I. GENERALISATIONS OF IQHE

To discuss generalisations of the IQHE we need first to consider symmetry classes for disordered conductors. In the following we aim to give a simple-minded account of how the classes arise.

A. Symmetry Classes

The notion of symmetry classes for random Hamiltonians first arose in nuclear physics when random matrices were studied as models for the statistical properties of highly excited states in nuclei. At that time three 'Wigner-Dyson' symmetry classes were identified, according to whether a system has no time-reversal symmetry, or has time-reversal symmetry without or with Kramers degeneracy. These are known respectively as the unitary, orthogonal and sympletic classes. Recall that Kramers degeneracy arises in systems with half odd-integer spin, and that time-reversal of spin vectors may be written in terms of the conventional Pauli matrices as $\tau_y \vec{\tau}^* \tau_y = -\vec{\tau}$. Then for a Hamiltonian $\mathcal{H}$ the time-reversal operations are

$$\text{orthogonal}: \quad \mathcal{H}^* = \mathcal{H}, \qquad \text{symplectic}: \quad \tau_y \mathcal{H}^* \tau_y = \mathcal{H}\,.$$

It is a feature of the Wigner-Dyson classes that no energy is special, and that statistical properties are unchanged under a shift in energy.

The so-called additional symmetry classes arise in systems that do have a special energy, which we take to be zero. They were first classified by Altland and Zirnbauer [Phys. Rev. B **55**, 1142 (1997)]. The significance of the special energy is that eigenvalues occur in $\pm E$ pairs for each realisation of the disordered Hamiltonian. In consequence, there is an operator X that transforms between states in the pair. Suppose

$$\mathcal{H}\Psi = E\Psi\,.$$

Then there are two possibilities. Either

$$\mathcal{H}(X\Psi) = -E(X\Psi), \qquad \text{implying} \quad X^{-1}\mathcal{H}X = -\mathcal{H},$$

or we need to use complex conjugation to generate the second state in the pair, and

$$\mathcal{H}(X\Psi^*) = -E(X\Psi^*), \qquad \text{implying} \quad X^{-1}\mathcal{H}^*X = -\mathcal{H}\,.$$

A simple example is the one-dimensional chain with disordered nearest neighbour hopping. With site labels n and eigenfunction amplitudes ψ_n, the eigenvalue equation is

$$E\psi_n = t_{n,n+1}\psi_{n+1} + t_{n,n-1}\psi_{n-1}$$

and the transformation X takes the form $(X\psi)_n = (-1)^n\psi_n$.

1. *Chiral classes*

We would now like to enumerate the additional symmetry classes. Instead of following the original route via Cartan's classification of symmetric spaces, I present some informal arguments that I first heard from T. Senthil. In general, because eigenvalues appear in pairs, we expect the Hamiltonians we are concerned with to have a 2×2 structure. Let σ_α be the Pauli matrices acting in this space (the σ's are distinct from the τ's). We can write an arbitrary Hamiltonian in the form

$$\mathcal{H} = h_0 \mathbb{1} + \vec{h}\cdot\vec{\sigma},$$

where h_0, h_x, h_y and h_z are Hermitian matrices.

We want to understand what the restrictions are on these matrices. We will deal separately with the two possibilities for transforming between states in the pair. Consider first $X^{-1}\mathcal{H}X = -\mathcal{H}$. This implies that X is an element of $SU(2)$ so that we can parameterise it as $X = e^{i\alpha\hat{n}\cdot\vec{\sigma}}$. Moreover, since $X^2 \propto \mathbb{1}$ we can set $\alpha = \pi/2$. The direction of $\hat{n}$ is now a matter of convention. We pick $\hat{n} = \hat{z}$ giving $X = i\sigma_z$. The symmetry relation then reads

$$\sigma_z\mathcal{H}\sigma_z = -\mathcal{H}\,,$$

implying that $h_0 = h_z = 0$. In this way we arrive at the chiral ensembles, with

$$\mathcal{H} = \begin{pmatrix} 0 & h_x - ih_y \\ h_x + ih_y & 0 \end{pmatrix}.$$

The same questions about time-reversal symmetry arise for the chiral classes as in the Wigner-Dyson cases, and so we have chiral orthogonal, unitary and symplectic classes.

