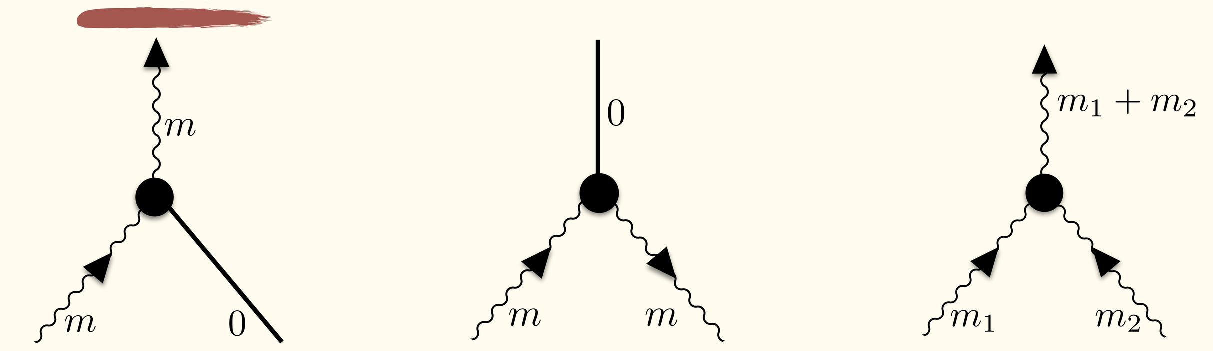
ABSTRACT

Particle-relabeling symmetry of inviscid fluid equations, equivalent in the case of incompressible fluids to the infinite dimensional group of volume-preserving diffeomorphisms, is broken by the quasi-linear approximation. Instead the equations of motion are invariant under an infinite U(1) symmetry as the phase of each wave number can be independently varied, reflecting the absence of wave + wave —> wave interactions. The symmetry is analogous to the infinite U(1) symmetry of Landau Fermi liquids as quasiparticles near the Fermi surface do not scatter. We investigate the physical implications of this infinite U(1) symmetry and attempt to identify corresponding conserved quantities.

Nonlinear Dynamics

Linear Approximation



Quasi-Linear Approximation

General Formalism

Barotropic Vorticity Equation

$$\partial_t \omega = J\Big(\omega + f, \psi\Big)$$

Reynolds Decomposition

$$\omega(\lambda,\phi,t) = \bar{\omega}(\phi,t) + \omega'(\lambda,\phi,t)$$

Decomposition in terms of zonal modes

$$\omega'(\lambda,\phi,t) = \sum_{m=1}^{\infty}\Big(\omega^m(\phi,t)e^{im\lambda} + \omega^{m*}(\phi,t)e^{-im\lambda}\Big),$$

where $\omega^m(\phi,t) = \sum_{\ell \geq |m|} \omega^m_\ell(t)\, P^m_\ell(\sin\phi)$

Hamiltonian for Linear Waves [1,2]

$$\mathscr{H}_{\text{linear}} = \frac{1}{2}\int\Big(|\nabla\psi'|^2 + \frac{d\bar{\psi}}{d(\bar{\omega}+f)}\omega'^2\Big)a^2 d\Omega$$