2. *BdG classes*

Consider next the alternative transformation $X^{-1}\mathcal{H}^*X = -\mathcal{H}$. As before we use the parameterisation $X = e^{i\alpha\hat{n}\cdot\vec{\sigma}}$. We also have $XX^* \propto \mathbb{1}$, and we treat separately the two cases $XX^* = +\mathbb{1}$ and $XX^* = -\mathbb{1}$.

In the first case we have $X^* = X^{-1}$, which implies that $n_y = 0$. We pick $\hat{n} = \hat{x}$ and $\alpha = \pi/2$, getting the condition

$$\sigma_x \mathcal{H}^* \sigma_x = -\mathcal{H}\,, \tag{1}$$

which has the solution $h_x^* = -h_x$, $h_y^* = -h_y$, $h_z^* = h_z$ and $h_0^* = -h_0$, so that

$$\mathcal{H} = \begin{pmatrix} h_0 + h_z & h_x - ih_y \\ h_x + ih_y & h_0 - h_z \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & -a^{\mathrm{T}} \end{pmatrix}. \tag{2}$$

Next we treat the second of our alternatives, $XX^* = -\mathbb{1}$, which implies $X = i\sigma_y$ and hence

$$\sigma_y \mathcal{H}^* \sigma_y = -\mathcal{H}\,,$$

with the solution $h_0^* = -h_0$ and $\vec{h}^* = \vec{h}$. Then

$$\mathcal{H} = \begin{pmatrix} h_0 + h_z & h_x - ih_y \\ h_x + ih_y & h_0 - h_z \end{pmatrix} = \begin{pmatrix} a & b \\ b^* & -a^{\mathrm{T}} \end{pmatrix}. \tag{3}$$

We can find realisations of the Hamiltonians of Eqns (2) and (3) as Bogoliubov de-Gennes (BdG) Hamiltonians for superconductors, for spinless and spin-singlet systems respectively. To see this, let's review the form taken by such superconductor Hamiltonians.

In the spinless case, in terms of fermion creation and annihilations operators c_α and $c_\alpha^\dagger$ for orbitals α, $\beta \ldots$, we have

$$\mathcal{H} = \sum_{\alpha,\beta} \left[h_{\alpha,\beta} c_\alpha^\dagger c_\beta + \frac{1}{2} \left(\Delta_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger + \Delta_{\alpha\beta}^* c_\beta c_\alpha \right) \right].$$

Note that Hermiticity and fermion anticommutation relations imply $h^\dagger = h$ and $\Delta^{\mathrm{T}} = -\Delta$. We can re-write this Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2} \left(\mathbf{c}^\dagger, \mathbf{c} \right) \begin{pmatrix} h & \Delta \\ -\Delta^* & -h^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \mathbf{c}^\dagger \end{pmatrix} + \mathrm{const}, \tag{4}$$

which matches that of Eq. (2).

In the singlet case, a generic form for the BdG Hamiltonian is

$$\mathcal{H} = \sum_{\alpha\beta} \left[h_{\alpha\beta} \left(c_{\alpha\uparrow}^\dagger c_{\beta\uparrow} + c_{\alpha\downarrow}^\dagger c_{\beta\downarrow} \right) + \left(\Delta_{\alpha\beta} c_{\alpha\uparrow}^\dagger c_{\beta\downarrow}^\dagger + \Delta_{\alpha\beta}^* c_{\beta\downarrow} c_{\alpha\uparrow} \right) \right].$$

Spin rotation symmetry requires $\Delta^{\mathrm{T}} = \Delta$. We can make conservation of spin explicit by doing a particle-hole transformation on one spin direction, taking $\gamma_{\alpha\uparrow}^\dagger = c_{\alpha\uparrow}^\dagger$ and $\gamma_{\alpha\downarrow}^\dagger = c_{\alpha\downarrow}$. Then the singlet BdG Hamiltonian takes the form

$$\begin{aligned}\mathcal{H} &= \sum_{\alpha\beta}\left[h_{\alpha\beta}\left(\gamma^{\dagger}_{\alpha\uparrow}\gamma_{\beta\uparrow}-\gamma^{\dagger}_{\beta\downarrow}\gamma_{\alpha\downarrow}\right)+\left(\Delta_{\alpha\beta}\gamma^{\dagger}_{\alpha\uparrow}\gamma_{\beta\downarrow}+\Delta^{*}_{\alpha\beta}\gamma^{\dagger}_{\beta\downarrow}\gamma_{\alpha\uparrow}\right)\right]. \\ &= \left(\boldsymbol{\gamma}^{\dagger}_{\uparrow},\boldsymbol{\gamma}^{\dagger}_{\downarrow}\right)\begin{pmatrix} h & \Delta \\ \Delta^{*} & -h^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\boldsymbol{\gamma}_{\uparrow} \\ \boldsymbol{\gamma}_{\downarrow}\end{pmatrix}+\text{const},\end{aligned}$$

which matches that of Eq. (3). Each of these BdG Hamiltonians (spinless or spin singlet) gives rise to two symmetry classes – with or without time-reversal symmetry.

3. Summary

In summary, we have a grand total of 10 classes: the 3 Wigner-Dyson ones, the 3 chiral ones, 2 for spineless superconductors, and 2 for spin singlet superconductors. The 10 classes are listed using standard nomenclature in the table, and the three cases showing versions of the quantum Hall effect are picked out. One of the criteria for showing a QHE is of course that the symmetry class should not have time reversal symmetry. This selects one class from each of the three groupings termed Wigner-Dyson, spineless BdG and singlet BdG. A further criterion is that RG flow at zero generalised Hall conductance should be to an Anderson insulator, which is true for these three examples but not in the chiral unitary class. Hence we end up with two generalisations of the IQHE — in classes C and D.

Wigner-Dyson	Orthogonal	AI	
	Unitary	A	IQHE
	Symplectic	AII	
Chiral	Orthogonal	BDI	
	Unitary	AIII	
	Symplectic	CII	
Spinless BdG	with TRS	DIII	
	no TRS	D	Majorana QHE
Singlet BdG	with TRS	CI	
	no TRS	C	SQHE

Some remarks about the BdG versions of the QHE are in order. First we should ask what transport effect will show a Hall plateau. We are not concerned with charge transport, since the BdG Hamiltonians do not conserve quasiparticle number, and anyway quasiparticle transport will be short-circuited by the superconducting condensate. Instead, we consider the quantities that are conserved, and so expect plateaus in the thermal Hall conductivity for both class C and class D, and additionally in the spin conductivity for class C.

To make contact with other problems of current interest, it is useful to re-visit out discussion of spinless superconductors, picking up the discussion from Eq. (1), which reads $\sigma_x \mathcal{H}^* \sigma_x = -\mathcal{H}$. Making the transformation $H = s^\dagger \mathcal{H} s$, where $s^2 = \sigma_x$, we have an alternative representation satisfying $H^* = -H$, or (since $H^\dagger = H$), $H^{\mathrm{T}} = -H$. This is exactly the condition arising for Majorana Hamiltonians of the form $\frac{i}{2}\sum_{\alpha\beta} H_{\alpha\beta} b_\alpha b_\beta$ with Majorana operators b_α, b_β satisfying the standard relations: $b_\alpha^\dagger = b_\alpha$, $\{b_\alpha, b_\beta\} = 0$ for $\alpha \neq \beta$ and $(b_\alpha)^2 = 1$.

B. Network models

Two main ingredients in the definition of the network model are the amplitudes z_l on links and the phases $e^{i\phi_l}$ acquired in propagation along a link. An obvious extension is the allow the amplitudes to be N-component vectors. Then the link phases are replaced by $N \times N$ matrices W.

Since the link phases arise during propagation, we can think of them as given in terms of a Hamiltonian $\mathcal{H}$ via $W = e^{i\mathcal{H}}$. Then the Wigner-Dyson unitary symmetry class for $N \times N$ matrices $\mathcal{H}$ generates $U(N)$ link phases. This generalisation does not change the symmetry. It therefore gives N copies of the IQHE transition as the node parameter varies between $\alpha = 0$ and $\alpha = \pi/2$, with each transition occurring at a distinct value of α.

Alternatively, we can consider the BdG symmetry classes. For class C the condition $\sigma_y \mathcal{H}^* \sigma_y = -\mathcal{H}$ implies that W belongs to $Sp(2n)$, and in the simplest case $Sp(2) \sim SU(2)$. Thus we can model a plateau transition in symmetry class C using a network model with two-component amplitudes on the links and $SU(2)$ link phases. This does indeed give a transition in a different universality class from the standard IQHE.

For class D, using the representation $H^* = -H$ we have link phases $W = e^{iH}$ that are $O(N)$ matrices. In the simplest case, $N = 1$, they are just real phases $W = \pm 1$. Again, this gives behaviour distinct from the standard IQHE (and from class C), with both a plateau transition and a metallic phase, depending on details of the model.