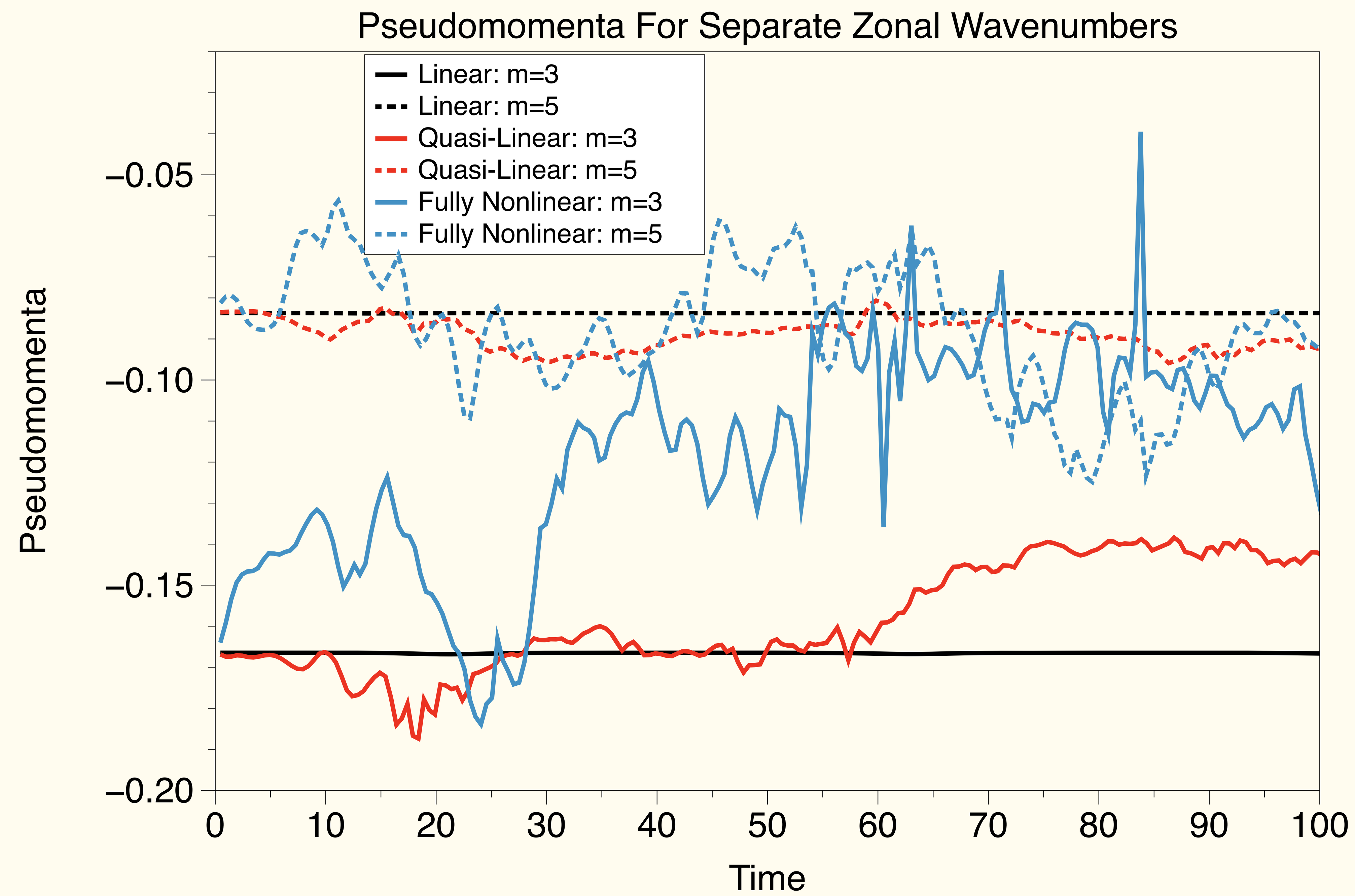
Noether's Theorem

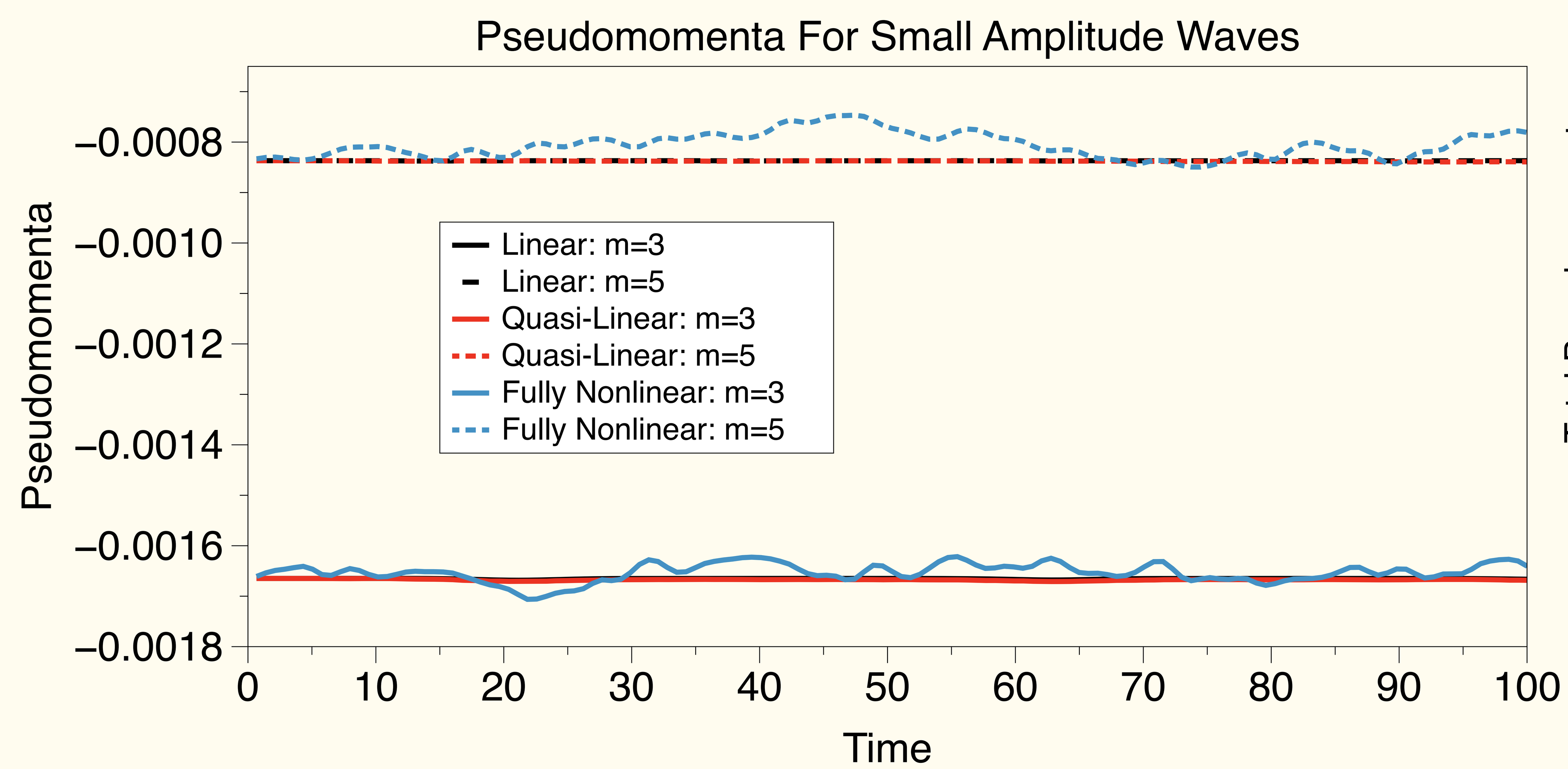
SYMMETRY

↓

CONSERVATION

Emmy Noether





Pseudomomenta for Each Mode [3]

$$\mathscr{M}_m = \int \frac{|\omega^m(\phi,t)|^2}{\partial_\phi(\bar{\omega}+f)}\cos^2\phi\, d\phi$$

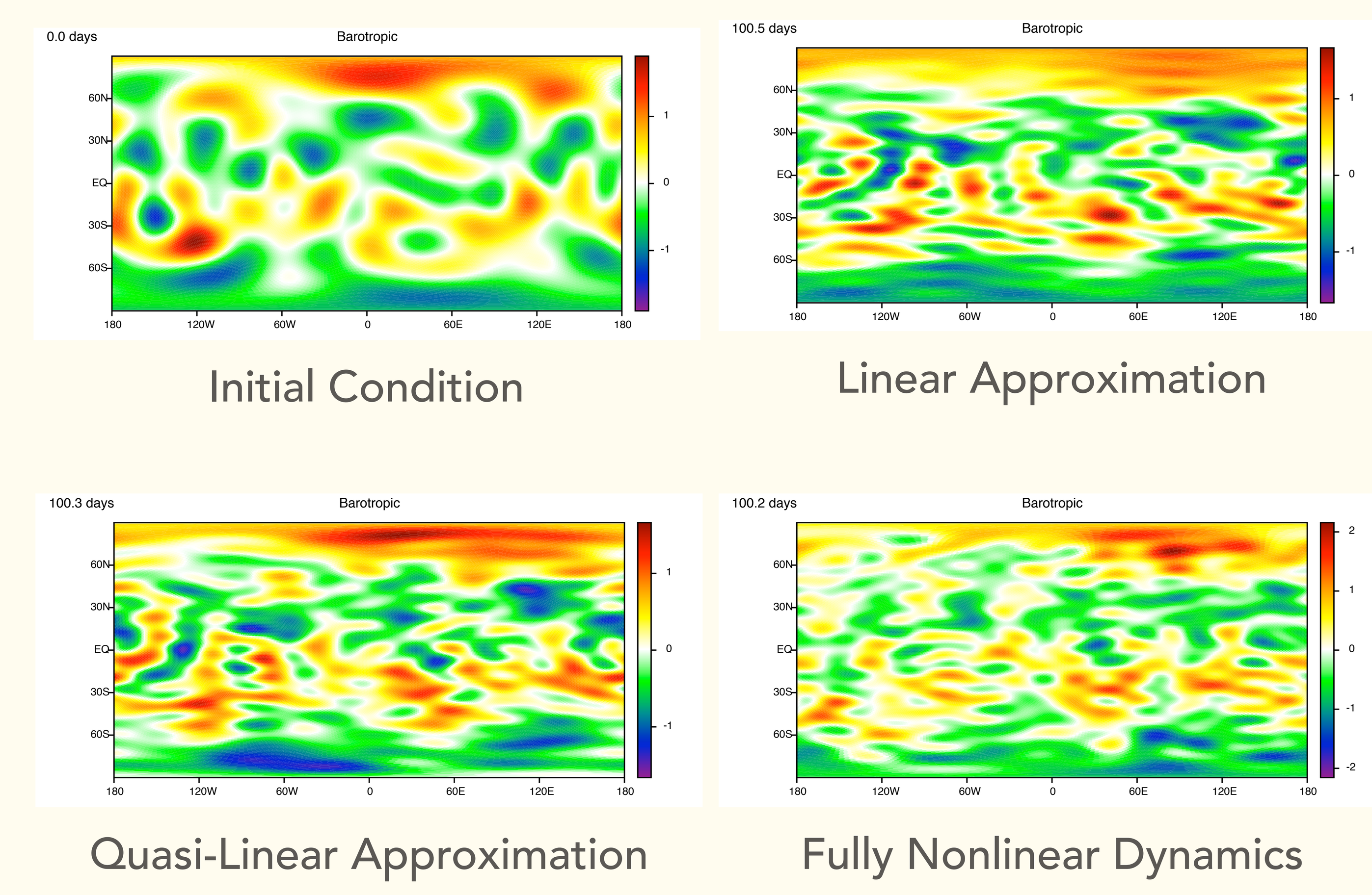
References

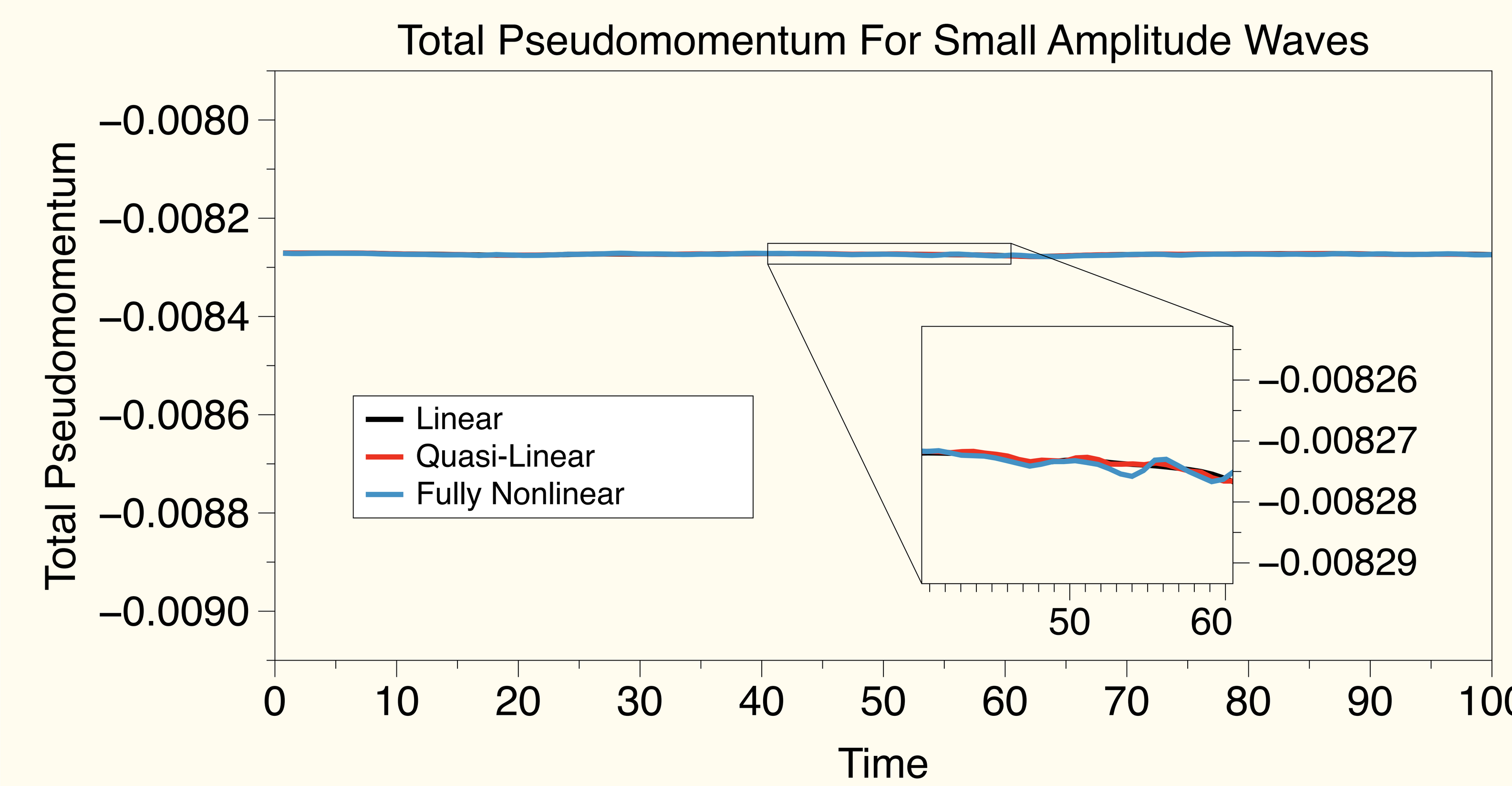
[1] T. G. Shepherd. Symmetries, conservation laws, and hamiltonian structure in geophysical fluid dynamics. In Advances in Geophysics, volume 32, pages 287–338. Elsevier, 1990.

[2] V. I. Arnol'd. On conditions for non-linear stability of plane stationary curvilinear flows of an ideal fluid. In Doklady Akademii Nauk, volume 162, pages 975–978. Russian Academy of Sciences, 1965.

[3] I. M. Held and Peter J Phillips. Linear and nonliear barotropic decay on the sphere. Journal of the atmospheric sciences, 44(1):200–207, 1987.

Barotropic Dynamics of Vorticity on a Sphere





Discussion

The infinite U(1) symmetry of linear waves manifests, by Noether's theorem, as separate conservation of the pseudomomenta for each zonal wavenumber. The pseudomomenta are approximately conserved for quasilinear dynamics due to the separation in time scales between the evolution of the zonal mean and the waves. Whether or not an action principle or a Hamiltonian can be found that generates the quasilinear dynamics remains an open question; if one can be found then it should be possible to find exactly conserved pseudomomenta as the quasilinear system retains the infinite U(1) symmetry. Pseudomomenta are not conserved by the fully nonlinear dynamics.

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